Geometric Algebra 3
Dirac Theory and
Multiparticle Systems

Chris Doran
Astrophysics Group
Cavendish Laboratory
Cambridge, UK
Dirac Algebra

- Dirac matrix operators are
  \[
  \hat{\gamma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}, \quad \hat{\gamma}_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
  \]

- These act on 4-component Dirac spinors
  \[
  |\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix}
  \]

- These spinors satisfy a first-order wave equation
  \[
  i\gamma^\mu \partial_\mu |\psi\rangle = m |\psi\rangle
  \]
STA Form

- Adapt the map for Pauli spinors
  \[|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix} \iff \psi = \phi + \eta \sigma_3\]

- Action of the various operators now
  \[
  \begin{align*}
  \hat{\gamma}_\mu |\psi\rangle & \iff \gamma_\mu \psi \gamma_0 \\
i |\psi\rangle & \iff \psi I\sigma_3 \\
\hat{\gamma}_5 |\psi\rangle & \iff \psi \sigma_3
  \end{align*}
  \]

  Imaginary structure still a bivector

Dirac equation
  \[\nabla \psi I\sigma_3 - eA\psi = m\psi \gamma_0\]
Comments

• Dirac equation based on the spacetime vector derivative
• Same as the Maxwell equation, so similar propagator structure
• Electromagnetic coupling from gauge principal
• Plane wave states have

\[ p\psi = m\psi\gamma_0 \]

A boost plus a rotation
Observables

- Observables are
  1. Gauge invariant
  2. Transform covariantly under Lorentz group

\[ \psi \mapsto R\psi \]

<table>
<thead>
<tr>
<th>Bilinear covariant</th>
<th>Standard form</th>
<th>STA equivalent</th>
<th>Frame-free form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar</td>
<td>( \langle \bar{\psi}</td>
<td>\psi \rangle )</td>
<td>( \langle \psi \bar{\psi} \rangle )</td>
</tr>
<tr>
<td>Vector</td>
<td>( \langle \bar{\psi}</td>
<td>i\gamma_\mu</td>
<td>\psi \rangle )</td>
</tr>
<tr>
<td>Bivector</td>
<td>( \langle \bar{\psi}</td>
<td>i\gamma_{\mu \nu}</td>
<td>\psi \rangle )</td>
</tr>
<tr>
<td>Pseudovector</td>
<td>( \langle \bar{\psi}</td>
<td>\gamma_\mu \gamma_5</td>
<td>\psi \rangle )</td>
</tr>
<tr>
<td>Pseudoscalar</td>
<td>( \langle \bar{\psi}</td>
<td>i\gamma_5</td>
<td>\psi \rangle )</td>
</tr>
</tbody>
</table>
Current

- Main observable is the Dirac current
  \[ J = \psi \gamma_0 \overline{\psi} \]
- Satisfies the conservation equation
  \[ \nabla \cdot J = 0 \]
- Understand the observable better by writing
  \[ \psi \overline{\psi} = \rho e^{i\beta} \quad \psi = \rho^{1/2} e^{i\beta/2} R \]
  \[ J = \rho R \gamma_0 \overline{R} \]

A Lorentz transformation
Current 2

• The Dirac current has a wider symmetry group than U(1).
• Take $\psi \mapsto \psi M$
• Require $M \gamma_0 \tilde{M} = \gamma_0$
• Four generators satisfy this requirement
  \[ I\sigma_1, \quad I\sigma_2, \quad I\sigma_3, \quad I \]
• Arbitrary transform $\psi \mapsto \psi e^{Ib} e^{I\phi}$

SU(2) \hspace{1cm} \text{U(1)}
Streamlines

- The conserved current tells us where the probability density flows
- Makes sense to plot current streamlines
- These are genuine, local observables
- Not the same as following a Bohmian interpretation
- No need to insist that a ‘particle’ actually follows a given streamline
- Tunnelling is a good illustration
Tunnelling

- Time component of probability current
- Time component of probability current
- Time component of probability current
- Time component of probability current

[Graphs showing different time components of probability current]
Streamlines

Only front of the packet gets through
Spin Vector

• The Dirac spin observable is
  \[ s = \psi \gamma_3 \tilde{\psi} = \rho R \gamma_0 \tilde{R} \]

• Same structure as used in classical model
• Use a 1D wavepacket to simulate a spin measurement
• Magnetic field simulated by a delta function shock
• Splits the initial packet into 2
Streamlines
Spin Orientation
Multiparticle Space

• Now suppose we want to describe $n$ particles.
• View their trajectories as a path in $4n$ dimensional configuration space
• The vectors generators of this space satisfy

$$\gamma^a_\mu \gamma^b_\nu + \gamma^b_\nu \gamma^a_\mu = \begin{cases} 0 & a \neq b \\ 2\eta_{\mu\nu} & a = b \end{cases}$$

• Generators from different spaces anticommute
• These give a means of projecting out individual particle species
N-Particle Bivectors

• Now form the relative bivectors from separate spaces \( \sigma_i^a = \gamma_i^a \gamma_0^a \)

• These satisfy

\[
\sigma_i^1 \sigma_j^2 = \gamma_i^1 \gamma_0^1 \gamma_j^2 \gamma_0^2 = \gamma_i^1 \gamma_j^2 \gamma_0^2 \gamma_0^1 \\
= \gamma_j^2 \gamma_0^2 \gamma_i^1 \gamma_0^1 = \sigma_j^2 \sigma_i^1
\]

• Bivectors from different spaces commute

• This is the GA implementation of the tensor product

• ‘Explains’ the nature of multiparticle Hilbert space
Complex Structure

• In quantum theory, states all share a single complex structure.

• So in GA, 2 particle quantum states must satisfy

\[ \psi (I\sigma_3)^1 = \psi (I\sigma_3)^2 \]

\[ \psi = -\psi (I\sigma_3)^1 (I\sigma_3)^2 = \psi \frac{1}{2} (1 - (I\sigma_3)^1 (I\sigma_3)^2) \]

• Define the 2 particle correlator

\[ E = \frac{1}{2} \left( 1 - (I\sigma_3)^1 (I\sigma_3)^2 \right) \]

\[ E^2 = E \]
2-Particle States

• Correlator ensures that 2-particle states have 8 real degrees of freedom

• A 2-particle direct-product state is

\[ |\psi, \phi\rangle \leftrightarrow \psi^1 \phi^2 E \]

• Action of imaginary is

\[ i |\psi, \phi\rangle \leftrightarrow \psi^1 \phi^2 E (I\sigma_3)^1 = \psi^1 \phi^2 E (I\sigma_3)^2 = \psi^1 \phi^2 J \]

• Complex structure now generated by \( J \)

\[ J^2 = -E \]
Operators

- Action of 2-particle Pauli operators

\[ \hat{\sigma}_k \otimes \hat{I} |\psi\rangle \leftrightarrow -(I\sigma_k)^1 \psi J \]
\[ \hat{\sigma}_k \otimes \hat{\sigma}_l |\psi\rangle \leftrightarrow -(I\sigma_k)^1 (I\sigma_l)^2 \psi E \]
\[ \hat{I} \otimes \hat{\sigma}_k |\psi\rangle \leftrightarrow -(I\sigma_k^2) \psi J \]

- Inner product

\[ \langle \psi | \phi \rangle \leftrightarrow (\psi, \phi)_q = 2\langle \phi E \bar{\psi} \rangle - 2\langle \phi J \bar{\psi} \rangle \]

- Examples

\[ \langle \psi | \hat{\sigma}_k \otimes \hat{I} |\psi\rangle \leftrightarrow -2(I\sigma_k)^1 \cdot (\psi J \bar{\psi}) \]
\[ \langle \psi | \hat{\sigma}_j \otimes \hat{\sigma}_k |\psi\rangle \leftrightarrow -2\left((I\sigma_j)^1 (I\sigma_k)^2\right) \cdot (\psi E \bar{\psi}) \]
Density Matrices

- A normalised 2 particle density matrix can be expressed as

\[ \hat{\rho} = \frac{1}{4} \left( \hat{I} \otimes \hat{I} + a_k \hat{\sigma}_k \otimes \hat{I} + b_k \hat{I} \otimes \hat{\sigma}_k + c_{jk} \hat{\sigma}_j \otimes \hat{\sigma}_k \right) \]

- So, for example

\[ a_k = \text{tr} \left( \hat{\rho} (\hat{\sigma}_k \otimes \hat{I}) \right) = -2(I\sigma_k)^1 \cdot (\psi J\tilde{\psi}) \]

- All of the information in the density matrix is held in the observables

\[ \psi E\tilde{\psi} \quad \psi J\tilde{\psi} \]
Inner products and traces

• Can write the overlap probability as

\[ P(\psi, \phi) = |\langle \psi | \phi \rangle|^2 = \text{tr}(\hat{\rho}_\psi \hat{\rho}_\phi) \]

• So have, for n-particle pure states

\[ P(\psi, \phi) = 2^{n-2} \left( \langle (\psi E \bar{\psi})(\phi E \bar{\phi}) \rangle - \langle (\psi J \bar{\psi})(\phi J \bar{\phi}) \rangle \right) \]

• The **partial trace** operation corresponds to forming the observables, and throwing out terms

• Clearly see how this is removing information

• For mixed states, can correlate on pseudoscalar
Schmidt Decomposition

• General way to handle 2-particle states is to write as a matrix and perform an SVD

\[ |\psi\rangle = e^{i\chi} \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) e^{\frac{i\tau}{2}} \\ \sin\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \end{pmatrix} \otimes \begin{pmatrix} \cos\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \\ \sin\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \end{pmatrix} + \sin\left(\frac{\alpha}{2}\right) e^{-\frac{i\tau}{2}} \begin{pmatrix} \sin\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \\ -\cos\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \end{pmatrix} \otimes \begin{pmatrix} \sin\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \\ -\cos\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \end{pmatrix} \]

• Only works for bipartite states
GA Form

- Define the local states / operators

\[ R = \psi(\theta_1, \phi_1)e^{I\sigma_3^1 \tau/4}, \quad S = \psi(\theta_2, \phi_2)e^{I\sigma_3^2 \tau/4} \]

- Result of the Schmidt decomposition can now be written

\[ \psi = \rho R^1 S^2 \left( \cos(\alpha/2) + \sin(\alpha/2) I\sigma_2^1 I\sigma_2^2 \right) e^{Jx E} \]

- Now have a form which generalises to arbitrary particles

Local unitaries  Entangling term
3-Particle States

• The GHZ state is
  \[ \psi = \frac{1}{\sqrt{2}} (|000\rangle - |111\rangle) \]

• GA equivalent
  \[ \psi = \exp\left(\frac{\pi}{4} (I\sigma_2)^1 (I\sigma_2)^2 (I\sigma_2)^3\right) \]

• The W-state is more interesting
  \[ |W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \]

• Goes to
  \[ W = R_1 R^2 R^3 T_{12} T_{13} T_{23} T_{123} (\pi/4) \]
  \[ R_i = e^{\frac{\pi}{4} I\sigma_2} \]
  \[ T_{12} = \cos(\pi/12) + \sin(\pi/12) (I\sigma_2)^1 (I\sigma_2)^2 \]
Singlet State

- An example of an entangled, or *non-local*, state is the 2-particle *singlet* state

\[ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \leftrightarrow \psi = \frac{1}{\sqrt{2}}(I\sigma_2^1 - I\sigma_2^2)E \]

- This satisfies the identity

\[ M^1_\varepsilon = \bar{M}^2_\varepsilon \]

- Gives straightforward proof of invariance

\[ R^1_\varepsilon R^2_\varepsilon = R^1_\varepsilon \bar{R}^1_\varepsilon = \varepsilon \]

- Observables are

\[ 2\varepsilon E\bar{\varepsilon} = 1 + (I\sigma_k)^1 (I\sigma_k)^2 \]
Relativistic States

• All of the previous considerations extend immediately to relativistic states
• Can give physical definitions of entanglement for Dirac states
• Some disagreement on these issues in current literature
• Has been suggested that relative observers disagree on entanglement and purity
• More likely that an inappropriate definition has been adopted
Relativistic Singlet

• Can extend the non-relativistic state to one invariant under boosts as well

\[ \eta = \varepsilon (1 - I^1 I^2) \]

• This satisfies

\[ R^1 R^2 \eta = R^1 \tilde{R}^1 \eta = \eta \]

A Lorentz rotor

• This state plays an important role in GA versions of 2-spinor calculus and twistor theory
Multiparticle Dirac Equation

- Relativistic multiparticle quantum theory is a slippery subject!
- Ultimately, most issues sorted by QFT
- Can make some progress, though, e.g. with Pauli principle

\[ I_P = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3, \quad \Gamma_\mu = \frac{1}{\sqrt{2}} (\gamma^1_\mu + \gamma^2_\mu) \]

- Antisymmetrised state constructed via

\[ \psi_-(x) = \psi(x) + I_P \psi(I_P x I_P) I_P \]
Current

• For equal mass particles, basic equation is
  \[ \nabla \psi J = m\psi (\gamma_0^1 + \gamma_0^2) \]

• Get a conserved current in 8D space
  \[ \mathcal{J} = \langle \psi (\gamma_0^1 + \gamma_0^2) \rangle_1 \]

• Pauli principle ensures that
  \[ I_P \mathcal{J} (I_P x I_P) I_P = \mathcal{J}(x) \]

• Ensures that if 2 streamlines ever met, they could never separate
Plots

In both cases the packets pass through each other
Exercises

• Verify that the overlap probability between 2 states is

\[ P(\psi, \phi) = \frac{\langle (\psi E\tilde{\psi})(\phi E\tilde{\phi}) \rangle - \langle (\psi J\tilde{\psi})(\phi J\tilde{\phi}) \rangle}{2\langle \psi E\tilde{\psi} \rangle \langle \phi E\tilde{\phi} \rangle} \]

• Now suppose that one state is the singlet, and the other is separable. Prove that

\[ P(\psi, \phi) = \langle \frac{1}{2} (1 - P^1 Q^2) \frac{1}{2} (1 + I \sigma^1_k I \sigma^2_k) \rangle = \frac{1}{4} (1 - \cos \theta) \]

Angle between the spin vectors, or between measuring apparatus