Recent Applications of Conformal Geometric Algebra

Anthony Lasenby

Cavendish Laboratory, University of Cambridge, Cambridge CB3 0HE, UK,
a.n.lasenby@mrao.cam.ac.uk,
WWW home page: http://www.mrao.cam.ac.uk/~anthony

Abstract. We discuss a new covariant approach to geometry, called conformal geometric algebra, concentrating particularly on applications to projective geometry and new hybrid geometries. In addition, a new method of working, which can achieve similar results, but using only one extra dimension instead of two, is also discussed.

1 Introduction

Our aim is to show how useful the covariant approach is in geometry. Like Klein, we will consider a particular geometry as a class of transformations which leave a particular object invariant. However, by starting with null vectors and using a conformal representation (as in the conformal representation of Euclidean geometry introduced by Hestenes [1]), we will see that covariance (retention of same physical content after transformation) is also of fundamental importance. Keeping covariance, and using not just null vectors but the whole representation space, we will be able to step outside conformally flat geometries to include projective geometry as well. This occurs in a fashion different from that which Klein envisaged and retains, at least in this approach, a fundamental role for null vectors. In the final sections, we discuss a new approach to the area of conformal geometric algebra, which needs only one extra dimension in which to carry out computations, as against the two needed in the Hestenes approach. This has potential savings in computational time, and in physics applications, seems a more natural framework in which to work. The examples we use are mainly drawn from geometry and physics. However, as already demonstrated in [2, 3] and other papers, the conformal geometric algebra approach has many applications in computer graphics and robotics. Thus the methods developed here may similarly be expected to be useful in application to this area, and we illustrate with the specific example of interpolation.

In order to allow space here to adequately describe new material, we refer the reader to [3] (this volume), for a summary of the conformal geometric algebra approach to Euclidean geometry. We also assume that the reader is familiar with the extension of this to non-Euclidean spaces. This was first carried out in [4] and [5–8]. The full synthesis, whereby exactly the same techniques and notation can be used seamlessly in all of spherical, hyperbolic and Euclidean geometry, has been described in [2, 9–11].

Despite not giving a beginning summary of the mathematics, it is important, to avoid any confusion, to stress what notation we will be using. We follow here the notation for the two additional vectors that are adjoined to the base space, which was used by Hestenes & Sobczyk in [12], Chapter 8. These two vectors are e, with $e^2 = 1$, and $\bar{e}$, with $\bar{e}^2 = -1$. The null vectors which are formed from them are $n = e + \bar{e}$ and $\bar{n} = e - \bar{e}$, and in the conformal approach to Euclidean geometry play the roles of the point at infinity, and the origin, respectively. This differs from the notation used subsequently by Hestenes in [1].

We should note, as a global comment, that no pretence of historical accuracy or completeness is made as regards the descriptions of geometry and geometrical methods given here. Almost certainly, all the geometrical material contained here, even when described as 'new', has been discovered and discussed many times before, in a variety of contexts. The aim of this contribution is therefore to try to show the utility, and unification of approach, that conformal geometric algebra can bring.

2 Comparison of Kleinian and Covariant Views of Geometry

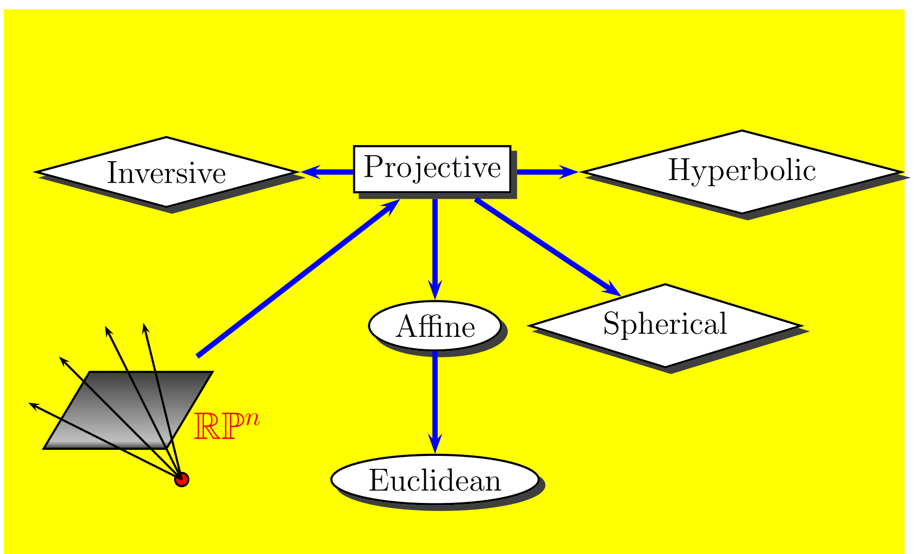

Fig. 1. The Kleinian view of geometry.

The Kleinian view of geometry is illustrated schematically in Fig. 1. Here the primary geometry, from which others are derived, is projective geometry, as exemplified for example by the geometry of the real projective plane in n dimensions, $\mathbb{RP}^n$ (see e.g. [13]). Projective geometry is primary, in that the largest group of transformations is allowed for it. Other geometries are then derived from it via restrictions which are imposed on the allowed transformations. For example spherical geometry is the subgroup of projective geometry which preserves the Euclidean angle between two lines through the origin, while affine and then Euclidean geometry represent increasing degrees of restriction in the behaviour under transformation of 'ideal points' at infinity (see [13–15]). The view of geometry which we begin with here, however, puts Euclidean, spherical and hyperbolic in a different relationship. As stated above, let us denote by e and $\bar{e}$, the additional vectors which we adjoin to the three basis vectors, e_1, e_2 and e_3 of ordinary Euclidean space, in order to construct the conformal geometric algebra, and let $n = e + \bar{e}$. Then, in the null vector representation

$$\begin{aligned} \text{Euclidean geometry} &\leftrightarrow \text{set of transformations which preserve } n \\ \text{Hyperbolic geometry} &\leftrightarrow \text{set of transformations which preserve } e \\ \text{Spherical geometry} &\leftrightarrow \text{set of transformations which preserve } \bar{e} \\ \text{Inversive geometry} &\leftrightarrow \text{set of transformations of the form } S\mathcal{O}S, \end{aligned} \tag{1}$$

where in the last line S represents a sphere in one of the above geometries, and $\mathcal{O}$, the object being reflected in the sphere is one of a point, line, circle, plane or sphere (see [2]). Euclidean, hyperbolic and spherical geometry thus appear as much more similar to each other in this approach — effectively on the same level as each other, rather than being quite different specialisations of projective geometry. Also inversive geometry is reduced to the study of generalised reflections in each space.

What is the significance of *covariance* in this approach? The key idea is that we can freely write down expressions composed of elementary objects in the geometry, such as lines, planes, spheres and circles (including non-Euclidean varieties of each), and that these remain geometrically meaningful after a transformation of the appropriate kind for the geometry. An example can illustrate this. Let P be a point and L a line. Consider the new point $Q = LPL$, the (possibly non-Euclidean) reflection of P in the line L. Suppose we carry out a transformation in the geometry, given by the rotor R (e.g. this might be a translation, rotation, or dilation) to get a new point $P' = RP\tilde{R}$, and line $L' = RL\tilde{R}$. The question is, does the new point we obtain by carrying out the reflection process with the transformed P' and L', correspond to the transformation of the old Q? That is, is the following true:

$$RQ\tilde{R} = L'P'L'? \tag{2}$$

The r.h.s. is $(RL\tilde{R})(RP\tilde{R})(RL\tilde{R})$, which since the algebra is associative and $R\tilde{R} = \tilde{R}R = 1$ for a rotor, means that we obtain $RQ\tilde{R}$ as required. This may seem trivial, but is the essence of the advantages of conformal geometric algebra for geometry. We can compose 'objects' into expressions in the same way as words are composed into sentences, and the expressions will be geometrically meaningful if they are covariant.

3 Extending the Framework

We have seen that with this conformal framework, where null vectors in our 5d algebra represent points, we can nicely represent Euclidean, spherical, inverse and hyperbolic geometry. But, is there any way of extending it to encompass Affine and Projective geometries? This would be very desirable in many fields. For example, in computer vision, projective geometry is used a great deal, partly for the obvious reason that it is related to the images projected down from the 3-dimensional world to the camera screen or retina, but also used because a projective approach allows the linearization of transformations that would otherwise be non-linear. In particular, by adding a third coordinate to the usual two in a camera plane, translations in 2-space become homogeneous linear transformations in 3-space. Also, incidence relations between points, lines and planes can be expressed in an efficient manner. However, using a purely linear algebra projective approach, one cannot extend these incidence relations to include conic sections, and moreover *metric* relations between objects, for example relations between conic sections and sums and differences of lengths, are lost, since a metric is not accessible, except by going to a Euclidean specialization.

If it were possible to extend the conformal approach to include projective and affine geometry, however, it might be possible to remedy these deficiencies, and to include incidence relations between lines and conic sections, and metric relations between points. We now show, that this is indeed possible, and that the method of achieving it has a natural structure which emphasizes the covariance properties of objects. The resulting theory does not quite achieve all we might ideally want, since although lines can be intersected with conic sections in a useful way, conic sections cannot be intersected with each other in the same fashion which is possible for the intersection of circles and spheres. An extension of this kind does not seem to be possible. However, that we have still done something significant is clear when one applies the principles we set up in the section, with an underlying Euclidean geometry, to the case where the underlying geometry is hyperbolic or spherical. Amazingly, all the same constructions and types of results still hold, thus generating what appear to be completely new geometries, with possible applications in many areas.

To start with then, we consider how to encode projective geometry in the null-vector description of Euclidean space already summarised.

Clearly we cannot embed projective geometry in our conformal setup as it stands. The essence of conformal mappings is that they preserve angles, and the essence of projective and affine transformations is that they do not preserve angles! Thus we have to step outside the framework we have set up so far. This framework has at its core the assumption that points in a Euclidean space $\mathbb{R}^{(p,0)}$ are represented by null vectors in the space $\mathbb{R}^{(p+1,1)}$, two dimensions higher, A projective transformation cannot be represented by a function taking null vectors to null vectors in this space, since then it would be a conformal transformation in the Euclidean space. The only possible conclusion, is that projective transformations must involve non-null vectors. But then how can we relate the results back to points in Euclidean space?

The key to the resolution of this is the following result. Let Y be any vector in $\mathbb{R}^{(p+1,1)}$ such that $Y{\cdot}n \neq 0$. Then there exists a unique decomposition of Y of the form

$$Y = \alpha X + \beta n \tag{3}$$

where α and β are scalars, X is null, and $X{\cdot}n = -1$. The proof of this is constructive. Assume it can be done, then

$$\begin{aligned} Y^2 &= 2\alpha\beta X{\cdot}n = -2\alpha\beta \\ Y{\cdot}n &= \alpha X{\cdot}n = -\alpha \end{aligned} \tag{4}$$

This entails

$$\begin{gathered} \alpha = -Y{\cdot}n, \qquad \beta = \frac{Y^2}{2Y{\cdot}n} \\ X = -\left(\frac{Y}{Y{\cdot}n} - \frac{Y^2}{2(Y{\cdot}n)^2}n\right) = -\frac{YnY}{2(Y{\cdot}n)^2} \end{gathered} \tag{5}$$

The values just given for α, β and X, satisfy $X{\cdot}n = -1$ and $\alpha X + \beta n = Y$, and so this proves the decomposition can be carried out and that the answer is unique. Up to scale, we see that X is the reflection of the point at infinity n, in the (generally non-null) vector Y.

What we have found then, is a unique mapping from any non-null vector Y satisfying $Y{\cdot}n \neq 0$, to a null vector X. Thus any non-null vector of this kind can be taken as representing a point in the finite Euclidean plane. Projective transformations can therefore work by mapping null vectors representing Euclidean points to non-null vectors — thereby avoiding the problem that 'null' $\mapsto$ 'null' would be a conformal transformation — and then these non-null vectors can be taken as representing Euclidean points, thus overall achieving a mapping from points to points. Note if the vector Y is such that $Y{\cdot}n = 0$, so that we cannot employ the above construction, then we will need some supplementary rule for what to do, and this is discussed further below.

The mappings we use will be linear mappings h taking $\mathbb{R}^{(p+1,1)} \mapsto \mathbb{R}^{(p+1,1)}$. Our main assertion is as follows:

Projective transformations correspond to linear functions which preserve the point at infinity up to scale, i.e. for which $h(n) \propto n$.

Using functions of this kind, points in Euclidean space are mapped to points in Euclidean space via

$$h(X) = Y = \alpha X' + \beta n \rightarrow X' \tag{6}$$

which defines a mapping from null X to null X'. If the value of α is independent of X, the mapping is an *Affine* transformation.

The mapping we have described,

$$h(X) = \alpha X' + \beta n \tag{7}$$

is implicit as regards obtaining X', but using the result above we can get a properly normalised point explicitly via

$$X' = -\frac{1}{[h(X){\cdot}n]^2}h(X)nh(X) \tag{8}$$

i.e. (up to scale) we just need to reflect the point at infinity in $h(X)$. This is just an easy way of carrying out the construction; the key question is the covariance of the $h(X) = \alpha X' + \beta n$ mapping itself. As already described, covariant means: do we get the same result if we transform first then carry out our mapping as if we map first then transform? One finds:

$$\begin{aligned} Rh(X)\tilde{R} &= \alpha RX'\tilde{R} + \beta Rn\tilde{R} \\ &= \alpha RX'\tilde{R} + \beta n \end{aligned} \tag{9}$$

So this works due to the invariance of the point at infinity under the allowed rotors. We note that we could allow dilations as well, which rescale n. These lead to the same output vector $RX'\tilde{R}$, since β would then just be rescaled.

3.1 Boundary points

Before giving examples of how all this works, we first consider the question raised above: what null vector should we associate with a non-null Y if $Y \cdot n = 0$, and what then happens to our projective mapping?

Now in the case of hyperbolic geometry, we know that the boundary of the space (the edge of the Poincaré disc in 2d) is given by the set of points satisfying the covariant constraint $X \cdot e = 0$. This is covariant since the allowed rotors keep e invariant in the hyperbolic case. Thus the boundary points are mapped into each other under translations and rotations. In [10] it is shown that for this hyperbolic case, as well as for de Sitter and anti-de Sitter universes, the boundary points can be represented by null vectors. For example, for the Poincaré disc, the boundary points are of the form

$$X = \lambda(\cos\phi e_1 + \sin\phi e_2 + \bar{e}) \tag{10}$$

where λ is a scalar, and we have a whole set of them — one for each 'direction' in the underlying space.

In the Euclidean case, however, we have not yet achieved an equivalent to this, having only the single vector n available to represent points at infinity. The resolution to this is now clear — the vectors representing the boundary points must be the set of vectors satisfying the covariant constraint $Y \cdot n = 0$, but we must admit as solutions non-null as well as null vectors. Thus the boundary points are the set of vectors

$$Y = a + \beta n \tag{11}$$

where a is a Euclidean vector. If we impose the further (covariant) constraint $Y^2 = 1$, then this reduces the set of vectors to just a direction in Euclidean space, plus an arbitrary multiple of n. This seems a very sensible definition for a set of 'points at infinity'.

We can summarize this setup as follows.

$$\begin{aligned} &\text{Interior points satisfy:} \quad X \cdot n = -1, \qquad X^2 = 0 \\ &\text{Boundary points satisfy:} \quad Y \cdot n = 0, \qquad Y^2 = 1 \end{aligned} \tag{12}$$

There is quite a neat symmetry in this. Note that the Euclidean transformations of rotation, translation and even dilation, when carried out using the rotor approach, all preserve the condition $Y^2 = 1$. (This is because only rotation affects the Euclidean a part of the vector.) In the case of non-Euclidean geometry, e.g. for the Poincaré disc, the boundary points, being null, are only preserved by the allowed transformations up to scale.

It is now obvious what to do if the mapping $h(X)$ produces a point Y such that $Y{\cdot}n = 0$. We just accept this non-null point Y as the output of the mapping, instead going through a further stage of finding an associated null vector X', for which the construction is impossible in this case. The non-null point Y contains more information than simply assigning a null vector output of n, since as well as seeing that we are at infinity, we can also tell from what direction infinity is approached.

3.2 Specialization to Affine Geometry

Above we made the brief comment that restricting α to a constant in equation (6), was the same as restricting the projective transformations to affine transformations. Armed with the new concept of 'boundary points' we can now understand this further. We give an alternative definition of affine transformations as follows:

> Affine transformations correspond to the subset of projective transformations which map boundary points to boundary points, up to scale. Thus h is affine if $Y{\cdot}n = 0 \implies h(Y){\cdot}n = 0$.

In this form we can recognize the classical approach to restricting projective transformations to affine form. E.g. in 2d, one would ask that the so called 'line at infinity' be preserved , while in 3d it would be the 'plane at infinity' [15, 14]. These objects correspond to our 'boundary points'.

To see the link with our previous version of an affine transformation, we can argue as follows. Let Z be a vector in $\mathbb{R}^{(p+1,1)}$ which is general except that it satisfies $Z{\cdot}n = 0$. A fully general vector in $\mathbb{R}^{(p+1,1)}$ can be derived from this by adding an arbitrary multiple of $\bar{n}$, and we set $Y = Z + \theta\bar{n}$, where θ is a scalar. We then have the following results:

$$
\begin{aligned}
h(Y) &= h(Z) + \theta h(\bar{n}) \\
h(Y){\cdot}n &= \theta h(\bar{n}){\cdot}n \\
Y{\cdot}n &= 2\theta \\
h(Y){\cdot}n &= \left(\frac{1}{2}h(\bar{n}){\cdot}n\right) Y{\cdot}n
\end{aligned}
\tag{13}
$$

We note the pre-factor in front of $Y{\cdot}n$ in the last equation is independent of Y for a given h. Now in equation (6), we can see that $\alpha = (h(X){\cdot}n)/(X'{\cdot}n)$. Thus providing X and X' are both normalised, which we are assuming, we can deduce that for an affine transformation α is constant, as stated.

3.3 Specialization to Euclidean geometry

An aspect of the conventional approach to projective geometry which often appears rather mysterious, particularly in applications to computer vision, is the use of ideal points, i.e. points lying in the line or plane at infinity, with complex coordinates. These are used in particular for encoding the specialization of affine geometry to Euclidean geometry, where the latter is defined to include the set of similitudes, as well as rotations and translations.

In this conventional approach, the Euclidean geometry of the plane is recovered from projective geometry as the set of affine transformations which leave the points $(1, i, 0)$ and $(1, -i, 0)$ invariant. These are called the ideal points $\mathcal{I}$ and $\mathcal{J}$ (where we use calligraphic letters to avoid confusion with our pseudoscalar I).

Similarly the Euclidean geometry of 3d space, is recovered by asking that an affine transformation leaves invariant a certain quadratic form involving complex ideal points, known as the *absolute conic* [15, 14]. This form is

$$\sum_{i=1}^{4} x_i^2 = 0 \tag{14}$$

where $x_4 = 0$ in order to place the set of points in the ideal plane at infinity.

The form of geometric algebra we use restricts itself to an underlying real space only, and does not make use of an uninterpreted scalar imaginary i. By taking this approach, one can often find a geometrical meaning for the various occurrences of i in the conventional approaches — a meaning that is often obscured without geometric algebra. It is thus interesting to see if the geometric algebra approach can shed light on the points $\mathcal{I}$ and $\mathcal{J}$, and give some increased understanding of the absolute conic.

Since $h(n)$ is proportional to n and maps boundary points to boundary points (since it is assumed affine), and also since the length of $a + \beta n$ does not depend on β when a is Euclidean, we can specialize our discussion of the ideal points $\mathcal{I}$ and $\mathcal{J}$ to a simple linear function f say, taking $\mathbb{R}^2$ to $\mathbb{R}^2$, which is the restriction of h to these spaces.

In terms of the orthogonal basis $\{e_1, e_2\}$, this has the matrix equivalent

$$f_{ij} = f(e_i)\cdot e_j \tag{15}$$

We now write down the conventional matrix relation for the ideal points and reconstitute from this what it means in a GA approach.

Saying that $\mathcal{I}$ is invariant under f means conventionally that

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = z \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{16}$$

where z is some complex number. Now complex numbers are the spinors of the geometric algebra of $\mathbb{R}^2$, and we accordingly represent $z = x + iy$ by $Z = x + I_2 y$

where $I_2 = e_1e_2$ is the pseudoscalar in 2d. Our translation of the invariance condition is thus that

$$\begin{aligned} f(e_1)\cdot e_1 + I_2 f(e_1)\cdot e_2 &= Z \\ f(e_2)\cdot e_1 + I_2 f(e_2)\cdot e_2 &= ZI_2 \end{aligned} \tag{17}$$

Now $I_2 f(e_1)\cdot e_2 = \langle I_2 f(e_1) e_2\rangle_2 = \langle -f(e_1) I_2 e_2\rangle_2 = -f(e_1)\wedge e_1$. We deduce

$$e_1 f(e_1) = Z, \quad e_1 f(e_2) = ZI_2 \tag{18}$$

Premultiplying by e_1 we obtain

$$f(e_1) = \tilde{Z}e_1 \quad \text{and} \quad f(e_2) = \tilde{Z}e_2 \tag{19}$$

from which we can deduce that

$$f(a) = \tilde{Z}a = aZ \quad \forall \text{ Euclidean vectors } a \text{ in } \mathbb{R}^2 \tag{20}$$

A general spinor Z or $\tilde{Z}$ applied to the left or right of a dilates a by $|Z|$ and rotates it through $\tan^{-1}(\langle Z\rangle_2/\langle Z\rangle_0)$, so we have indeed found that demanding that the ideal point $\mathcal{I}$ is preserved is equivalent to specializing to Euclidean transformations. We note that requiring that $\mathcal{J}$ is preserved as well as $\mathcal{I}$, is redundant (which it is also in the conventional approach, in fact) since the invariance condition

$$\begin{aligned} f(e_1)\cdot e_1 - I_2 f(e_1)\cdot e_2 &= Z' \\ f(e_2)\cdot e_1 - I_2 f(e_2)\cdot e_2 &= -Z'I_2 \end{aligned} \tag{21}$$

reconstitutes to give

$$f(a) = a\tilde{Z}' = Z'a \quad \forall \text{ Euclidean vectors } a \text{ in } \mathbb{R}^2 \tag{22}$$

just identifying Z' as $\tilde{Z}$, and giving no new information.

So having seen that (16) is directly equivalent to (20) can we say whether the GA formulation offers any improvement over the conventional one? We think the answer is yes, in that (20) more directly reveals the geometrical origins of what is going on, and also clearly refers homogeneously to all vectors in the space, rather than singling out particular points as being preserved. It also avoids the need for introducing complex coordinates of course.

The Absolute Conic We now consider a similar process in 3d. We will again let f be the specialization of h to the Euclidean portion of the space, and we will let z_i and z_i' $(i = 1, 2, 3)$ be general complex numbers. In the conventional approach, 'leaving the absolute conic invariant' amounts to the following:

The matrix $||f_{ij}||$ is such that the output vector (z_1', z_2', z_3') in the relation

$$\begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \tag{23}$$

satisfies $(z_1')^2 + (z_2')^2 + (z_3')^2 = 0$ if $z_1^2 + z_2^2 + z_3^2 = 0$.

Framed like this, the statement looks rather odd, since we have clearly introduced a non-Hermitian metric on the space of complex vectors. In the GA approach, we can make more sense of things since we have available entities which are naturally null. For example, consider the vector and bivector combination $e_1 + I_3 e_2$, where I_3 is the pseudoscalar for $\mathbb{R}^3$. Then $(e_1 + I_3 e_2)^2 = 0$. Such combinations arise naturally in electromagnetism for example, where e_1 would be interpreted as a 'relative vector' and $I_3 e_2$ as a relative bivector', with both actually being bivectors in spacetime. (For example, in a spacetime basis $\{\gamma_0, \gamma_i\}$ we would identify $e_i = \gamma_i \gamma_0$ and $I_3 = e_1 e_2 e_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$.) Taking the sum of them is natural in forming the Faraday electromagnetic bivector F, where the two parts would then correspond to the electric and magnetic fields, and the null condition is that $F^2 = 0$.

Following this line, our reformulation of the notion of leaving the absolute conic invariant is as follows.

> Let a and b be Euclidean vectors. Then we require the linear function f to satisfy the relation
>
> $$\left(f(a) + I_3 f(b)\right)^2 = 0 \Longleftrightarrow (a + I_3 b)^2 = 0 \tag{24}$$

In this version it is easy to see the link with specializing to Euclidean transformations, since

$$\begin{aligned} \left(f(a) + I_3 f(b)\right)^2 &= f^2(a) - f^2(b) + 2 I_3 f(a) \cdot f(b) \\ (a + I_3 b)^2 &= a^2 - b^2 + 2 I_3 a \cdot b \end{aligned} \tag{25}$$

$(a + I_3 b)^2$ is thus only zero for vectors a and b which are the same length and orthogonal, and f is required to preserve both these features. This pins down f to a combination of rotation and dilatation.

To see how (23) is equivalent to (24) is also easy. If we set $z_i = x_i + i y_i$, $(i = 1, 2, 3)$, we simply need to make the identification $a = x_i e_i$, $b = y_i e_i$, and we see the same result is achieved in both cases.

To judge here whether the GA formulation is superior to the conventional one, is more difficult than in the 2d case, since the availability of a pseudoscalar which squares to -1 and commutes with all vectors in 3d, means that the GA and conventional formulations are closely parallel. However, the GA formulation does reveal that the complex points made use of in the notion of the absolute conic, which many have found rather mysterious, are actually vector plus bivector combinations, and that the idea of these being null takes on a familiar look when we see the parallel with electromagnetism, where the analogue of the second equation of (25) is

$$F^2 = 0, \quad \text{with } F = \boldsymbol{E} + I_3 \boldsymbol{B} \quad \Longrightarrow \quad \boldsymbol{E}^2 - \boldsymbol{B}^2 = 0 \quad \text{and} \quad \boldsymbol{E} \cdot \boldsymbol{B} = 0 \tag{26}$$

which is the statement that the Faraday bivector is null if the electric and magnetic fields have the same magnitude and are orthogonal (as in an electromagnetic plane wave for example).

4 Lines and Conics

Having considered the specialization to Affine and Euclidean geometry in some detail, we now give a short, schematic account of some remaining major features of projective geometry. A more comprehensive account will be given in a future publication, and some applications and further details are given in [3].

We need to see what happens to lines under an h transformation. This works well, since the requirement $h(n) = n$ means that if we are transforming the line $L = A \wedge B \wedge n$, we obtain

$$\begin{aligned} h(L) &= h(A) \wedge h(B) \wedge n \\ &= \alpha_A \alpha_B A' \wedge B' \wedge n, \end{aligned} \tag{27}$$

with only α appearing since the wedge kills off the β parts.

Thus we get a line through the transformed points, while the factor $\alpha_A \alpha_B$ contains information that can be expressed via cross-ratios.

For conics, if C is a multivector representing a circle or sphere in Euclidean space, then the equation $Y \wedge h(C) = 0$, where $Y = \alpha X' + \beta n$, and $h^{-1}(Y)$ is null, successfully produces a solution space corresponding to a general conic, which we can think of as the image of C under h. However, one cannot use this approach to intersect two different conics, unless it happens that they correspond to the same h function acting on different initial circles or spheres.

h itself for an object transforms covariantly as

$$h(X) \mapsto R h(\tilde{R} X R) \tilde{R}. \tag{28}$$

This means that we can use rotors to achieve transformation of any conic, or quadric in 3d, thus achieving one of our main objectives. For example rotation and translation of a quadric in 3d via use of rotors, is shown in Fig. 2(a) and (b). The rotors can encompass all of rotation, translation and dilation, and thus we can position and scale any conic or quadric using rotor techniques.

The other success of this method is that one can use the machinery which enables intersection of any line or plane with a circle or sphere, to look at intersections of lines and planes with conics and quadrics. There are many applications of this in computer graphics.

The way we can do this is to use reflection of $h^{-1}(X')$ in n to bring back points X' to where they originated before application of h:

$$X = -\frac{1}{[h^{-1}(X') \cdot n]^2} h^{-1}(X') n h^{-1}(X') \tag{29}$$

A line transforms back under this to another line, whilst a conic transforms back to the underlying circle or sphere. We can then intersect the new line with the resulting circle or sphere, using the standard intersection techniques in conformal geometric algebra, and then forward transform again to find the solution to the original problem. An example of this is shown schematically Fig. 3, which whilst not looking very exciting, was worked out using this method, and thus constitutes proof that we can indeed intersect lines with arbitrary conics.

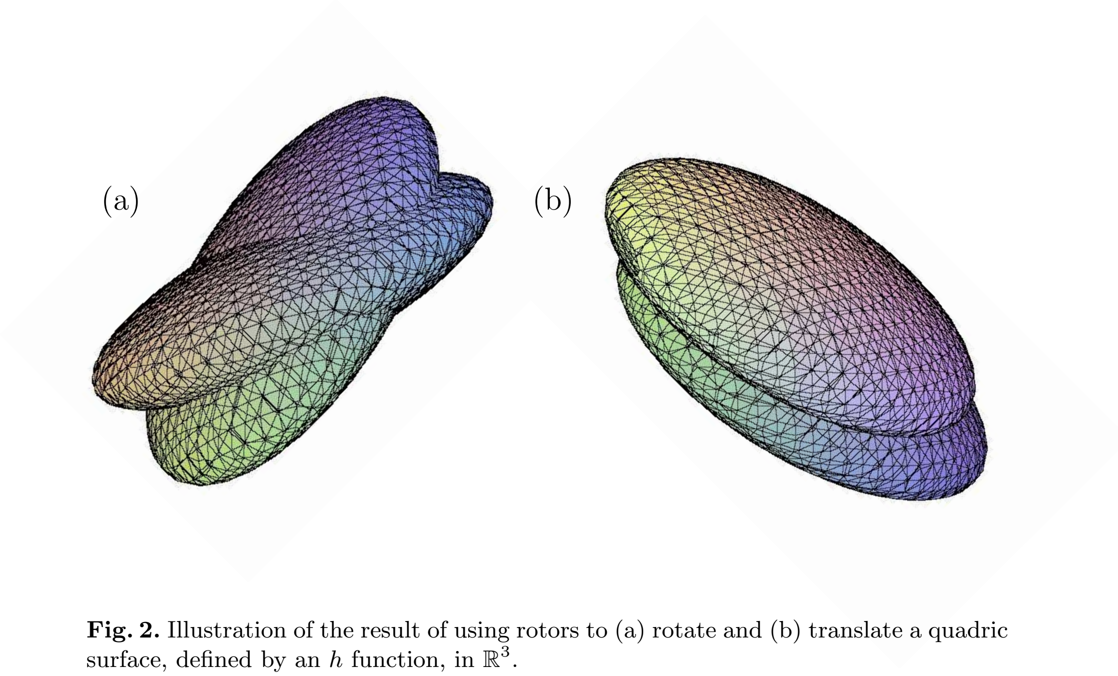

Fig. 2. Illustration of the result of using rotors to (a) rotate and (b) translate a quadric surface, defined by an h function, in $\mathbb{R}^3$.

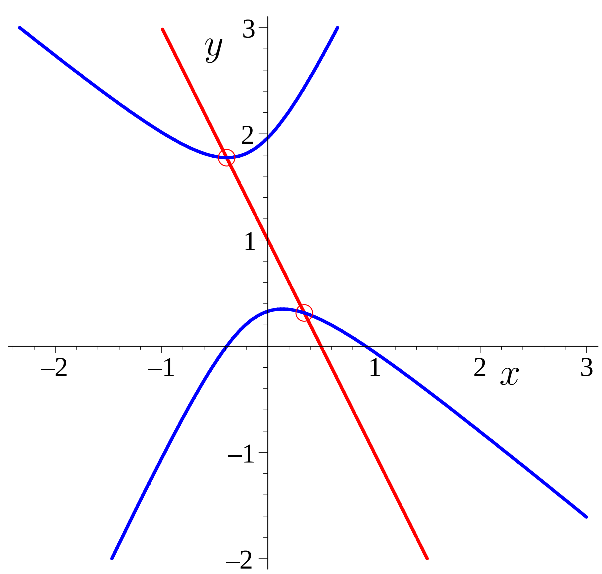

Fig. 3. Intersection of a line and a hyperbola using conformal method described in text.

5 Creating New Geometries

We now reach our second overall topic in this brief survey. This is the use of conformal geometry to create new, hybrid geometries, which combine the features of both projective and non-Euclidean geometries. In particular, using this method can we carry out projective transformations *within* hyperbolic or spherical spaces. This is quite different from, for example, the representation of hyperbolic geometry in terms of projective geometry, as in the Klein approach.

The way we do this is illustrated in Fig. 4. We apply linear functions which

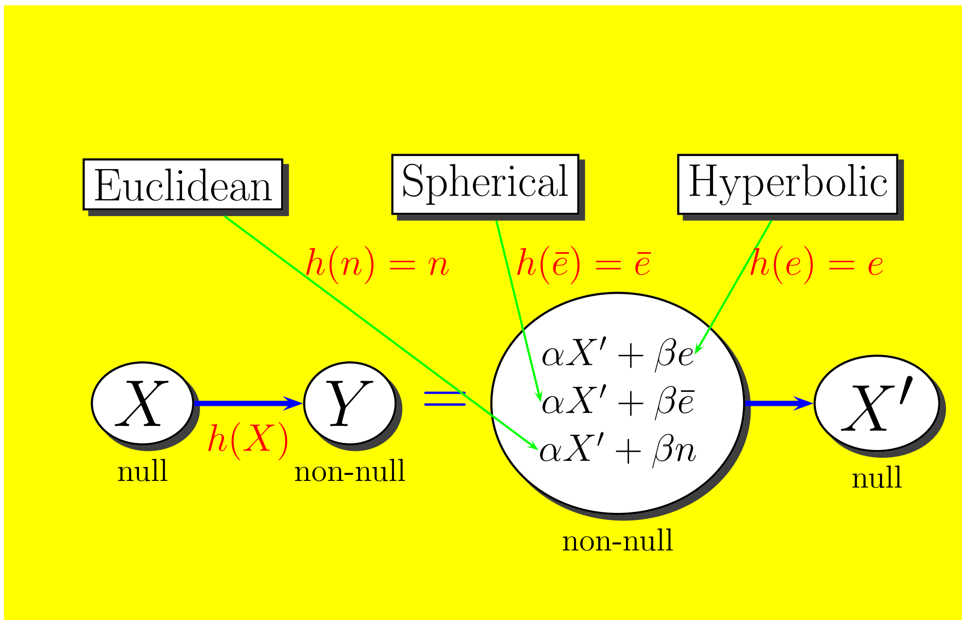

Fig. 4. Schematic diagram showing how the new hybrid geometries are created.

now preserve e, or $\bar{e}$, rather than n, and wind back to null vectors using

$$h(X) = \alpha X' + \beta e \tag{30}$$

for the hyperbolic case, and

$$h(X) = \alpha X' + \beta \bar{e} \tag{31}$$

for the spherical case. Considering for example the hyperbolic case, we see that (30) implicitly defines a null vector X' for each X, and that this happens in a fully covariant manner, since now e is the object preserved under the relevant rotor operations.

So what might we expect to be able to do using this structure? Most importantly, we will want to see if it is possible to obtain a non-Euclidean version of the family of conics (quadrics in 3d), which are familiar in projective geometry as applied in Euclidean space. For the latter there are two separate ways of defining them. For example, an ellipse may be defined as the locus of points which have a constant summed distance to two fixed points (the focii) or alternatively via the projective transformation of a circle. It is a theorem for conics that these two approaches lead to the same object. Ideally then, for non-Euclidean conics, we will want both of the following being true and leading to the same object:

1. The sum or difference of non-Euclidean lengths being constant defines the locus;
2. The shape arises via a projective transformation (linear mapping in one dimension higher) of e.g. a circle.

So can we form for example a non-Euclidean ellipse? This would have constant sum of non-Euclidean distance to two given points (the non-Euclidean foci). Such an object can of course be defined, and we will end up with a certain definite locus. But will it also correspond to a projective transformation applied to a circle? Here is an example. Let us take as our h-function:

$$h(a) = a + \frac{1}{2}\delta\, a{\cdot}e_1\bar{e} \tag{32}$$

where δ is a scalar. This looks very simple, but the actual expressions implied by it are quite complicated. Temporarily writing $\boldsymbol{x} = (x, y)$ for the real space coordinates, and $\boldsymbol{x}' = (x', y')$ for those they are transformed to via (30), then, taking the appropriate branches of the square root functions, one finds:

$$\boldsymbol{x}' = 2\lambda^2\boldsymbol{x}/\left(x^2 + y^2 + \lambda^2 + \delta\,\lambda x + \sqrt{x^4 + 2\,x^2y^2 - 2\,\lambda^2x^2 + 2\,x^3\delta\,\lambda + y^4 - 2\,\lambda^2y^2 + 2\,y^2\delta\,\lambda x + \lambda^4 + 2\,\lambda^3\delta\,x + \delta^2\lambda^2x^2}\right) \tag{33}$$

Thus there is a radial dilation, but with a factor which is a complicated function of position. Choose $\delta = 2.25$, and map the circle $r = 1/4$ using the above h function in the basic construction

$$h(X) = \alpha X' + \beta e \tag{34}$$

This gives the result shown in Fig.5.

This particular case is actually a limiting case of an non-Euclidean ellipse — one of the foci has migrated to infinity! Thus it should be regarded as a non-Euclidean parabola. We will not substantiate this here, but instead, to get to a more easily understandable version of an ellipse, let us try a slightly different form of transformation which still preserves e, namely

$$h(a) = a + \frac{1}{2}\delta\, a{\cdot}e_1\, e_1 \tag{35}$$

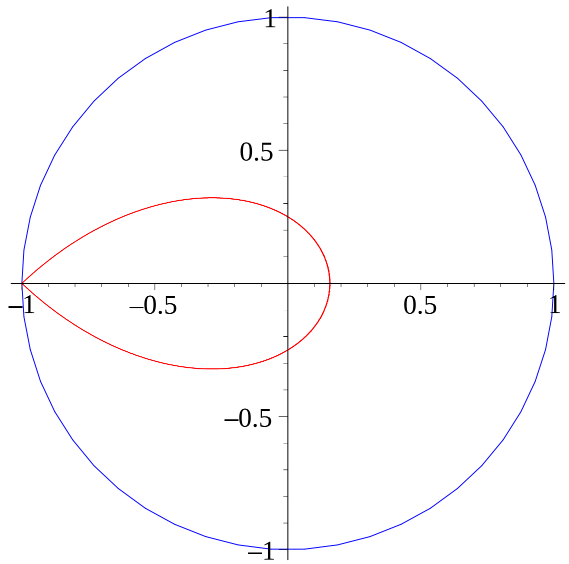

Fig. 5. Mapping of the circle $r = 1/4$ using the h-function of equation (32) with $\delta = 2.25$. As discussed in the text, this is actually a non-Euclidean parabola.

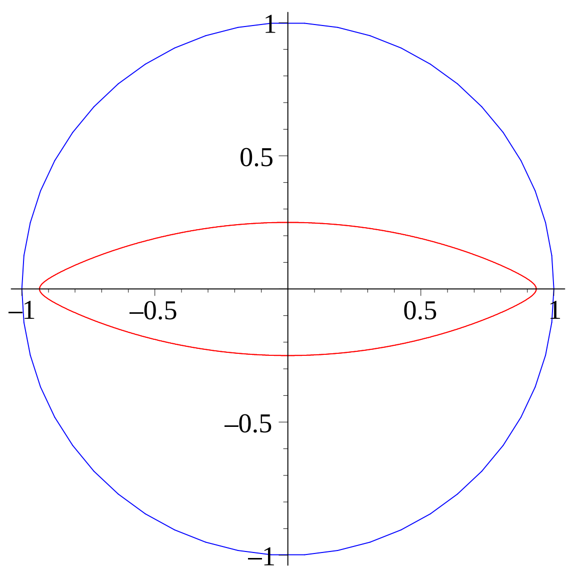

Fig. 6. Mapping of the circle $r = 1/4$ using the h-function of equation (35) with $\delta = 2.24$. This is a non-Euclidean ellipse.

This form leads to centered objects. As an example, choose $\delta = 2.24$, and map the circle $r = 1/4$ again. This leads to the locus shown in Fig.6. The explicit equations for (x', y') in this case are

$$
\begin{aligned}
x' &= \lambda^2 x(2+\delta)/\left(x^2+y^2+\lambda^2\right.\\
&\quad \left.+\sqrt{x^4+2x^2y^2-2x^2\lambda^2+y^4-2\lambda^2y^2+\lambda^4-4x^2\lambda^2\delta-\delta^2\lambda^2x^2}\right)\\
y' &= 2\lambda^2 y/\left(x^2+y^2+\lambda^2\right.\\
&\quad \left.+\sqrt{x^4+2x^2y^2-2x^2\lambda^2+y^4-2\lambda^2y^2+\lambda^4-4x^2\lambda^2\delta-\delta^2\lambda^2x^2}\right)
\end{aligned}
\tag{36}
$$

Now can we tie these structures into the concept of constant sum of non-Euclidean distances from two foci positions? If so where are they? One finds that indeed foci can be found such that sums of non-Euclidean distances agree with the h function method. However, this is quite complicated to prove, so we shall not go through this here, but instead just illustrate in the context of the example just given. The foci are in fact at $x = \pm 0.925$, whilst the intersections of the 'ellipse' with the x axis are at $x = \pm 0.934$. The relationship of the foci to the ends of the axes is illustrated in Fig.7, and of course in terms of our normal view of the relationship between an ellipse and its foci looks rather odd. However, the total sum of non-Euclidean distances to the two foci is genuinely constant (at 3.372).

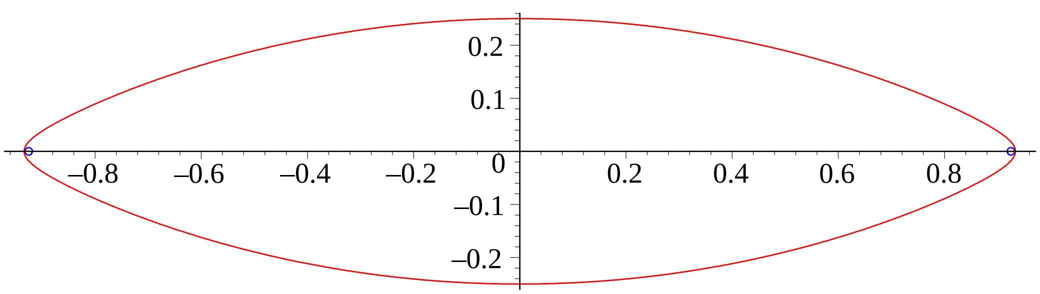

Fig. 7. The same non-Euclidean ellipse as in the previous figure, but with the locations of the non-Euclidean foci marked.

The same procedures can also be carried out for spherical, as against hyperbolic geometry. In this case we need to preserve $\bar{e}$ rather than e. Also, here it is possible to visualise the results rather more easily, by directly lifting the loci up onto a sphere for display purposes. Shown in Fig. 8 is the typical type of locus on the sphere that can result. A very interesting feature of such curves on the sphere is that they are simultaneously ellipses and hyperbolae! That is, the curve shown (in multiple views) in Fig. 8 satisfies simultaneously that the sum of distances (in a great circle sense on the surface of the sphere) to two given points is constant, and also that the difference of distances to two points is also constant. One can see that indeed, from some viewpoints the curve looks like an ellipse, whilst for others it looks like a hyperbola. Defining these curves this way, can of course be carried out without any recourse to geometric algebra. What is new here, is that these curves are actually constructed via a 'projectivity' acting on a circle in a non-Euclidean spherical space. The particular projectivity used corresponds to an h function

$$h(a) = a + \frac{1}{2}\delta\, a \cdot e_1 e \tag{37}$$

and the curve plotted is the image of a circle of radius 1/4, with $\delta = 10$.

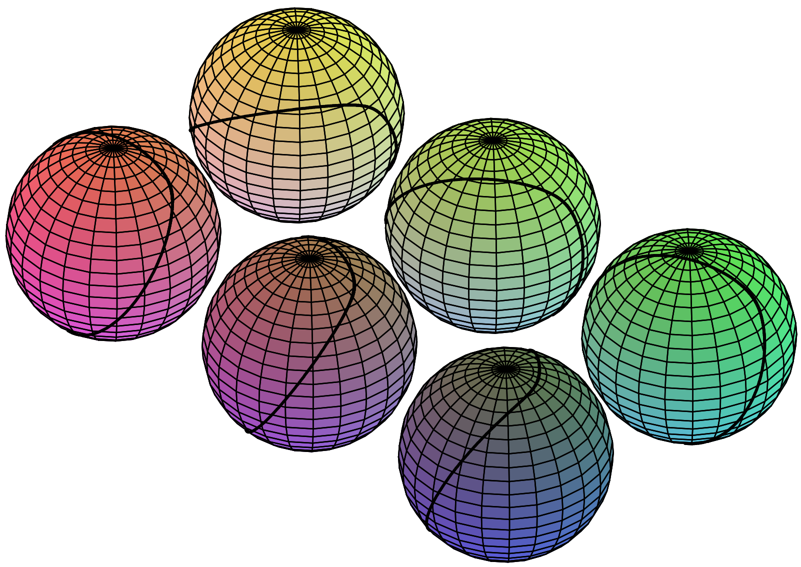

Fig. 8. A non-Euclidean ellipse/hyperbola displayed on the sphere. The curve is the image of the circle r=1/4, and uses $\delta = 10$ in the h-function given in equation (37).

Overall, as regards projective geometry, we see that the GA approach provides a flexible framework enabling a wide range of geometries to be treated in a unified fashion. Everything said here extends seamlessly to 3d and higher dimensions, although a lot of work remains to be done to explore the structures which become available this way.

6 The '1d up' approach to conformal geometry

In the approach to conformal geometry advocated by David Hestenes, two extra dimensions are necessary. However, do we really need these? From investigations that are still ongoing, it appears that for physics problems, just one extra dimension may be sufficient, and indeed more natural, while for pure geometry aspects, occasionally having in mind a further dimension is useful for abstract manipulation, but again one may not need it for actual calculation.

In terms of the physics applications, what makes this possible is using de Sitter or anti-de Sitter space as the base space, rather than Lorentzian spacetime. The rotor structure in these spaces allows us to do rotations and translations as for the 'Euclidean' (here Lorentzian) case, so that as regards the physics we can do everything we want, aside perhaps from dilations. (These, however, can be taken care of by an h-function, if necessary.) This can be achieved using only one extra dimension, namely $\bar{e}$ in the de-Sitter case, and e in the anti-de Sitter case, as we show below.

However, in the current conformal geometric algebra approach to Euclidean space, then conceptually one is always using both extra dimensions, since $\bar{n}$ is required as the origin, and n is integral in the translation formulae.

It may be objected that we do not wish to work with e.g. de Sitter space as a base space, since then there is intrinsic curvature in everything, and while this may be alright, maybe even desirable for physics purposes, it does not make any sense for engineering applications.

However, it turns out that in applications we can take a Euclidean limit at the end of the calculations, such that the curvature is removed, but we have nevertheless derived what we wanted whilst still only going up 1 dimension, so actually this is not a problem.

To be more specific on what we are going to do here, we shall start with a physical context, and embed spacetime in a 5d space, in which conceptually we are keeping e constant, therefore it is de Sitter space. However, we never actually need to bring e in to do the physics. Our basis vectors are thus

$$e_0(\equiv \gamma_0), \quad e_1(\equiv \gamma_1), \quad e_2(\equiv \gamma_2), \quad e_3(\equiv \gamma_3), \quad \text{and} \quad \bar{e} \tag{38}$$

with squares $+1$, -1, -1, -1, -1 respectively.

Let $Y = y^\mu e_\mu$ be a general vector in this space. For our purposes here, it is sufficient to restrict ourselves to the part of Y space in which $Y^2 < 0$. Our fundamental representation can be expressed via the null vector X used in our previous 6d embedding space for the de Sitter case [10]. This was scaled so that $X{\cdot}e = -1$. Our new representation works via

$$\hat{Y} = X + e \tag{39}$$

which then via X corresponds to a point in 4d. $\hat{Y}$ is a scaled version of Y satisfying

$$\hat{Y}^2 = -1 \tag{40}$$

We note that $\hat{Y}{\cdot}e = X{\cdot}e + e{\cdot}e = -1 + 1 = 0$, so that indeed $\hat{Y}$ does not contain e. Moreover,

$$\hat{Y}^2 = (X + e)^2 = 2X{\cdot}e + e^2 = -1 \tag{41}$$

consistent with the definition of $\hat{Y}$.

Of course we could also go directly from a general Y point to a point in 4d de Sitter space, without using X as an intermediary. The formulae for this, which we give assuming $y^4 > 0$, are quite simple:

$$x^\mu = \frac{\lambda y^\mu}{l + y^4}, \qquad \mu = 0, 1, 2, 3 \tag{42}$$

where $l = \sqrt{-Y^2}$. We can see that the x^μ so defined are homogeneous of degree 0 in the y^μ, and therefore depend only on the direction of Y, $\hat{Y}$, in agreement with the definition via X.

6.1 Geometry

As a start, we need to reassure ourselves that the conformal geometry we carried out in 6 dimensions in e.g. [10], is all still possible and works here in 5d.

We have seen so far that points in spacetime are no longer represented by null vectors, but by *unit* vectors, the $\hat{Y}$ discussed above. However, null vectors still turn out to be very important — these are the vectors which represent the boundary points. In physical terms, they are *null momenta*, and we shall label boundary points as P to emphasise this. To convince ourselves that the boundary points are null, as in the treatment in [10], even though the interior points have unit square (actually negative unit square, but the same principle), we can take a limit as an interior point approaches the boundary.

Using the notation for coordinates as in [10], consider as an example the $\hat{Y}$ corresponding to the point

$$x = (\mathsf{t}, \mathsf{x}, \mathsf{y}, \mathsf{z}) = (-\lambda(1-\delta), 0, 0, 0) \tag{43}$$

This is

$$\hat{Y} = \frac{1}{\delta(2-\delta)}((2 - 2\delta + \delta^2)\bar{e} - 2(1-\delta)e_0) \tag{44}$$

It is easy to verify that this indeed satisfies $\hat{Y}^2 = -1$. Taking the limit as $\delta \to 0$ we obtain

$$\hat{Y} \to \frac{1}{\delta}(\bar{e} - e_0) \tag{45}$$

as the first term of an expansion of $\hat{Y}$ in δ. This tells us that indeed $\hat{Y}$ is tending to a null vector as x tends to the lower boundary. We decide on a choice of scale for this particular null vector by writing

$$P = P_0 = \bar{e} - e_0 \tag{46}$$

for this point. We will define the corresponding point on the top boundary by $P_0' = \bar{e} + e_0$ (see Fig. 9).

One of the main features of the Hestenes approach to conformal geometry, is that it encompasses *distance geometry* [16], in that inner products of null vectors correspond to Euclidean distances between points. In [10] this was extended to

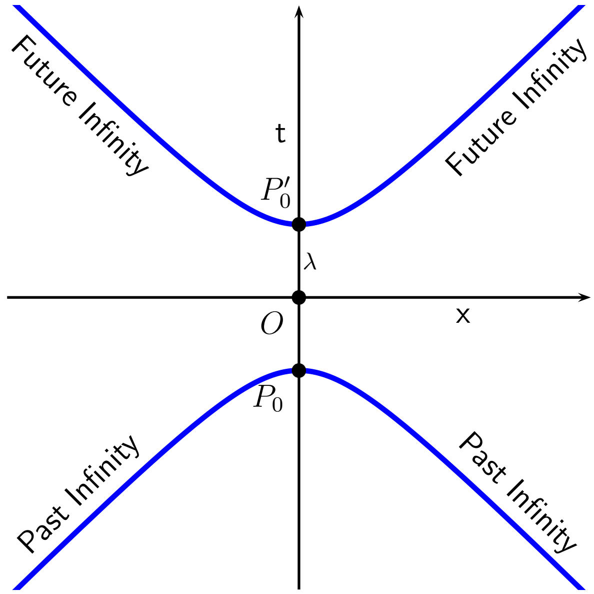

Fig. 9. The boundaries of de Sitter space (blue lines) in our conformal setup. The positions of the 4d points corresponding to the null momenta $P_0 = \bar{e} - e_0$ and $P_0' = \bar{e} + e_0$ are also shown.

the non-Euclidean case. He we can achieve exactly the same by noting that for two points $\hat{Y}_1$ and $\hat{Y}_2$, with corresponding null vectors X_1 and X_2, we have from equation (39) that

$$\begin{aligned} \hat{Y}_1 \cdot \hat{Y}_2 &= (X_1 + e) \cdot (X_2 + e) \\ &= X_1 \cdot X_2 - 1 \end{aligned} \tag{47}$$

Thus the new version of equation (3.6) from [10] is

$$d(x_1, x_2) = \begin{cases} \lambda \sinh^{-1} \sqrt{-(\hat{Y}_1 \cdot \hat{Y}_2 + 1)/2} & \text{if} \quad \hat{Y}_1 \cdot \hat{Y}_2 < -1, \\ 0 & \text{if} \quad \hat{Y}_1 \cdot \hat{Y}_2 = -1, \\ \lambda \sin^{-1} \sqrt{(\hat{Y}_1 \cdot \hat{Y}_2 + 1)/2} & \text{if} \quad \hat{Y}_1 \cdot \hat{Y}_2 > -1. \end{cases} \tag{48}$$

Note that $\hat{Y}_1 \cdot \hat{Y}_2 = -1$ is the condition for x_1 and x_2 to lie on each others light cones.

We now need to decide what operations can be carried out on points in our new representation. Equation (39) shows that any operation in the old null vector approach which preserves e will transfer covariantly to the new approach. This means that the three basic operations discussed in [10] will all work here, namely:

$$\begin{aligned} &\text{Rotations, carried out with rotors with bivector generators} \quad e_\mu e_\nu \\ &\text{Translations, carried out with rotors with bivector generators} \quad \bar{e}\, e_\nu \\ &\text{Inversions, carried out via} \quad \text{‘object’} \mapsto e\,\text{‘object’}\, e \end{aligned} \tag{49}$$

(Here μ and ν belong to the set $\{0,1,2,3\}$.) We see that in terms of their effect on a point, inversions correspond to $\hat{Y} \mapsto -\hat{Y}$. The geometrical significance of this is important physically, but would take us too far afield to discuss further here.

In terms of building up a geometry, the next step is to find out what lines (geodesics) correspond to. In the conformal geometry approach using two extra dimensions, a d-line in de Sitter space through the points with null vector representatives A and B is $A\wedge B\wedge e$ (see [10]). For us, lines become bivectors, and if A and B are now the *unit* vector representatives, then the (un-normalised) line through them is represented by

$$L = A\wedge B \tag{50}$$

In particular, if the point Y lies on the line, then

$$Y\wedge L = Y\wedge A\wedge B = 0 \tag{51}$$

This corresponds to how we would represent a line in projective geometry. However, even though we only use the same dimensionality as would be used in projective geometry, the availability of the operations in equation (49) means we can go far beyond the projective geometry in terms of how we can manipulate lines and points. E.g. the rotor for translation through a timelike 4d vector a is (see [10])

$$R_T = \frac{1}{\sqrt{\lambda^2 - a^2}} (\lambda + \bar{e}a) \tag{52}$$

Applying this to L, we get a translated line, in exactly the same way as we would get if we applied it to the null vector version, except here we only have to work in 5d geometry rather than 6d. This is a computational advantage to having lines as bivectors rather than trivectors. Two more profound advantages are as follows.

A problem of the null vector representation is that if two null vectors are added together, then we do not in general get another null vector. Thus we cannot form weighted sums of points, in the same way as one could in projective geometry. However, here we can. If we take

$$Y_3 = \alpha\hat{Y}_1 + \beta\hat{Y}_2 \tag{53}$$

then the resulting Y_3 is generally normalisable, so that we can define a new point this way, exactly as in the projective case. The set of points formed this way (i.e. as α and β are varied over some range) corresponds to the line $\hat{Y}_1\wedge\hat{Y}_2$, since of course

$$(\alpha\hat{Y}_1 + \beta\hat{Y}_2)\wedge(\hat{Y}_1\wedge\hat{Y}_2) = 0 \tag{54}$$

for all α and β. Exactly the same comments apply to *d-planes*, formed via

$$\Phi = \hat{Y}_1 \wedge \hat{Y}_2 \wedge \hat{Y}_3 \tag{55}$$

and linear combinations of the form

$$Y_4 = \alpha \hat{Y}_1 + \beta \hat{Y}_2 + \gamma \hat{Y}_3 \tag{56}$$

etc.

The second advantage of the new approach is as follows. Consider a d-line L formed from two normalised points A and B as in equation (50). Let us contract L with a point on the line, say A. We obtain

$$L \cdot A = (A \wedge B) \cdot A = (B \cdot A) A - A^2 B = (B \cdot A) A + B \tag{57}$$

We claim this is the *tangent vector* to the line at A. The point is that if we add a multiple of this vector to A, we will get a point along the line joining A to B. This is a very direct definition of tangent vector. We shall see below that if we normalise things appropriately, then with the path along the geodesic L written as $\hat{Y}(s)$, with s a scalar parameter, then

$$\dot{\hat{Y}} = \frac{d\hat{Y}(s)}{ds} = L \cdot \hat{Y}(s) \tag{58}$$

gives a unit tangent vector at each point of the path, which we can take as the *velocity* of traversing the path. Thus we can see that the line itself acts as a *bivector generator of motion along the line*. Clearly one will also be able to form a rotor version of this equation as well. A novelty here is that *the bivector generator acts on the position to give the velocity*. This is in contrast to similar equations in 4d, e.g. the Lorentz force law, where the bivector generator (the electromagnetic Faraday) acts on the velocity to give the acceleration.

Within the null vector approach to conformal geometry, if we attempt to take a tangent to a line we obtain a bivector which must be wedged with e again before it becomes meaningful. This then recovers the line itself again [2]. This is sensible, at one level, but precludes anything to do with the bivector generator approach, which we have found to be very useful in physical applications.

We next return to the boundary points. These give a very good way of encoding geodesics. Let P be a point on the lower boundary, and P' a point on the upper boundary, as in Fig. 10.

If we form a linear combination of P and P', with positive coefficients, say

$$Y = \alpha P + \beta P' \tag{59}$$

then this defines an interior point Y on the geodesic (d-line) joining them. Furthermore, the bivector representing this line is $P \wedge P'$. Since P and P' are null, then we see that the tangent vector at Y is given up to scale by

$$T_Y = \beta P' - \alpha P \tag{60}$$

We will discuss elsewhere how to relate the direction of this tangent vector in Y space, with the direction of the corresponding tangent vector (e.g. as shown on the plot) in x space, but note here that it works in a simple and straightforward manner.

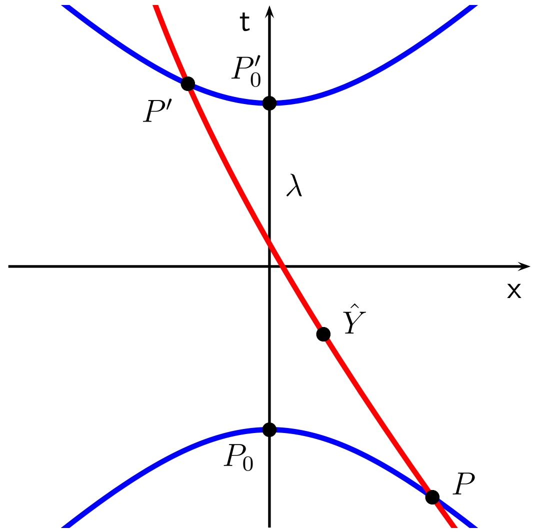

Fig. 10. Plot illustrating the formation of an interior point by linear combination of two boundary points. For the example shown, $Y = 3P + P'$. The part of the geodesic between the boundary points is given by considering the set of points $\alpha P + \beta P'$ for all positive α and β. The direction in Y space of the tangent vector at $\hat{Y}$, is given in general by $\beta P' - \alpha P$ (see text), and so is in the direction $P' - 3P$ at the point shown.

6.2 Non-Euclidean circles and hyperbolae

Obviously a key feature of the Hestenes approach to conformal geometry is the way it deals with circles and spheres. This is discussed in the context of non-Euclidean geometry in, for example, [2, 9]. This approach is obviously very useful, and may be preferable to what we are going to discuss here in some applications. However, what we will show here, is that the functionality of this approach can be effectively reproduced using one less dimension for the actual computations. Taking the specific case of de Sitter spacetime, we show that while the availability of the basis vector e is very useful for conceptual and analytic purposes, it is actually better not to have any e component in the vectors representing the points. This is because it is then possible to separate out cleanly the parts of an equation containing e and those that do not, and then results can be obtained from each part separately. These tend to give expressions which achieve what we want, but with many fewer computations than in the 'two dimensions up' case.

To take a specific case, we consider finding the centre and radius of a non-Euclidean circle in the $(\mathsf{x}, \mathsf{y}, \mathsf{z})$ hyperplane within a de Sitter spacetime. (This transfers through directly to the approach to 3d Euclidean space which we discuss below.)

We take the circle to be defined by three unit vector points A, B and C. We therefore know from the null vector approach that the circle is represented by

$$\Sigma = (A - e) \wedge (B - e) \wedge (C - e) \tag{61}$$

and that points $\hat{Y}$ on the circle must satisfy

$$(\hat{Y} - e) \wedge (A - e) \wedge (B - e) \wedge (C - e) = 0 \tag{62}$$

Expanding out (61) we obtain

$$\Sigma = V - eS \tag{63}$$

where

$$V = A \wedge B \wedge C \quad \text{and} \quad S = A \wedge B + B \wedge C + C \wedge A \tag{64}$$

Now none of $\hat{Y}$, V or S contains e. Thus (62) rearranges to

$$\hat{Y} \wedge V - eV + e\hat{Y} \wedge S = 0 \tag{65}$$

and thus we get the two conditions

$$\hat{Y} \wedge V = 0 \tag{66}$$

$$\hat{Y} \wedge S = V \tag{67}$$

In fact (67) implies (66), so we need just (67), plus the fact that $\hat{Y}$ is a unit vector, for the circle to be specified.

Now, to get the centre of the circle passing through A, B and C, we know that in the null vector approach the quantity $\Sigma e \Sigma$ will be important. This is a covariant construction yielding a vector, but of course not a null vector, and finding the explicit null vector representing the centre from this is slightly messy (see [2, 9]). Here we know straightaway that the point we want will be the 'non-e' part of $\Sigma e \Sigma$. Since S and V commute, we get

$$\Sigma e \Sigma = e(S^2 - V^2) - 2VS \tag{68}$$

from which we can deduce that the unit vector representing the non-Euclidean centre of the circle is

$$D = -\frac{SV}{|S||V|} \tag{69}$$

(The choice of sign here corresponds to keeping $y^4 > 0$.) Furthermore, it is easy to show that the non-Euclidean 'radius' (in the sense of minus the dot product of a point on the circle with the centre), is given by

$$\rho = -\hat{Y} \cdot D = \frac{|V|}{|S|} \tag{70}$$

($\hat{Y}$ being any point on the circle).

These operations (particularly the one for finding the centre) seem likely to be much easier to compute than the equivalent ones in the 'two dimensions up' approach. Moreover, they are susceptible of a further conceptual simplification. The fact that V and S commute is due to V acting as a pseudoscalar within the (non-Euclidean) plane spanned by the triangle. Viewed this way, we can see that *the centre of the circumcircle is the 'dual' of the sum of the sides*. This is a nice geometrical result.

More generally, for any point $\hat{Y}$ in the plane of the triangle, we can form its dual in V, which will be a d-line, and for any d-line we can get a corresponding dual point. To see this, let us take e.g. the dual of the 'side' $A\wedge B$. We will denote this by C', i.e.

$$C' = -A\wedge B\, E_V \tag{71}$$

where $E_V = V/|V|^2$ and we have adopted a convenient normalisation for the dual point. We see that $\hat{Y}\cdot C' = 0$ for any $\hat{Y}$ on the d-line $A\wedge B$. Thus all points on this line are equidistant from C', which provides another definition of this dual point. Defining further

$$A' = -B\wedge C\, E_V \quad \text{and} \quad B' = -C\wedge A\, E_V \tag{72}$$

we can see that D satisfies

$$D = \rho(A' + B' + C') \tag{73}$$

i.e., in order to find the circumcircle centre, we take a linear combination, with equal weights, of the points dual to the sides.

A yet further useful way to understand this, is via the notion of *reciprocal frame*. Defining

$$\begin{aligned} A_1 &= A, \quad A_2 = B, \quad A_3 = C \\ A^1 &= A', \quad A^2 = B', \quad A^3 = C' \end{aligned} \tag{74}$$

then we have

$$A^i\cdot A_j = -\delta^i_j \quad \text{for} \quad i,j = 1,2,3 \tag{75}$$

which (modulo a sign) is the usual defining relation for a reciprocal frame. This relation immediately makes clear how (73) is consistent with (70). For example, dotting (73) with A, we immediately get $A\cdot D = -\rho$, as expected.

This then suggests perhaps the simplest way of finding the centre and radius of a non-Euclidean circle defined by three points. Let the three points be called A_1, A_2 and A_3, and define a reciprocal frame to these via (75). Then the centre and radius of the (non-Euclidean) circle through the three points are given by

$$D' = A^1 + A^2 + A^3 \tag{76}$$

and

$$\rho^2 = -\frac{1}{D'^2} \tag{77}$$

respectively.

The calculations to get the reciprocal frame are of course the same as in (71) and (72), but this route seems very clean conceptually, and gets both the centre and radius in one go. Moreover, we can immediately generalise to a sphere and its centre.

Let A_i, $i = 1 \ldots 4$, be four points through which we wish to find a non-Euclidean sphere. Form the reciprocal frame A^i and then let

$$D' = \sum_{i=1}^{4} A^i \tag{78}$$

Then with $\rho^2 = -1/D'^2$, we have that ρ is the radius of the sphere, and $\rho D'$ is the (normalised) point representing its centre. At this point we have a method which is entirely independent of the 'null vector' approach and of the use of e. This statement follows since the result can be proved immediately in its own terms — in particular, by virtue of the definition of the reciprocal frame, the D' defined by (78) is clearly equidistant from each A_i, with 'distance' given by ρ.

7 Electromagnetism, point particle models and interpolation

The remark after equation (58) suggests an analogy with the Lorentz force law of electromagnetism in GA form: this is $dv/ds = F \cdot v$ where v is the particle 4-velocity and F is the Faraday bivector $\boldsymbol{E} + I\boldsymbol{B}$ (the I here being the pseudoscalar for spacetime). The details will be the subject of a further paper, but it is possible to set up electromagnetism in an interesting way in the 5d space described above. Using this approach, standard problems take on an intriguingly geometrical character, and then one can move down to 4d spacetime in order to find their physical consequences. An important feature is that solutions can be 'moved around' with rotors in 5d, which then correspond to translated and rotated solutions in 4d. This can be expected to be useful e.g. in numerical calculations of the scattered fields from conductors which can be decomposed into surface facets. Fields can then be moved round between facets, themselves described with conformal geometric algebra, by using appropriate rotors in 5d.

We content ourselves with one illustration here, which shows clearly the geometrical character which electromagnetism takes on in 5d. After this we show how the type of point particle model which this view leads to, can give us an insight into something more directly relevant for computer graphics, namely interpolation.

It is possible to set up a point particle Lagrangian which leads to an equation for the position of a particle that is same as for d-line motion above, namely $dY/ds = L \cdot Y$ where L is the line the particle is currently moving along. This would be constant d-line in the absence of any forces. The effect of an electromagnetic field is that now the line L, instead of being constant, satisfies

$$\frac{dL}{ds} = F \times L \tag{79}$$

where $\times$ is the GA commutator product ($A \times B = (1/2)(AB - BA)$) and F is the 5d Faraday bivector. As an example of the latter, the field at position Y due to a point charge moving along a geodesic (d-line) L' is (ignoring some constant factors)

$$F = \frac{Y \cdot (Y \wedge L')}{|Y \wedge L'|^3} \tag{80}$$

This is clearly singular, as one would expect, if Y lies on the line L' itself. In the last two equations we see a very interesting feature of the physics that emerges in 5d, which is that it is not points which are important, but the whole world line which a particle travels over.

One can also set up a rotor version of the above in which the rotor encodes the rotation of a fixed set of fiducial axes, in a similar way to the GA formulation of rigid body mechanics. The fixed axes are $\bar{e}$ (the origin) and e_0 (the reference timelike velocity of the body). These get rotated to $\hat{Y}$ (the particle position) and $\hat{T}$ (the instantaneous velocity), via

$$\hat{Y} = \psi \bar{e} \tilde{\psi}, \quad \hat{T} = \psi e_0 \tilde{\psi} \tag{81}$$

where ψ is the rotor. It turns out that ψ itself then satisfies the rotor equation

$$\frac{d\psi}{ds} = \frac{1}{2} L\psi - \frac{1}{2} F(\hat{Y}) \cdot \hat{T} \hat{T} \psi \tag{82}$$

where $L = \hat{Y}\hat{T}$. This is very novel. We now have a axis pointing out of the particle, $\hat{Y}$, which represents where it is, rather than its velocity. An actual example of such a computation is shown in Fig. 11. This is for the 5d version of a constant electric field in the x direction. At the lower edge of the diagram is the past infinity surface from which the particle emanates. The geodesic motion that would be implied by its initial motion is shown as the 'line corresponding to past infinity'. Similarly its asymptotic final motion is shown as the 'line corresponding to future infinity'. Inbetween, one can see the actual path of the particle, which suffers acceleration in the x direction, due to the action of the electric field. A very intriguing feature of this diagram, is that the asymptotic motion of the particle towards past and future follows free-particle geodesic lines, rather than becoming ever more accelerated, which is what happens in the standard approach for motion under a constant field in 4d.

The point particle model we are using prompts the question of whether one could use bivector interpolation to interpolate motion between any two ψs. This will then be performing interpolation of position and rotation simultaneously, but just using one dimension up. In addition to reduced computational complexity, we suggest that this removes an ambiguity which would otherwise be present in the 2d up approach. In particular, we claim that the '1d up' approach has exactly the right number of degrees of freedom to simultaneously represent the position and rigid body attitude of a body in n dimensions. We see this as follows. Starting with e.g. a 3d body we have that the rigid body attitude accounts for 3 d.o.f., while rigid body position accounts for another 3, implying

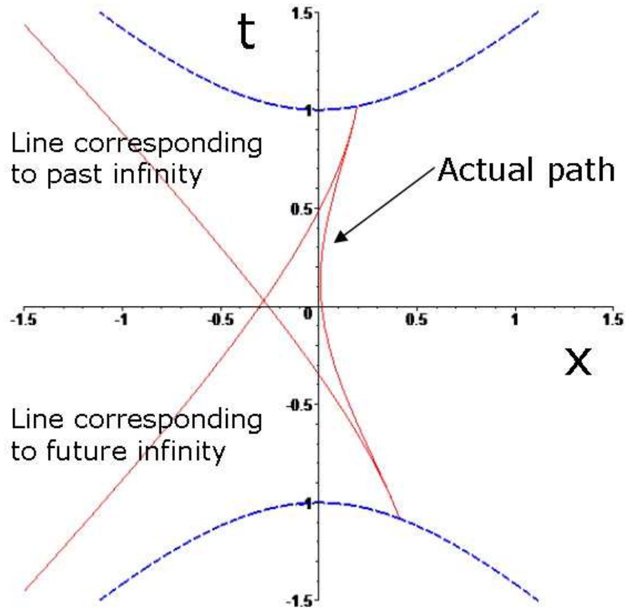

Fig. 11. Motion of a charged particle in the 5d version of a constant electric field. The particle emanates from past infinity, initially along a d-line (free-particle geodesic), but is then accelerated by the field onto a different d-line before reaching future infinity.

overall 6. This is the same as the number of components of a 4d bivector and thus of a rotor in 4d. Generally, to describe rotation in n dimensions, we need $1/2n(n-1)$ components to describe rotation, whilst position needs another n. Therefore the total is $1/2n(n+1)$, which is the same as the number of bivector (and therefore independent rotor) components in $n+1$ dimensions. In the 2d up approach, rotors have more components than we need, and thus there is an inherent ambiguity in linking rotors with the actual attitude and position of a rigid body. Of course, it may be objected that we have only achieved the possibility of working with one dimension extra by going to curved space. However, one can explicitly demonstrate for this case of interpolation, that one can bring everything back unambiguously to Euclidean geometry, by letting $\lambda \to \infty$ at the end of the calculations.

A further problem, which we believe this approach solves, is that so far no proper Lagrangian formulation or motivation has been found for interpolation in the '2d up' approach. A Lagrangian approach would be very desirable, since it would then give a rationale for preferring a particular method of interpolation over others. For modelling rigid body motion in the 1d up approach, we propose the rotor Lagrangian

$$\mathcal{L} = -\frac{1}{2}(-\tilde{\psi}\dot{\psi} + \dot{\tilde{\psi}}\psi) \cdot (-\tilde{\psi}\dot{\psi} + \dot{\tilde{\psi}}\psi) + \theta(\psi\tilde{\psi} - 1) \tag{83}$$

Here θ is a Lagrange multiplier enforcing the rotor character of ψ. This Lagrangian is the specialisation of that for rigid body dynamics given on page 426 of [9], to the case where the inertia tensor is spherically symmetric. Its first term is effectively the 'rotational energy' of the spinning point particle we are using to represent the axes in our higher dimensional space. The differentiation is w.r.t a path parameter s. One finds the solution to the equations of motion is $\psi = \exp(Bs)$, where B is a constant bivector. This therefore indeed gives bivector interpolation as the way in which one set of axes transforms into another. Two examples of this type of motion are shown in Fig. 12. In these examples, we

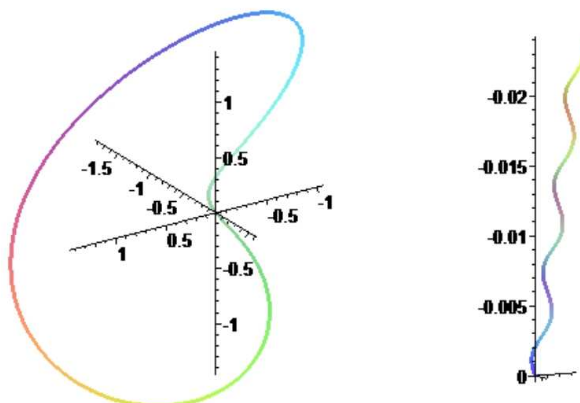

Fig. 12. Examples of interpolation based upon the 'spinning point particle' Lagrangian (83). The position of the rigid body, based upon the direction of the $\hat{Y}$ axes, is shown. Both are calculated in 4d 'spherical space', and at the left a complete closed path is shown. At the right, the Euclidean limit is shown, resulting in a helical path near the origin.

are dealing with rigid body motion in 3d, and using a rotor in 4d. In dealing with 3d space and using the '1d up' approach, we have a choice of signature of the extra axis. Assuming we take the 3 space axes as having positive square, then if we add an extra axis with negative square, we will be working with hyperbolic 3d space, whilst if we add an axis with positive square, we will have spherical space. The example given above for finding the centre of a (non-Euclidean) sphere through 4 points, was in a spherical space, which is the natural restriction of our version of de Sitter space to spatial directions only. For some other applications, hyperbolic space might be more appropriate. The question would then be, for engineering applications, whether the Euclidean limit differs, depending on whether one works in spherical or hyperbolic space in 1d up. Both the examples shown in Fig. 12 are for spherical space, as is evident in the picture at the left, which shows a complete closed curve as the orbit under $\psi = \exp(Bs)$ interpolation. At the right, we see a different case, which is in the Euclidean limit in which $\lambda \rightarrow \infty$. Here the position interpolation reduces to a constant pitch helix, and one can show (details will be given elsewhere), that exactly the same helix is obtained whether one reduces to the Euclidean limit from spherical space or hyperbolic space, which is reassuring as to the meaningfulness of the result.

8 Non-Euclidean projective geometry revisited, and the role of null vectors

Having established the utility of the 1d up approach, it is worth returning to what we did in Section 1 and asking whether the role of null vectors in 2d up is crucial in this. The answer is no! We can achieve all the results given in that section by only moving 1d up. This is to some extent obvious from the fact that we insisted that $h(n)$, $h(e)$ and $h(\bar{e})$, left n, e or $\bar{e}$ invariant, respectively, in the Euclidean, hyperbolic or spherical cases. We can thus 'throw away' e or $\bar{e}$ as appropriate, and work only in the 1d up space left. As a specific example, let us say that we want to carry out projective geometry in the hyperbolic 2d plane (the Poincaré disc). Then our basis vectors in the 1d up conformal representation are e_1, e_2 and $\bar{e}$, with $e_1^2 = 1$, $e_2^2 = 1$ and $\bar{e}^2 = -1$. We let $h(a)$ be a general, constant linear transformation in this 3d space, and define for a general point Y, that its image under the 'projectivity' h is

$$Y' = h(Y) \tag{84}$$

There is no need to try to go back to a null vector here. Assuming Y^2 is non-zero and has the right sign, a general point Y immediately represents a point in the disc, and if it is null it represents a point on the boundary. It is then easy to show that an h which leaves the boundary invariant corresponds to 'rigid' (rotor) motions in the interior, whilst if it does not, it corresponds precisely to the non-Euclidean projective transformations we studied above. As an example, the h given in equation (32) can be used directly in (84) and again yields the non-Euclidean figure shown in Fig. 5. We can see that it will be a projectivity, since e.g. the boundary point $e_1 + \bar{e}$ is mapped to the (non-null) vector representing an interior point $e_1 + (1 + (1/2)\delta)\bar{e}$.

Where then, can one see a natural role for the 2d up approach, and the null vectors it uses? The most natural way this seems to appear is shown in the example of Fig. 13. Here we have the 'Poincaré ball' in 3d. The boundary of this consists of a set of null vectors (the surface of the sphere). This boundary is a 2d space, corresponding with the set of null vectors (with scale ignored) in the 4d space spanned by e_1, e_2, e_3, $\bar{e}$. Thus in this boundary space we are automatically working with null vectors in the 2d up approach! The example shown of this, is for the intersection of two d-planes within the unit ball. Each d-plane is specified by 3 boundary points, A, B, C and A', B', C' respectively. When we intersect the d-planes, we get a d-line in the interior, which intersects the boundary at the points labelled P and Q. On the boundary itself, the interpretation is that

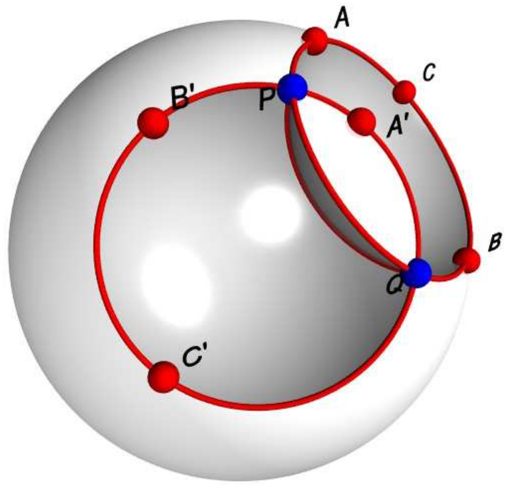

Fig. 13. Diagram illustrating the relation between intersection of hyperplanes in the Poincaré ball, and the intersection of circles, defined via null vectors, in the boundary surface of the ball. See text for details.

we have intersected the two *circles* $A \wedge B \wedge C$ and $A' \wedge B' \wedge C'$ and the result is the bivector formed from the points of intersection P and Q. This construction throws some light on the role of wedges of two null points, which occurs very frequently in the 2d up approach. We see that we can reinterpret them as the end points of a d-line in one dimension higher, and in fact this bivector is precisely the representation of the d-line in the 1d up approach.

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