A State-Space Based Approach To Quantum Field Theory In Classical Background Fields

Carl E. Dolby

Cavendish Astrophysics Group
and
Trinity College, Cambridge.

A dissertation submitted for the degree of Doctor of Philosophy in the University of Cambridge.

December 2000

Preface

This dissertation is the result of work carried out in the Astrophysics Group of Cavendish Laboratory between October 1997 and December 2000. Except where explicit reference is made to the work of others, the work contained in this dissertation is entirely my own. Although previously unpublished, much of it is currently being prepared for publication[1, 2, 3, 4, 5, 6]. None of it has been submitted in any form for a degree, diploma or other qualification at this or any other university. This dissertation does not exceed 60 000 words in length.

Carl E. Dolby

Acknowledgements

My most grateful thanks go to Dr Steve Gull for his numerous insightful comments, our many helpful discussions, and his timely words of encouragement. Thanks also go to Dr A. N. Lasenby and Dr C. J. L. Doran for helpful discussions, Dr Nando de Freitas for proof reading this manuscript, and Trinity College, Cambridge for financial support. Special thanks go to Jim Reeds of AT&T Labs, USA, for his help with the Cauchy-Binet theorem, and the MacMahon Master theorem, to Professor Alan Guth for helpful discussions during his stay at the Isaac Newton Institute for Mathematical Sciences, and to Professor Emil Mottola for helpful discussions during our time at the Institute for Nuclear Theory, University of Washington. I would like to thank the friendly staff of the Institute for Nuclear Theory, for their hospitality during my visit there. Also, thanks to all my friends, house-mates, and other drinking partners, who have kept me from straying too far from my sanity. Finally, I would like to dedicate this dissertation to Dr Henry Levy, from whom I learnt special relativity as an undergraduate at the University of Otago. His enthusiasm for k-calculus made 'radar time' seem inevitable.

Abstract

This dissertation is concerned with a new formulation of fermionic quantum field theory in classical (electromagnetic or gravitational) backgrounds, which uses methods analogous to those used in conventional multiparticle quantum mechanics. Emphasis is placed on the states of the system, described in terms of Slater determinants, rather than on the field operator, $\hat{\psi}(x)$. The vacuum state 'at time τ', defined as the Slater determinant of a basis for the span of the negative spectrum of the 'first quantized' Hamiltonian $\hat{H}_1(\tau)$, provides a concrete realisation of the Dirac Sea. The general S-matrix element of the theory is derived in terms of time-dependent Bogoliubov coefficients, demonstrating that this follows directly from the definition of inner product between Slater determinants. The process of 'Hermitian extension', inherited directly from conventional multiparticle quantum mechanics, allows second quantized operators to be defined without appealing to a complete set of orthonormal modes, and provides an extremely straightforward derivation of the general expectation value of the theory.

By using the concept of 'radar time' originally made popular by Bondi in his work on k-calculus, the particle interpretation is generalised to an arbitrarily moving observer in curved spacetime. I show that this particle interpretation depends *only* on the choice of observer and the background present, not on the choice of coordinates, the choice of gauge (in electromagnetic backgrounds) or the detailed construction of the observer's particle detector. It is also the first definition that does not rely on the spacetime possessing any convenient symmetries. I show that in the cases of a uniformly accelerating observer in flat space (Unruh effect), and a comoving observer in an exponentially inflating universe, my definition reduces to previously accepted definitions. Although this definition is necessarily non-local (no local definition of particle could possibly be consistent with the Unruh effect) I demonstrate with a simple example that this non-locality is only significant on scales of the order of the Compton wavelength $\lambda_c = \frac{h}{mc}$ of the particle concerned. Applications of the formalism to pair creation in spatially uniform electric fields, and to the treatment of discrete symmetries, are also presented.

Contents

1 Introduction **1**

2 Fermions in an Electromagnetic Background **5**

2.1 The State Space 6
2.1.1 Preliminaries 6
2.1.2 The Full Fock Space Over $\mathcal{H}$ 7
2.1.3 Operators on Fock Space 10
2.1.4 Evolution of States, and S-Matrix Elements 11
2.2 A Particle Interpretation of State Space 12
2.2.1 The Positive/Negative Energy Split and the Vacuum State 12
2.2.2 Particle States Built On The Vacuum 15
2.2.3 Vacuum Subtraction and Physical Operators 17
2.2.4 Non-locality, and Limitations of the Particle Concept 21
2.3 Bogoliubov Coefficients and Their Application 23
2.3.1 Some Definitions 23
2.3.2 Properties of the Evolved Vacuum 26
2.3.3 S-Matrix Elements in General 31
2.4 Back-Reaction and the Mean Field Approximation 34

3 Gravitational Backgrounds **37**

3.1 Dirac Equation in a Gravitational Background 37
3.2 An Observer and Their Particle Interpretation 38
3.2.1 The Travelling Twin. 41
3.2.2 Uniformly Accelerating Observer. 43
3.2.3 DeSitter Space. 44
3.3 Bogoliubov Coefficients, S-Matrix Elements Etc. 45

4 Relationship to Conventional Methods **47**

4.1 Preliminaries 47
4.2 S-Matrix Elements 50
4.3 Expectation Values and Vacuum Subtraction 52
4.4 On Hamiltonian Diagonalisation 55
4.5 Greens Functions 59
4.6 Conclusion 64

5 Continuous and Discrete Symmetries **65**
5.1 Gauge Invariance and Lorentz Covariance 65
5.2 Parity Reversal 68
5.3 Time Reversal 71
5.4 Charge Conjugation 74

6 Some Perturbative Calculations **79**
6.1 Pair Creation To Lowest Order 81
6.2 Equivalence To Conventional Methods 82

7 Application to Uniform Electric Fields **85**
7.1 A Complete Set of Solutions 86
7.2 Energy Eigenstates 89
7.3 Finite Time Transition Probabilities 91
7.4 Constant Electric Fields 93
7.4.1 Number Density In The Evolved Vacuum 94
7.4.2 Vacuum Current, and Induced Electric Field 98
7.4.3 Non-Vacuum Initial Conditions 102

8 Bosonic Systems **107**
8.1 The Equations 107
8.2 Bogoliubov Coefficients and S-Matrix Elements 109
8.3 Expectation Values and Vacuum Subtraction 114
8.4 Gravitational Backgrounds 115

9 Discussion and Outlook **118**

List of Figures

3.1 Schematic of the definition of 'radar time' $\tau(x)$. 39

3.2 Hypersurfaces of simultaneity of the 'travelling twin' in the well-known 'twin paradox' (schematic). 42

3.3 Hypersurfaces of simultaneity of a uniformly accelerating observer. 43

3.4 DeSitter space in (t, r) coordinates. The particle horizon $r_\infty(t) = \frac{e^{-\lambda t}}{\lambda}$ is shown in mid grey. The light grey lines represent ingoing and outgoing photon trajectories, while the black lines are the observers hypersurfaces of simultaneity for various τ_0. 45

7.1 Number density in an infinitely evolved vacuum, as a function of $p_z = \frac{p^3_{out}}{\sqrt{eE}}$ for $M = 1$ and $p = 0$ (top curve) , 0.5 (middle curve), 1 (bottom curve). 95

7.2 Number density in an infinitely evolved vacuum, as a function of p for $M = 1$ and $\frac{p^3_{out}}{\sqrt{eE}} = -1$ (bottom curve) , 0 (outermost curve), 1 (top curve) and 5 (innermost curve). A plot of $e^{-\pi\lambda_p}$ is also included, which coincides almost exactly with the curve for $\frac{p^3_{out}}{\sqrt{eE}} = 5$. . 96

7.3 Number density in the evolved vacuum, as a function of $\frac{p^3_{out}}{\sqrt{eE}}$, for $M = 1 = p$, and $\tau = 7$ (right), 4, 2 and .5 (single peak). A graph for $\tau = \infty$ is included, which is seen to fit the left peak of the $\tau = 7$ line, while becoming constant at large $\frac{p^3_{out}}{\sqrt{eE}}$. 97

7.4 Total particle number (in a volume $(\triangle l)^3$) in the evolved vacuum, as a function of τ, for $\frac{eE}{m^2} = \frac{1}{2}$ (bottom curve), 1 and 2 (top curve). 98

7.5 Vacuum - vacuum transition probability for $|E| = E_c$ and $V = 10\lambda_c^3$. The initial burst of pair creation cause the exact transition probability (bottom curve) to be significantly lower than Schwinger's prediction (top curve) at early times. 99

7.6 Current density in an infinitely evolved vacuum, as a function of p_z, for $\lambda_p = 2$. The $|\beta|^2$ contribution is also shown (the smoother of the two curves). The $\Re(\beta\alpha)$ term produces rapid oscillations about the mean contribution. 100

7.7 Current density in an evolved vacuum, after a time $\triangle t = 7\sqrt{eE}$, as a function of p_z, for $\lambda_p = 2$. The $|\beta|^2$ contribution in also shown (the darker curve). Again, the $\Re(\beta\alpha)$ term produces rapid oscillations about the mean contribution. 101

7.8 Current density in an evolved vacuum, after a time $\triangle t = \sqrt{eE}$, as a function of p_z, for $\lambda_p = 2$. The $|\beta|^2$ contribution in also shown (the darker curve). We can no longer say that the $\Re(\beta\alpha)$ term produces rapid oscillations about the mean contribution. 102

7.9 Current density in an evolved vacuum, as a function of τ, for $p_z = 5$, and $\lambda_p = 2$. The $|\beta|^2$ contribution in also shown. 103

7.10 Total current (in a volume $(\triangle l)^3$) in an evolved vacuum, as a function of τ. The $|\beta|^2$ contribution in also shown, which provides an accurate approximation except at very early times. The late time approximation of (7.76) is also included, which gets the final gradient correct, but misses the initial 'burst' of creation, just as was the case with the total number of created particles. 104
7.11 $J(u_{\mathbf{p},\lambda;t_0}(\mathbf{x}, t_0 + \frac{\tau}{\sqrt{eE}}))$ as a function of τ, for $\mathbf{p}_{in} = 0$ and $|E| = E_c$. Also included is the classically expected current for a particle accelerated from rest for a time $\frac{\tau}{\sqrt{eE}}$)). . 105
7.12 $J(u_{\mathbf{p},\lambda;t_0}(\mathbf{x}, t_0 + \frac{\tau}{\sqrt{eE}}))$ as a function of τ, for $\mathbf{p}_{in} = 0$ and $|E| = \frac{E_c}{10}$. Also included is the classically expected current for a particle accelerated from rest for a time $\frac{\tau}{\sqrt{eE}}$)). We see, as expected, that the full result more closely resembles the classical result in this case. We also calculated the result for $|E| = \frac{E_c}{100}$, and found no discernible difference between the full result and the classical result in this case. 105

Chapter 1

Introduction

Since its invention in the 1920's, quantum mechanics has grown to become probably the most important theory of the twentieth century, finding applications throughout physics and chemistry. Its relativistic counterpart, quantum field theory, has been successfully applied to the calculation of scattering cross-sections and S-Matrix elements in the context of perturbation theory. However, going beyond perturbation theory has proved a formidable task, preventing quantum field theory from enjoying the vast range of applicability of its non-relativistic counterpart, and causing it to rely largely on high-energy collision experiments for verification. It has become apparent in recent years that there are many situations of physical interest for which the perturbative scattering formalism of quantum field theory is completely inadequate. Among these are the quark confinement problem, astrophysical plasmas, very high temperature phase transitions, strong field electrodynamics and anomalies, as well as various issues in quantum gravity. These situations require an initial value formulation in which we can track the entire time-dependent evolution of the state of the system, just as is done in quantum mechanics, rather than simply comparing the asymptotically free 'in' and 'out' states of the conventional scattering formalism. Meanwhile the standard 'textbook approach' to (non-relativistic) multiparticle quantum mechanics, based on Slater determinants and the multiparticle Shrödinger equation, has no direct counterpart at the relativistic level.

In this dissertation I present an initial value formulation of fermionic quantum field theory in classical (electromagnetic or gravitational) backgrounds that can be seen as the natural relativistic generalisation of (non-relativistic) multiparticle quantum mechanics. Emphasis is placed on the (Shrödinger picture) states of the system, described in terms of Slater determinants of solutions of the Dirac equation, rather than on the field operator, $\hat{\psi}(x)$ (indeed the field operator proves to be completely unnecessary in this approach). The vacuum state 'at time τ' is defined as the Slater determinant of a basis for the span of the negative spectrum of the 'first quantized' Hamiltonian $\hat{H}_1(\tau)$, thus providing a concrete realisation of the Dirac Sea. As well as providing a conceptually transparent approach to the theory, and an extremely simple derivations of the general S-Matrix element and expectation value of the theory, it also allows us to provide a consistent particle interpretation for all time, without requiring any 'asymptotic niceness conditions' on the 'in' and 'out' states.

Perhaps the most surprising aspect of this description of relativistic multiparticle systems is the fact that it is essentially new. Even in 1932 when the 'Dirac Sea' concept was first introduced, the notion of using Slater determinants to describe the 'filling of energy levels' was already well-known, so the idea of describing the vacuum as the Slater determinant of the span of the negative spectrum of the Hamiltonian should have been natural even then. Indeed, the formulation of relativistic fermionic systems that is perhaps the closest to the formulation presented here (albeit not in a gravitational background, and not explicitly using Slater determinants) was that presented in 1934 by Furry and Oppenheimer[7] (and described from a more modern standpoint in the introduction to Weinberg's book[8]). However, the prevailing belief at the time was that the Dirac Sea was little more than a mathematical trick, and should be essentially stable and trivial to describe. Even Dirac[9] stated that "The vacuum is quite a trivial thing physically, and we should expect it to correspond to a trivial solution of the Schrödinger equation". Hence, the idea of carrying around an entire Dirac Sea was generally considered to be an unnecessary complication. Furry and Oppenheimer's theory was rapidly replaced by the now well-known Canonical approach, which treats the 'holes in the Dirac Sea' (rather than the occupants of the Dirac Sea) on the same footing as the electrons, while defining the vacuum only implicitly, by the fact that it 'contains no electrons or holes'.

A more recent formulation of fermionic QFT in classical backgrounds, which also actively incorporates the Dirac Sea concept, is that of Keifer et. al.[10, 11, 12], Floreanini et. al.[13] and others, on the 'functional Schrödinger equation', where the state of the system is described by a functional of Grassmann fields, in much the same way as was proposed originally by Berezin[15]. This has had considerable success in various applications[11, 12, 14], although questions regarding the particle interpretation, the choice of boundary conditions, and the foliation of spacetime remain unresolved.

The formulation of QFT presented in this dissertation successfully resolves these problems. With it I support the proposal that, when considering QFT in electromagnetic or gravitational backgrounds, where vacuum effects are of the utmost importance, working directly with a concrete representation of the Dirac sea provides the most calculationally efficient and conceptually transparent approach to the theory. Also since this approach is the natural relativistic generalisation of methods already familiar in conventional multiparticle quantum mechanics, I believe that it will often provide a favourable alternative to the canonical approach.

In this dissertation my attention will be restricted to quantum field theory in time dependent classical background fields, and the inclusion of back-reaction. This is the relativistic equivalent of using the Hartree Fock approximation in multiparticle quantum mechanics (albeit with the inclusion of vacuum effects). I will focus mainly on fermionic systems which possess a linear governing equation and a conserved inner product, although bosonic systems, non-linear systems, and even non-unitary systems (systems whose inner product is not conserved under evolution) will be discussed. For concreteness, my attention is often restricted further, to the consideration of (4-dimensional) Dirac quantum field theory in an electromagnetic or gravitational background.

The problem of quantum field theory in an electromagnetic background has received considerable attention in recent years[16, 17, 14, 18, 19, 20], partly in an effort to understand its gravitational counterpart, and partly because of the experimental successes with heavy-ion collisions, (see Chapter 13 of Greiner et. al.[21] for a review) which look set to make background quantum field theory an experimentally verifiable science. Interest in quantum field theory in gravitational backgrounds[22, 23, 24] peaked in the late 1970's, after the rapid discovery by Hawking[25], Gibbons[26] and Unruh[27] of three different gravitational analogs of the electromagnetic Schwinger[28] mechanism. Of course these papers were not the first examples of particle creation by gravitational backgrounds. The idea that expanding universes would lead to particle creation was proposed by Schrödinger[29] in 1939, and was investigated in significant detail in Parker's PhD thesis (much of which was published in a series of papers[30, 31, 32]) in 1969. However, the papers of Hawking[25], Gibbons[26] and Unruh[27] quickly lead to an apparent connection between black hole mechanics and thermodynamics, which continues to fascinate physicists today.

Numerous papers were published throughout the 70's and early 80's (see the extensive bibliographies in [22, 23, 21] for instance), investigating particle creation in various different gravitational fields. However, in most situations considered the results in the literature were inconsistent, due to people using different methods, and different 'definitions of particle'. There was little agreement between the predictions of these different definitions, and no known way of choosing which 'definition of particle' was appropriate to a given background. People rapidly became despondent about the possibility of finding a generally acceptable 'definition of particle' for an arbitrary observer in an arbitrary gravitational field, eventually concluding that such a definition was probably impossible, and even claiming that[33] "particles do not exist".

However, by appealing to the observers 'radar time' (see Section 3.2) I propose a generalisation of the concept of 'hypersurface of simultaneity', which can be applied to an arbitrarily moving observer in curved spacetime. I show how this construction naturally incorporates the concept of 'particle horizon', and I use it to develop a definition of particle appropriate to an arbitrarily moving observer in curved spacetime. I show that this definition depends *only* on the choice of observer (and the background present) not on the choice of coordinates, the choice of gauge (in electromagnetic backgrounds) or the existence of special symmetries. Although this definition is necessarily non-local (no local definition of particle could possibly be consistent with the Unruh effect) I demonstrate with a simple example that this non-locality is only significant on scales of the order of the Compton wavelength $\lambda_c = \frac{h}{mc}$ of the particle concerned. I show that in the cases of a uniformly accelerating observer in flat space, and a comoving observer in DeSitter space, my definition reduces to previously accepted definitions[21, 26].

In Chapter 2, I present the foundations of my approach to fermionic quantum field theory in electromagnetic backgrounds (as described in the rest frame of an inertial observer). In this case the vacuum state at each 'time t_0' is uniquely defined to be the Slater determinant of a basis for the span of the negative spectrum of the 'first quantized' Hamiltonian $\hat{H}(t_0)$, thus representing the 'Dirac Sea'. Time-Dependent Bogoliubov coefficients are presented, expressed in terms of solutions of the 'first quantized' equation. I use these to calculate general S-Matrix elements and expectation values at arbitrary times. 'Radar time' is introduced in Chapter 3, and used to generalise the particle interpretation to arbitrarily moving observers in curved spacetimes. In Chapter 4, I demonstrate how my formalism relates to more conventional methods, based on the field operator $\hat{\psi}(x)$. By extending a technique originally developed (for real scalar fields) by DeWitt[24] I rederive the general S-Matrix element of the theory, and compare this to the derivation presented in Chapter 2. I also demonstrate how my particle definition can be expressed in more conventional terms, and discuss the relation between this particle definition and the method of Hamiltonian diagonalisation.

In Chapters 5, 6 and 7 attention is restricted again to the case of inertial observers in electromagnetic backgrounds. In Chapter 5, I demonstrate that all the physical predictions of my theory are both gauge invariant and Lorentz covariant, and I present a treatment of discrete symmetries that does not make any reference to the 'free' theory, and is in no way dependent on using momentum eigenstates. Chapter 6 is devoted to some perturbative calculations, including a general proof that, in situations where the standard perturbation procedure is possible, it will agree with the results of my formalism. In Chapter 7, I consider a concrete application of the formalism to Dirac quantum field theory in a time varying, but spatially uniform electric field. In Chapter 8, I describe how the definition of particle can be applied to charged Bosonic systems, and I derive the general S-matrix element of the theory by generalising the technique developed by DeWitt[24]. In Chapter 9, I discuss the achievements that have been made so far, and the prospects for future work in this area.

Chapter 2

Fermions in an Electromagnetic Background

I present in this Chapter a formulation of fermionic quantum field theory in a time-dependent electromagnetic background, that is analogous to that used in (non-relativistic) multiparticle quantum mechanics, putting the emphasis on the Schrödinger picture states of the system, described in terms of Slater determinants, rather than on the field operator. Time-dependent Bogoliubov coefficients are presented, expressed in terms of solutions of the 'first quantized' equation, which allow us to calculate general S-Matrix elements and expectation values at arbitrary times. A time-dependent *vacuum subtraction* scheme has been introduced, which replaces normal ordering in this formalism. Although our attention has been largely restricted to the case of an electromagnetic background, this formalism applies equally well to any classical background - indeed to any fermionic system with a linear governing equation and a conserved inner product. Gravitational backgrounds (and non-inertial observers) are considered in Chapter 3.

Section 2.1 is devoted to the description of the full state space as the *anti-symmetric Tensor Algebra* over the state space of the 'first quantized' Dirac equation. In Section 2.2, I describe the time-dependent particle interpretation of state space, and discuss some of the properties of this definition. I describe how our formalism consistently combines the conventional 'Bogoliubov coefficient method' with the 'tunnelling method', thus resolving the gauge inconsistencies[34] that troubled each of these methods previously. The evolution of states is also described in Section 2.2, along with the *vacuum subtraction* procedure which replaces normal ordering in this formalism. *Time-dependent Bogoliubov Coefficients* are defined in Section 2.3, and used to calculate the General S-Matrix element of the theory (between arbitrary initial and final times) , along with the expectation value of physical operators on the time-dependent states of the system. A possible explanation of quantum anomalies is also presented, which explains the physical mechanism responsible for the anomalies, without requiring any reference to the UV-behaviour of the system.

Back-reaction is included in Section 2.4, at the level of the mean field approximation, and my approach is compared with that used by Cooper et. al.[18].

2.1 The State Space

2.1.1 Preliminaries

The Lagrangian density for the Dirac equation in an electromagnetic background $A_\mu(x)$ is:

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\nabla_\mu - m)\psi(x) \tag{2.1}$$

where the 'Dirac γ-matrices' satisfy:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_4 \tag{2.2}$$

(where I_4 is the 4×4 identity matrix) and the *covariant derivative* ∇_μ is defined by:

$$\nabla_\mu\psi(x) = \partial_\mu\psi(x) + ieA_\mu(x)\psi(x) \tag{2.3}$$

$\psi(x)$ is a 4-component spinor, $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$, and e is the charge of the fermion ($e = -|e|$ for electrons). I use signature $\eta^{\mu\nu} = diag(1, -1, -1, -1)$ and define $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$ in the obvious way.

The Lagrangian of (2.1) leads to the governing equation:

$$(i\gamma^\mu\nabla_\mu - m)\psi(x) = 0 \tag{2.4}$$

The inner product of two states $\psi(\mathbf{x}, t)$, $\phi(\mathbf{x}, t)$ is given by:

$$\langle\psi(\mathbf{x}, t)|\phi(\mathbf{x}, t)\rangle = \int d^3\mathbf{x}\psi^\dagger(\mathbf{x}, t)\phi(\mathbf{x}, t) \tag{2.5}$$

To show that this is conserved under evolution, recall that the conserved charge Q conjugate to changes in phase is:

$$Q = \int d^3\mathbf{x}J^0(\mathbf{x}, t) = \int d^3\mathbf{x}\psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}, t) \tag{2.6}$$

Let $\psi \to \psi + \phi$ and expand to get conservation of the real part of the inner product. Conservation of the whole inner product then follows from the linearity of (2.4) over $\mathbb{C}$. A 'first quantized' Dirac state is hence a spinor-valued function of space, which evolves with time. The 'first quantized' state space $\mathcal{H}$ is therefore the set of all spinor-valued functions of $\mathbb{R}^3$ whose norm under (2.5) is finite. That is, $\mathcal{H} \equiv L^2(\mathbb{R}^3)^4$.

(2.4) can be written in 'Hamiltonian form' as:

$$i\nabla_0\psi(\mathbf{x}, t) = \hat{H}_1(\mathbf{x}, t)\psi(\mathbf{x}, t) \tag{2.7}$$

$$\text{where } \hat{H}_1(\mathbf{x}, t)\psi(\mathbf{x}, t) = (-i\gamma^0\gamma^k\nabla_k + m\gamma^0)\psi(\mathbf{x}, t) \tag{2.8}$$

and k is to be summed over $k = 1, 2, 3$. This defines the (gauge covariant) *first quantized Hamiltonian operator* $\hat{H}_1(\mathbf{x}, t)$, which will play an important role in what follows. The expectation value of the operator $\hat{H}_1(\mathbf{x}, t)$ in the state $\psi(\mathbf{x}, t)$ is:

$$\langle\psi(\mathbf{x}, t)|\hat{H}_1(\mathbf{x}, t)|\psi(\mathbf{x}, t)\rangle = \int d^3\mathbf{x}\bar{\psi}(\mathbf{x}, t)(-i\gamma^k\nabla_k + m)\psi(\mathbf{x}, t) \tag{2.9}$$

By writing the energy-momentum tensor as

$$T^{\mu}{}_{\nu}(x) = i\bar{\psi}\gamma^{\mu}\nabla_{\nu}\psi - \delta^{\mu}_{\nu}\mathcal{L} \tag{2.10}$$

we see that

$$\langle\psi(\mathbf{x},t)|\hat{H}_1(\mathbf{x},t)|\psi(\mathbf{x},t)\rangle = \int d^3\mathbf{x}T^{0}{}_{0}(\mathbf{x},t) \tag{2.11}$$

That is, the expectation value of $\hat{H}_1$ can be recognised as the spatial integral of the '00-component' of the energy-momentum tensor. Of course, if we require that $\psi(\mathbf{x},t)$ be a solution of the Dirac equation, then we can drop the $\delta^{\mu}_{\nu}\mathcal{L}$ term in (2.10) and write $T^{0}{}_{0}(x) = i\bar{\psi}\gamma^0\nabla_0\psi$. However, by keeping the $\delta^{\mu}_{\nu}\mathcal{L}$ term we see that (2.11) is satisfied regardless of the dynamics of ψ, and that it can be calculated knowing only the value of ψ at the time of interest.

Notice also that it is not $i\frac{\partial\psi}{\partial t}$ on the left hand side of (2.7), but rather $i\nabla_0\psi = i\frac{\partial\psi}{\partial t} - eA_0(x)\psi$. If the left hand side had been $i\frac{\partial\psi}{\partial t}$, then the right hand side would have been $\hat{H}_{ev,1}(x) = \hat{H}_1(x) + eA_0(x)$, which is clearly dependent on gauge, so can have no physical significance. This is mentioned again in Section 2.2.3 and Section 5.1.

Finally, a point of notation is in order. The set of functions $\psi(\mathbf{x},t_0)$ (at time t_0) is of course only one possible representation of the state space $\mathcal{H}$. I will denote the state itself as $|\psi(t_0)\rangle$ or as $\overrightarrow{\psi}(t_0)$ and will sometimes use the suggestive[1] notation $\psi(\mathbf{x},t_0) = \langle\mathbf{x}|\psi(t_0)\rangle$.

2.1.2 The Full Fock Space Over $\mathcal{H}$

The *antisymmetric Fock Hilbert space* over the complex Hilbert space $\mathcal{H}$, denoted $\mathcal{F}_{\wedge}(\mathcal{H})$, and defined[35] in terms of the *antisymmetric Tensor Algebra* over $\mathcal{H}$, provides a natural and familiar construction in which to formulate a relativistic quantum theory of fermions. Firstly because it provides a concrete realisation of the *CAR-algebra* introduced by Jordan and Wigner[36] in 1928 for the purpose of quantization of the electron field, and secondly because it is the natural extension of the space of (non-relativistic) n-particle quantum mechanics to the case when n is unspecified. In order to define $\mathcal{F}_{\wedge}(\mathcal{H})$, let $\mathcal{H}$ be the Hilbert space of the previous Section, with inner product denoted by $\langle\ |\ \rangle$. Let $\otimes^n\mathcal{H}$ denote the direct product of n copies of $\mathcal{H}$, and let $\wedge^n\mathcal{H}$ denote the restriction of $\otimes^n\mathcal{H}$ to those 'states' which are completely antisymmetric under changes in the order of the elements $\overrightarrow{\psi} \in \mathcal{H}$ from which it is constructed. That is, given $\overrightarrow{\psi}_1 \otimes \cdots \otimes \overrightarrow{\psi}_n \in \otimes^n\mathcal{H}$ we can define $\overrightarrow{\psi}_1 \wedge \cdots \wedge \overrightarrow{\psi}_n \in \wedge^n\mathcal{H}$ by:

$$\overrightarrow{\psi}_1 \wedge \cdots \wedge \overrightarrow{\psi}_n \equiv \frac{1}{\sqrt{n!}}\sum_{\sigma} sign(\sigma)\overrightarrow{\psi}_{\sigma(1)} \otimes \cdots \otimes \overrightarrow{\psi}_{\sigma(n)} \tag{2.12}$$

[1] This is certainly not the first time that this suggestive notation is employed, although it should be mentioned here that this is not actually the inner product of the state $|\psi(t_0)\rangle$ and the 'state' $|\mathbf{x}\rangle$, since the 'basis states' $\{|\mathbf{x}\rangle; \mathbf{x} \in \mathbb{R}^3\}$ do not contain spin labels. This is of course why $\langle\mathbf{x}|\psi(t_0)\rangle$ is a 4-component spinor, and why the 'retarded propagator' $S_+(\mathbf{x},t;\mathbf{x}_0,t_0) = \langle\mathbf{x}|\hat{U}_1^+(t,t_0)|\mathbf{x}_0\rangle$ (see Section 4.5) is a 4×4 matrix.

where $\{\sigma(i), i = 1, \ldots, n\}$ is a permutation of $\{1 \ldots n\}$. $\mathcal{F}_\wedge(\mathcal{H})$ is now given by:

$$\mathcal{F}_\wedge(\mathcal{H}) = \oplus_{n=0}^{\infty} \wedge^n \mathcal{H}$$

where $\wedge^0\mathcal{H} \equiv \mathbb{C}$ and $\wedge^1\mathcal{H} \equiv \mathcal{H}$. (For a more detailed account of this construction, see Ottesen[35].) It is easy to see the connection between $\wedge^n\mathcal{H}$ and the state space of n non-relativistic fermions, by noticing that (2.12) is simply shorthand for the Slater determinant of the states $\vec{\psi}_1, \ldots, \vec{\psi}_n$.

States in $\mathcal{F}_\wedge(\mathcal{H})$ will generally be denoted by capital letters such as F or G, although the Dirac 'bra - ket' notation $|F\rangle$ and $\langle F|$ will also be used whenever it is felt that this notation would be more clear. States F which lie entirely within $\wedge^r\mathcal{H}$ for some r will generally be denoted F_r, and are said to be of grade r; $grade(F_r) \equiv r$. I will no longer use the direct product $\otimes$ explicitly, since I can work directly in terms of the outer, or 'wedge' product $\wedge$ instead. Indeed (2.12) can be used to define the outer product $F_r \wedge G_s$ between 'states' $F_r = \vec{\psi}_1 \wedge \cdots \wedge \vec{\psi}_r$ and $G_s = \vec{\phi}_1 \wedge \cdots \wedge \vec{\phi}_s$ as:

$$(\vec{\psi}_1 \wedge \cdots \wedge \vec{\psi}_r) \wedge (\vec{\phi}_1 \wedge \cdots \wedge \vec{\phi}_s) \equiv \vec{\psi}_1 \wedge \cdots \wedge \vec{\psi}_r \wedge \vec{\phi}_1 \wedge \cdots \wedge \vec{\phi}_s$$

from which the outer product of arbitrary states follows from linearity (and the definition $\lambda \wedge F = F \wedge \lambda = \lambda F$ for $\lambda \in \wedge^0\mathcal{H} = \mathbb{C}$).

Another useful operation to define on $\mathcal{F}_\wedge(\mathcal{H})$ is the 'inner derivative' (so called because of the obvious similarity between this relation and the equivalent relation in differential geometry) $i_{\vec{\psi}} : \wedge^n\mathcal{H} \to \wedge^{n-1}\mathcal{H}$. This is defined by:

$$i_{\vec{\psi}}(\vec{\phi}_1 \wedge \cdots \wedge \vec{\phi}_n) \equiv \sum_i (-)^{i+1} \langle \vec{\psi} | \vec{\phi}_i \rangle \vec{\phi}_1 \wedge \cdots \wedge \check{\vec{\phi}}_i \wedge \cdots \wedge \vec{\phi}_n \qquad (2.13)$$

where the check over $\vec{\phi}_i$ in (2.13) denotes that this vector is omitted from the product. $i_{\vec{\psi}} : \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$ is obtained from (2.13) by using linearity, and the definition $i_{\vec{\psi}}\lambda = 0$ for $\lambda \in \wedge^0\mathcal{H}$.

Essentially $i_{\vec{\psi}}$ removes the degree of freedom $\vec{\psi}$ from the state $G_n = \vec{\phi}_1 \wedge \cdots \wedge \vec{\phi}_n$. It is easy to see that if $\vec{\psi}$ is not in the span of $\{\vec{\phi}_1 \ldots \vec{\phi}_n\}$ then $i_{\vec{\psi}}G_n = 0$, and that $i_{\vec{\psi}}(i_{\vec{\psi}}F) = 0$ for all $F \in \mathcal{F}_\wedge(\mathcal{H})$. The operation $i_{\vec{\psi}}$ is denoted $a(\psi)$ in Ottesen[35], and plays the role of an annihilation operator. It will play a similar, although not identical role in the following construction.

We can extend this definition to i_F for any $F \in \mathcal{F}_\wedge(\mathcal{H})$ by defining:

$$\begin{aligned} &i_\lambda G \equiv 0 \quad \text{for} \quad \lambda \in \wedge^0\mathcal{H} \\ &i_{F_m} G \equiv i_{\vec{\psi}_m} \ldots i_{\vec{\psi}_2} i_{\vec{\psi}_1} G \quad \text{for} \quad F_m = \vec{\psi}_1 \wedge \cdots \wedge \vec{\psi}_m \in \wedge^m\mathcal{H} \end{aligned} \qquad (2.14)$$

and $i_{F_0+F_1+F_2+\ldots} = i_{F_0} + i_{F_1} + i_{F_2} + \ldots$. It is easy to see that (2.14) is well defined, since the right hand side is completely antisymmetric under changes in the order of the $\vec{\psi}_i$.

Finally, we need to define an *inner product on* $\mathcal{F}_\wedge(\mathcal{H})$. This is generally defined[35] by:

$$\langle \vec{\psi}_1 \wedge \cdots \wedge \vec{\psi}_n | \vec{\phi}_1 \wedge \cdots \wedge \vec{\phi}_m \rangle = \delta_{nm} \det[\langle \vec{\psi}_i | \vec{\psi}_j \rangle] \qquad (2.15)$$

where $\langle \overrightarrow{\psi}_i | \overrightarrow{\psi}_j \rangle$ refers to the inner product on $\mathcal{H}$. The reader familiar with multiparticle quantum mechanics can easily verify that this agrees with the inner product defined in terms of Slater determinants. This definition does not apply to states $\lambda \in \wedge^0\mathcal{H}$. For these I simply define $\langle \lambda | \mu \rangle = \bar{\lambda}\mu$ (where $\bar{\lambda}$ is the complex conjugate of λ) and $\langle \lambda | F_n \rangle = 0$ for $n > 0$. Notice that I use $\langle \ | \ \rangle$ to refer to the inner product on $\mathcal{H}$ and the inner product on $\mathcal{F}_\wedge(\mathcal{H})$, but I trust that it is clear from the context which inner product is being referred to. Also notice that this definition of inner product can be written in terms of the inner derivative as:

$$\langle F | G \rangle = (i_F G)_0 \tag{2.16}$$

where I have used the notation $(\ )_0$ to denote taking the scalar part of the state $i_F G$ (that is, the projection of $i_F G$ onto $\wedge^0\mathcal{H}$).

Now that I have defined the state space, it is worth making some comments regarding how it relates to conventional methods:

1. The full multi-particle state space that I have introduced is $\mathcal{F}_\wedge(\mathcal{H})$ where $\mathcal{H}$ is the state space of the ('first quantized') Dirac equation. As natural as this choice of state-space may seem, it is not the same as the choice made in conventional approaches to quantum field theory. Conventional methods[37, 38] use $\mathcal{F}_\wedge(\mathcal{H}_1)$ where $\mathcal{H}_1$ is by definition the 'set of all one-particle states'. Then $\wedge^n\mathcal{H}_1$ represents the set of all n-particle states. However, since the Dirac Hamiltonian contains just as many negative energy eigenstates as positive energy eigenstates, the Dirac equation cannot be used as a 1-particle equation. Hence, conventionally solutions of the Dirac equation must be modified in order to construct 1-particle states. Thaller[37] (page 275) for instance uses the set of 'positive energy solutions and charge conjugates of negative energy solutions' $\mathcal{H}_1 \equiv \mathcal{H}^+ + \hat{C}\mathcal{H}^-$, but only treats those cases where this choice can be made independently of time, while Wald[38] (page 103) uses a more general construction, that no longer makes reference to positive or negative energy states. In the standard textbook approach to second quantization in a classical background (see Birrell and Davies[22], Fulling[23], or DeWitt[24] for instance), where the state space is seen only through the creation and annihilation operators, this 'modification' is seen in the fact that the 'positive energy solutions' are associated with particle creation operators, while the 'negative energy solutions' are associated with antiparticle annihilation operators.

   The requirement that $\mathcal{H}_1$ be time-independent means that conventionally we are forced to define '1-particle state' in a time-independent way. This definition will not be physically reasonable if the Hamiltonian is time-dependent, a fact that has caused significant problems for quantum field theorists working in time-dependent classical backgrounds. Particle creation could still be described in this framework by using two different choices of Heisenberg picture Fock space, based on $\mathcal{H}_{1,in}$ and $\mathcal{H}_{1,out}$, each of which was in itself independent of time. However, this procedure is less satisfactory than that presented below, particularly if we wish to work in the Schrödinger picture, where we would be left with the problem of describing the evolution of a Schrödinger picture state when not only the state, but also the space it evolves in, is changing with time!

By choosing to use $\mathcal{H}$ rather than $\mathcal{H}_1$ we are led to a theory where the states can evolve in a fixed state-space. However, we must now abandon the requirement that $\wedge^n\mathcal{H}$ represent n-particle states, and hence we can no longer use $\wedge^0\mathcal{H}$ to represent the physical vacuum. We will see in the next Section that by choosing a vacuum state which directly represents the Dirac Sea (i.e. the Slater determinant of all negative energy states) we are led to a theory which can handle a time-dependent particle interpretation in a simple and uniquely defined fashion, and can do so entirely within the Schrödinger picture, without requiring any reference to the 'field operator' $\hat{\psi}(\mathbf{x},t)$.

2. The reader familiar with differential geometry will notice the similarity between the antisymmetric tensor algebra over $\mathcal{H}$, and the exterior algebra over $\mathcal{H}^*$ (see [39] for instance). Indeed, Wald[38] chooses to represent Fock space in terms of the antisymmetric multilinear functions of the exterior algebra. However, I have chosen to use the construction above, since it represents the most direct extension of the methods used in non-relativistic multiparticle quantum mechanics.

2.1.3 Operators on Fock Space

Let $\hat{A}_1 : \mathcal{H} \to \mathcal{H}$ be an operator on the space of Dirac states (the subscript 1 denotes 'first quantized'). We wish to construct from it an operator that can act on the entire state space. There are two useful ways of doing this, which I describe here (these are also described briefly in Ottesen[35]).

Define the *Hermitian extension* $\hat{A}_H : \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$ of $\hat{A}_1$ by the Leibnitz rule. That is:

$$\hat{A}_H(\overrightarrow{\psi}_1 \wedge \overrightarrow{\psi}_2 \wedge \cdots \wedge \overrightarrow{\psi}_N) \equiv \hat{A}_1(\overrightarrow{\psi}_1) \wedge \overrightarrow{\psi}_2 \wedge \cdots \wedge \overrightarrow{\psi}_N + \overrightarrow{\psi}_1 \wedge \hat{A}_1(\overrightarrow{\psi}_2) \wedge \cdots \wedge \overrightarrow{\psi}_N + \cdots + \overrightarrow{\psi}_1 \wedge \overrightarrow{\psi}_2 \wedge \cdots \wedge \hat{A}_1(\overrightarrow{\psi}_N) \tag{2.17}$$

It is straightforward to show that if $\hat{A}_1$ is (anti)Hermitian w.r.t. (2.5), then $\hat{A}_H$ is (anti)Hermitian w.r.t. (2.15). Now define the *Unitary extension* $\hat{A}_U : \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$ of $\hat{A}_1$ by:

$$\hat{A}_U(\overrightarrow{\psi}_1 \wedge \overrightarrow{\psi}_2 \wedge \cdots \wedge \overrightarrow{\psi}_N) \equiv \hat{A}_1(\overrightarrow{\psi}_1) \wedge \hat{A}_1(\overrightarrow{\psi}_2) \wedge \cdots \wedge \hat{A}_1(\overrightarrow{\psi}_N) \tag{2.18}$$

Again, it is straightforward to show that if $\hat{A}_1$ is unitary, then so is $\hat{A}_U$, and also that $(e^{\hat{A}_1})_U = e^{\hat{A}_H}$ so that if $\hat{U}_1 = e^{\hat{A}_1}$ on $\mathcal{H}$ then $\hat{U}_U = e^{\hat{A}_H}$ on $\mathcal{F}_\wedge(\mathcal{H})$ as would be expected.

Some Properties

1. $(\hat{A}_1 + \hat{B}_1)_H = \hat{A}_H + \hat{B}_H$, $[\hat{A}_H, \hat{B}_H] = [\hat{A}_1, \hat{B}_1]_H$ and $(\hat{A}_1\hat{B}_1)_U = \hat{A}_U\hat{B}_U$

2. If $\overrightarrow{\psi}_1, \overrightarrow{\psi}_2, \ldots \overrightarrow{\psi}_N$ are all eigenstates of $\hat{A}_1$, with eigenvalues $\lambda_1, \ldots \lambda_N$, then $\overrightarrow{\psi}_1 \wedge \overrightarrow{\psi}_N \cdots \wedge \overrightarrow{\psi}_N$ is an eigenstate of $\hat{A}_H$, with eigenvalue $\sum_{i=1}^N \lambda_i$.

3. If $\overrightarrow{\psi}_1, \overrightarrow{\psi}_2, \ldots \overrightarrow{\psi}_N$ are such that $\forall i, \hat{A}_1 \overrightarrow{\psi}_i$ is in the span of $\{\overrightarrow{\psi}_1, \overrightarrow{\psi}_2, \ldots \overrightarrow{\psi}_N\}$, then $F_N \equiv \overrightarrow{\psi}_1 \wedge \overrightarrow{\psi}_2 \cdots \wedge \overrightarrow{\psi}_N$ is an eigenstate of $\hat{A}_H$, (with eigenvalue $\frac{\langle F_N|\hat{A}_H|F_N\rangle}{\langle F_N|F_N\rangle}$).

4. If $\overrightarrow{\psi}_1, \overrightarrow{\psi}_2, \ldots \overrightarrow{\psi}_N$ are orthogonal, then

$$\frac{\langle \overrightarrow{\psi}_1 \wedge \overrightarrow{\psi}_2 \cdots \wedge \overrightarrow{\psi}_N|\hat{A}_H|\overrightarrow{\psi}_1 \wedge \overrightarrow{\psi}_2 \cdots \wedge \overrightarrow{\psi}_N\rangle}{\langle \overrightarrow{\psi}_1 \wedge \overrightarrow{\psi}_2 \cdots \wedge \overrightarrow{\psi}_N|\overrightarrow{\psi}_1 \wedge \overrightarrow{\psi}_2 \cdots \wedge \overrightarrow{\psi}_N\rangle} = \sum_{i=1}^N \frac{\langle \overrightarrow{\psi}_i|\hat{A}_1|\overrightarrow{\psi}_i\rangle}{\langle \overrightarrow{\psi}_i|\overrightarrow{\psi}_i\rangle}$$

2.1.4 Evolution of States, and S-Matrix Elements

Although I have not yet described what the states represent in terms of a particle interpretation, I can describe how these states evolve with time, and how S-matrix elements are calculated. Define the evolution operator $\hat{U}_1(t, t_0)$ on $\mathcal{H}$ by

$$\hat{U}_1(t, t_0)\overrightarrow{\psi}_{t_0} \equiv \overrightarrow{\psi}_{t_0}(t) \tag{2.19}$$

where $\overrightarrow{\psi}_{t_0}$ represents the chosen initial conditions $\psi_{t_0}(\mathbf{x})$ at time t_0, and $\overrightarrow{\psi}_{t_0}(t)$ represents the solution $\psi_{t_0}(\mathbf{x}, t)$ of the Dirac equation which satisfies these initial conditions. That is, (2.19) simply means: *solve the Dirac equation with initial data* $\psi_{t_0}(\mathbf{x})$ *at time* t_0.

The evolution operator $\hat{U}(t, t_0) : \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$ is now defined simply to be the unitary extension of $\hat{U}_1(t, t_0)$. That is:

$$\hat{U}(t, t_0) = \hat{U}_{1,U}(t, t_0) \tag{2.20}$$

To see how straightforward this evolution is, suppose we had an initial state $F_{t_0} = \overrightarrow{\psi}_{1,t_0} \wedge \cdots \wedge \overrightarrow{\psi}_{n,t_0}$. Then this state will evolve to:

$$F_{t_0}(t) = \hat{U}(t, t_0)F_{t_0} = \overrightarrow{\psi}_{1,t_0}(t) \wedge \cdots \wedge \overrightarrow{\psi}_{n,t_0}(t) \tag{2.21}$$

That is:

> *solving the multiparticle evolution equations has been reduced to simply solving the Dirac equation for each of the initial conditions* $\psi_{i,t_0}(\mathbf{x})$, *and taking the Slater determinant of the solutions!*

This is trivially grade preserving, and reveals how the unitarity of $\hat{U}(t, t_0)$ follows immediately from the unitarity of the first quantized Dirac equation. It may seem at this stage that an evolution this simple could not possibly describe pair creation, and the other vacuum processes. However, we will see in the next Section, when I define my particle interpretation, that this evolution describes perfectly pair creation processes, and that grade preservation simply corresponds to conservation of charge. In relation to non-relativistic multiparticle quantum mechanics, this is precisely how a system of fermions would evolve if we ignored fermion - fermion interactions, and only considered the interaction of the fermions with the background field.

S-Matrix Elements

We are interested in finding out the probability that a system, prepared in state $|F_{t_0}\rangle$ at time t_0 will be found at some later time t_1 to be in some state $|G\rangle$. This is of course given by the *Transition Probability*:

$$\mathcal{P}_{F\to G} = \frac{|\langle G|F_{t_0}(t_1)\rangle|^2}{\langle F|F\rangle\langle G|G\rangle} \tag{2.22}$$

where $\langle F|F\rangle$ denotes the norm of the state $F_{t_0}(t)$, which is independent of t by virtue of (2.20). This motivates defining the *S-Matrix Element*

$$S_{F\to G} \equiv \langle G|F_{t_0}(t_1)\rangle = \langle G|\hat{U}(t_1,t_0)|F_{t_0}\rangle \tag{2.23}$$

In order to give physical significance to this transition probability, the states $|F_{t_0}\rangle$ and $|G\rangle$ must be specified according to their physical properties. That is, $|F_{t_0}\rangle$ will be specified relative to a complete set of commuting operators at time t_0, while since the measurement is made at time t_1, $|G\rangle$ should be defined relative to a complete set of commuting operators at time t_1 (it will generally be denoted $|G_{t_1}\rangle$ from now on, to represent this). This procedure allows S to be specified up to some arbitrary phase, which has no physical effect, so can be ignored. Lets turn our attention now to the particle interpretation.

2.2 A Particle Interpretation of State Space

I have already mentioned that $\mathcal{H}$ does not represent the set of '1-particle states', so that $\wedge^n\mathcal{H}$ does not represent the set of n-particle states, and in particular, $\wedge^0\mathcal{H}$ does not represent the vacuum. In this Section, I describe how states of definite particle number are represented within this formalism, and how this depends on the background present.

2.2.1 The Positive/Negative Energy Split and the Vacuum State

Consider the action of $\hat{H}_1(t_0)$ on $\mathcal{H}$ (at some fixed time t_0). Since $\hat{H}_1(t_0)$ is Hermitian, then $\mathcal{H}$ can be parametrised in terms of eigenvectors of $\hat{H}_1(t_0)$. From this I define $\mathcal{H}^\pm(t_0)$ by:

$\mathcal{H}^+(t_0)$ is the span of the positive spectrum of $\hat{H}_1(t_0)$

$\mathcal{H}^-(t_0)$ is the span of the negative spectrum of $\hat{H}_1(t_0)$

That is, $\mathcal{H}^+(t_0)$ is the set of all positive energy states, and $\mathcal{H}^-(t_0)$ is the set of all negative energy states, as defined at time t_0. I can now define the *vacuum at time* t_0, $|vac_{t_0}\rangle$ by:

$|vac_{t_0}\rangle$ is the Slater determinant of any basis of $\mathcal{H}^-(t_0)$,
normalised so that $\langle vac_{t_0}|vac_{t_0}\rangle = 1$.

This specifies $|vac_{t_0}\rangle$ uniquely up to an arbitrary phase. It is the state in which all negative energy degrees of freedom are full, and hence it is the physical realisation of the Dirac Sea.

In order to illustrate the meaning of this, consider first a model example, where I assume for convenience that the system contains only N positive, and N negative energy degrees of freedom. Let $\{u_{i,t_0}; i = 1, \ldots N\}$ be an orthonormal basis for $\mathcal{H}^+(t_0)$ at some time t_0, and let $\{v_{i,t_0}; i = 1 \ldots N\}$ be an orthonormal basis for[2] $\mathcal{H}^-(t_0)$. Then $\{u_{i,t_0}, v_{i,t_0}; i = 1, \ldots N\}$ will form an orthonormal basis for $\mathcal{H}$ (since $\mathcal{H}^+(t_0)$ and $\mathcal{H}^-(t_0)$ are mutually orthogonal spaces). The vacuum at time t_0 can be written as:

$$|vac_{t_0}\rangle = v_{1,t_0} \wedge \cdots \wedge v_{N,t_0} \tag{2.24}$$

This state is independent of the choice of basis for $\mathcal{H}^-(t_0)$ (up to phase) because of the complete antisymmetry of the Slater determinant. If the operator $\hat{H}_1(t)$ depends on time, then so will the space $\mathcal{H}^-(t)$, so that the vacuum $|vac_t\rangle$ will be different at different times. If $\{u_{i,t_1}; i = 1, \ldots N\}$, $\{v_{i,t_1}; i = 1, \ldots N\}$ are orthonormal bases for $\mathcal{H}^+(t_1)$ and $\mathcal{H}^-(t_1)$ respectively, for some time $t_1 > t_0$, then we may write the vacuum at time t_1 as:

$$|vac_{t_1}\rangle = v_{1,t_1} \wedge \cdots \wedge v_{N,t_1} \tag{2.25}$$

We may consider qualitatively now, the evolution from time t_0 to time t_1, of the state $|vac_{t_0}\rangle$. The evolved state $|vac_{t_0}(t_1)\rangle$ obtained from this is simply:

$$|vac_{t_0}(t_1)\rangle = v_{1,t_0}(t_1) \wedge \cdots \wedge v_{N,t_0}(t_1) \tag{2.26}$$

where $v_{i,t_0}(t_1)$ denotes the state obtained from v_{i,t_0} under evolution to time t_1, and will not, in general, be contained in $\mathcal{H}^-(t_1)$. Physically, if we were to measure the properties of this evolved state, we would be measuring this relative to the Fock space constructed on $|vac_{t_1}\rangle$. We could for instance calculate the probability that $|vac_{t_0}(t_1)\rangle$ will still be in the vacuum state:

$$\mathcal{P}_{vac\to vac} = |\langle vac_{t_1}|vac_{t_0}(t_1)\rangle|^2 \tag{2.27}$$

$$\text{where } \langle vac_{t_1}|vac_{t_0}(t_1)\rangle = \det[\langle v_{i,t_1}|v_{j,t_0}(t_1)\rangle] \quad \text{from} \quad (2.15) \tag{2.28}$$

The case of standard Dirac theory in an electromagnetic background, where $\mathcal{H}^\pm(t)$ are infinite dimensional spaces, can still be treated as above. Although the Slater determinant of an uncountably infinite number of states is not yet mathematically well defined, the inner product of two such Slater determinants is simply a Fredholm determinant, which is well defined, and can often be straightforwardly calculated. In this case, the time dependence of $\hat{H}(t)$ is due to the time dependence of the background $A^\mu(\mathbf{x}, t)$. Hence the state $|vac_{t_0}\rangle$ represents the vacuum in the presence of a background $A^\mu(\mathbf{x}, t_0)$. That is, it represents an *empty room, in the presence of the background* $A^\mu(\mathbf{x}, t_0)$. The fact that $|vac_{t_0}\rangle \neq |vac_{t_1}\rangle$ in general reflects the fact that an 'empty room in the presence of $A(\mathbf{x}, t_0)$' is not the same as an 'empty room in the presence of $A(\mathbf{x}, t_1)$'. Notice that the state $|vac_{t_0}(t_1)\rangle$, sometimes called the *evolved vacuum* is not actually a vacuum state.

This description is clearly more powerful than most previous methods, since it allows the vacuum to be defined in any chosen background, rather than simply in the asymptotically 'well-behaved' regions, and it is also conceptually clearer, since we are comparing

[2] I will not need to use the notation $\overrightarrow{u}_{i,t_0}$ here.

the evolved state $|vac_{t_0}(t_1)\rangle$ (which is in general not vacuum) with the actual vacuum $|vac_{t_1}\rangle$, rather than comparing two different 'choices' of Heisenberg picture (hence time independent) vacuum states, each of which is to be considered as a vacuum state, but with respect to interpretations which become 'asymptotically more realistic' at early and late times respectively.

Before considering non-vacuum states it is worth considering a couple of technical details:

- Implicit in the construction just presented is the assumption that the 'in-state' is prepared on the spacelike Cauchy surface $t = t_0$ for some t_0, and that the 'out state' is to be measured on the spacelike Cauchy surface $t = t_1$ for some $t_1 > t_0$. We will see in the next Chapter, when I generalise the particle definition to an arbitrary observer in an arbitrary curved spacetime, that this assumption amounts to assuming that we are working in the rest frame of an inertial observer. Hence, although the explicit role played by t might seem quite unnatural and arbitrary here, its relevance will become clear within the context of the next Chapter.

- Secondly, notice that $\mathcal{H}^{\pm}(t_0)$ can alternatively be defined by the requirement that the projection operators $\hat{P}^{\pm}(t_0) : \mathcal{H} \to \mathcal{H}^{\pm}(t_0)$ must be orthogonal projections satisfying:

$$\langle \hat{P}^{+}(t_0)\psi|\hat{H}_1(t_0)|\hat{P}^{+}(t_0)\psi\rangle \geq \langle\psi|\hat{H}_1(t_0)|\psi\rangle \geq \langle \hat{P}^{-}(t_0)\psi|\hat{H}_1(t_0)|\hat{P}^{-}(t_0)\psi\rangle \qquad (2.29)$$

for all $\psi \in \mathcal{H}$. This definition is equivalent to the definition above (at least for $\hat{H}_1(t_0)$ non-degenerate), and since it does not make reference to the spectrum of $\hat{H}_1(t_0)$ it will prove more useful than the above definition in some cases. In particular, we can use (2.11) to express the requirement (2.29) in terms of the integral of the '00-component' of the energy-momentum tensor, and we can even, if we desire, use this to work directly with the 'space of solutions of the Dirac equation', splitting this space into the span of $\{v_{i,t_0}(\mathbf{x},t)\}$ and the span of $\{u_{i,t_0}(\mathbf{x},t)\}$. These facts will be used in Chapter 3 when I generalise the particle definition to arbitrarily moving observers in gravitational backgrounds, and show that the generalised definition depends only on the choice of observer, and not the choice of coordinates.

To see that these definitions are equivalent, first notice that $\mathcal{H}^{\pm}(t_0)$ defined in terms of the positive and negative spectrum of the non-degenerate, Hermitian operator $\hat{H}_1(t_0)$ will clearly satisfy (2.29), so that I need only prove the converse. Hence, let $\mathcal{H}^{\pm}(t_0)$ be orthogonal spaces defined by (2.29), let $\psi^+ \in \mathcal{H}^+(t_0)$, $\psi^- \in \mathcal{H}^-(t_0)$ and let $\psi = a\psi^+ + b\psi^-$ for a, b complex scalars. Then (2.29) reads:

$$|a|^2\langle\psi^+|\hat{H}_1(t_0)|\psi^+\rangle \geq |a|^2\langle\psi^+|\hat{H}_1(t_0)|\psi^+\rangle + 2Re(\bar{a}b\langle\psi^+|\hat{H}_1(t_0)|\psi^-\rangle) + |b|^2\langle\psi^-|\hat{H}_1(t_0)|\psi^-\rangle$$
$$\geq |b|^2\langle\psi^-|\hat{H}_1(t_0)|\psi^-\rangle$$

From this we have $|a|^2\langle\psi^+|\hat{H}_1(t_0)|\psi^+\rangle + 2Re(\bar{a}b\langle\psi^+|\hat{H}_1(t_0)|\psi^-\rangle) \geq 0$ for all a, b, which can only be true if $\langle\psi^+|\hat{H}_1(t_0)|\psi^-\rangle = 0$. That is, $\langle\psi^+|\hat{H}_1(t_0)|\psi^-\rangle = 0$ for all $\psi^+ \in \mathcal{H}^+(t_0)$ and $\psi^- \in \mathcal{H}^-(t_0)$, while considering $a = 0$ and $b = 0$ respectively gives

$\langle\psi^+|\hat{H}_1(t_0)|\psi^+\rangle \geq 0$ and $\langle\psi^-|\hat{H}_1(t_0)|\psi^-\rangle \leq 0$ for all $\psi^+ \in \mathcal{H}^+(t_0)$ and $\psi^- \in \mathcal{H}^-(t_0)$. Hence the matrix form of $\hat{H}_1(t_0)$ w.r.t. a basis for $\mathcal{H}^+(t_0)$, and $\mathcal{H}^-(t_0)$ is:

$$\boldsymbol{H}_1(t_0) = \begin{bmatrix} \binom{positive}{definite} & 0 \\ 0 & \binom{negative}{definite} \end{bmatrix}$$

so that $\mathcal{H}^+(t_0)$ is the span of the positive spectrum of $\hat{H}_1(t_0)$ and $\mathcal{H}^-(t_0)$ is the span of the negative spectrum of $\hat{H}_1(t_0)$ as required.

2.2.2 Particle States Built On The Vacuum

Now that we know what is meant by an 'empty room at time t_0' we can look at states corresponding to a given particle number (at time t_0). For this, let $\overrightarrow{\psi}^+_{t_0} \in \mathcal{H}^+(t_0)$, and let $\overrightarrow{\psi}^-_{t_0} \in \mathcal{H}^-(t_0)$. Then a *one-electron state at time* t_0 is of the form:

$$\overrightarrow{\psi}^+_{t_0} \wedge |vac_{t_0}\rangle$$

while a *one-positron state at time* t_0 is of the form:

$$i_{\overrightarrow{\psi}^-_{t_0}} |vac_{t_0}\rangle$$

That is, electrons are represented by the presence of positive energy degrees of freedom, while positrons are represented by the absence of negative energy degrees of freedom. Notice also that:

$$\overrightarrow{\psi}^-_{t_0} \wedge |vac_{t_0}\rangle = 0 = i_{\overrightarrow{\psi}^+_{t_0}} |vac_{t_0}\rangle$$

for all $\overrightarrow{\psi}^+_{t_0} \in \mathcal{H}^+(t_0)$ and $\overrightarrow{\psi}^-_{t_0} \in \mathcal{H}^-(t_0)$ Similarly two-electron states will be of the form:

$$\overrightarrow{\psi}^+_{1,t_0} \wedge \overrightarrow{\psi}^+_{2,t_0} \wedge |vac_{t_0}\rangle$$

two-positron states will be of the form

$$i_{\overrightarrow{\psi}^-_{1,t_0}} i_{\overrightarrow{\psi}^-_{2,t_0}} |vac_{t_0}\rangle$$

states consisting of one positron and one electron will be of the form

$$i_{\overrightarrow{\psi}^-_{1,t_0}}(\overrightarrow{\psi}^+_{2,t_0} \wedge |vac_{t_0}\rangle) = -\overrightarrow{\psi}^+_{2,t_0} \wedge (i_{\overrightarrow{\psi}^-_{1,t_0}} |vac_{t_0}\rangle) \tag{2.30}$$

and higher particle-number states are constructed similarly. It is convenient here to define the *Standard Form* of a state F_{t_0} to be

$$|F_{t_0}\rangle = F^+_{m,t_0} \wedge (i_{G^-_{n,t_0}} |vac_{t_0}\rangle) \tag{2.31}$$

where

$$\begin{aligned} F^+_{m,t_0} &= \overrightarrow{\psi}^+_{1,t_0} \wedge \cdots \wedge \overrightarrow{\psi}^+_{m,t_0} \\ G^-_{n,t_0} &= \overrightarrow{\phi}^-_{1,t_0} \wedge \cdots \wedge \overrightarrow{\phi}^-_{n,t_0} \end{aligned}$$

This represents a state of m electrons, and n positrons.

We can see how these states are related to the conventional versions by defining creation and annihilation operators as:

$$a(\overrightarrow{\psi}^+_{t_0}) = i_{\overrightarrow{\psi}^+_{t_0}} \qquad a^\dagger(\overrightarrow{\psi}^+_{t_0}) = \overrightarrow{\psi}^+_{t_0} \wedge \tag{2.32}$$

$$b(\overrightarrow{\psi}^-_{t_0}) = \overrightarrow{\psi}^-_{t_0} \wedge \qquad b^\dagger(\overrightarrow{\psi}^-_{t_0}) = i_{\overrightarrow{\psi}^-_{t_0}} \tag{2.33}$$

It is straightforward to verify that $a^\dagger(\overrightarrow{\psi}^+_{t_0})$ is indeed the Hermitian conjugate of $a(\overrightarrow{\psi}^+_{t_0})$, and similarly for $b^\dagger(\overrightarrow{\psi}^+_{t_0})$ and $b(\overrightarrow{\psi}^+_{t_0})$. These satisfy the Canonical Anticommutation Relations (CAR's):

$$\{a(\overrightarrow{\psi}^+_{1,t_0}), a^\dagger(\overrightarrow{\psi}^+_{2,t_0})\} = \langle \psi^+_{1,t_0} | \psi^+_{2,t_0} \rangle \hat{1} \tag{2.34}$$

$$\{b(\overrightarrow{\psi}^-_{1,t_0}), b^\dagger(\overrightarrow{\psi}^-_{2,t_0})\} = \langle \psi^-_{2,t_0} | \psi^-_{1,t_0} \rangle \hat{1} \tag{2.35}$$

$$= \overline{\langle \psi^-_{1,t_0} | \psi^-_{2,t_0} \rangle} \hat{1} \tag{2.36}$$

(where $\hat{1}$ is the identity operator on $\mathcal{F}_\wedge(\overrightarrow{\mathcal{H}})$) with all others zero. The Vacuum State $|vac_{t_0}\rangle$ can then be defined by the requirement that:

$$a(\overrightarrow{\psi}^+_{t_0})|vac_{t_0}\rangle = 0 = b(\overrightarrow{\psi}^-_{t_0})|vac_{t_0}\rangle \tag{2.37}$$

for all $\overrightarrow{\psi}^+_{t_0} \in \mathcal{H}^+(t_0)$ and $\overrightarrow{\psi}^-_{t_0} \in \mathcal{H}^-(t_0)$.

Equation (2.31) can now be written in terms of the creation and annihilation operators as:

$$F_{t_0} = a^\dagger(\overrightarrow{\psi}^+_{1,t_0}) \ldots a^\dagger(\overrightarrow{\psi}^+_{m,t_0}) b^\dagger(\overrightarrow{\phi}^+_{n,t_0}) \ldots b^\dagger(\overrightarrow{\phi}^+_{1,t_0})|vac_{t_0}\rangle$$

The inner product between two such states can then be written as:

$$\langle F_{1,t_0} | F_{2,t_0} \rangle = (i_{F^+_{m_1,t_0}} F^+_{m_2,t_0})_0 (i_{G^-_{n_2,t_0}} G^-_{n_1,t_0})_0 = \delta_{m_1 m_2} \delta_{n_1 n_2} \det[\langle \psi^+_{i,1,t_0} | \psi^+_{j,2,t_0} \rangle] \overline{\det[\langle \phi^-_{i,1,t_0} | \phi^-_{j,2,t_0} \rangle]} \tag{2.38}$$

To deduce (2.38) first use (2.31) and (2.15) to deduce that:

$$\langle F_{1,t_0} | F_{2,t_0} \rangle = \delta_{m_1 m_2} \det[\langle \psi^+_{i,1,t_0} | \psi^+_{j,2,t_0} \rangle] \langle vac_{t_0} | b(\overrightarrow{\phi}^-_{1,1,t_0}) \ldots b(\overrightarrow{\phi}^-_{n_1,1,t_0}) b^\dagger(\overrightarrow{\phi}^-_{n_2,2,t_0}) \ldots b^\dagger(\overrightarrow{\phi}^-_{1,2,t_0})|vac_{t_0}\rangle$$

(2.38) then follows by using

$$b(\overrightarrow{\phi}^-_{n_1,1,t_0}) b^\dagger(\overrightarrow{\phi}^-_{n_2,2,t_0}) \ldots b^\dagger(\overrightarrow{\phi}^-_{1,2,t_0})|vac_{t_0}\rangle = \sum (-)^{i+1} \overline{\langle \phi^-_{n_1,1,t_0} | \phi^-_{i,2,t_0} \rangle} b^\dagger(\overrightarrow{\phi}^-_{n_2,2,t_0}) \ldots \check{b}^\dagger(\overrightarrow{\phi}^-_{i,2,t_0}) \ldots b^\dagger(\overrightarrow{\phi}^-_{1,2,t_0})|vac_{t_0}\rangle$$

and induction on n.

Some comments are now in order:

1. Notice the apparent difference between this expression of the CAR's and that of Wald[38] or Ottesen[35] (which do not have the conjugation in (2.36)). This difference is because of the conjugation of negative energy solutions that is inherent in their definition of $\mathcal{H}_1$. This subtlety is described in further detail in Thaller[37]. It is this conjugation that appears in S-Matrix elements, leading to the apparent 'reversal of roles' of antiparticle in and out states. We will also see in Chapter 4 that although the definitions (2.32) and (2.33) above differentiate between +ve and -ve energy states in a way that other definitions do not, this is done in such a way that the field operator $\hat{\psi}(x)$ (defined in Chapter 4) does not depend on any +ve/-ve energy split. This essentially corresponds (when translated into the Heisenberg Picture) to the field operator defined in Dimock[40].

2. An interesting result which follows directly from the definitions (2.32), (2.33) and (2.17) is:

$$[\hat{A}_H, a^\dagger(\overrightarrow{\psi}^+)] = (\hat{A}_1\overrightarrow{\psi}^+)\wedge \qquad [\hat{A}_H, a(\overrightarrow{\psi}^+)] = -i_{\hat{A}_1^\dagger\overrightarrow{\psi}^+}$$
$$[\hat{A}_H, b^\dagger(\overrightarrow{\psi}^-)] = -i_{\hat{A}_1^\dagger\overrightarrow{\psi}^-} \qquad [\hat{A}_H, b(\overrightarrow{\psi}^-)] = (\hat{A}_1\overrightarrow{\psi}^-)\wedge$$

where $\hat{A}_H$ is the Hermitian extension of the linear operator $\hat{A}_1$. This contains equations such as $[\hat{P}_\mu, a^\dagger_\lambda(\mathbf{p})] = p_\mu a^\dagger_\lambda(\mathbf{p})$, $[\hat{Q}, a^\dagger_\lambda(\mathbf{p})] = a^\dagger_\lambda(\mathbf{p})$ etc.. as special cases, but its calculation is completely general, and does not require any of the tedious manipulations involved in expanding $\hat{P}_\mu$ in terms of creation and annihilation operators. In fact, the expansion of $\hat{P}_\mu$ in terms of creation and annihilation operators can now be seen as a straightforward consequence of the fact that $a^\dagger(\overrightarrow{\phi}^+)a(\overrightarrow{\psi}^+)$ is the Hermitian extension of the operator $|\phi^+\rangle\langle\psi^+|$ on $\mathcal{H}$.

2.2.3 Vacuum Subtraction and Physical Operators

Consider for the moment, the example of free field theory. If we extend the '1-particle' operators, as described in Section 2.1.3, and apply these to $|vac_f\rangle$ (the vacuum of free Dirac theory, which is simply the wedge product of all the negative energy plane-wave states, appropriately normalised) we find undesirable results. For instance, taking the continuum limit of property 2, Section 2.1.3, and applying this to the free Dirac Hamiltonian, $\hat{H}_0$, will give:

$$\hat{H}_0|vac_f\rangle = H_{vacuum}|vac_f\rangle$$

where:

$$H_{vacuum} = -4V\int \frac{d^3\mathbf{p}}{(2\pi)^3}\tfrac{1}{2}\sqrt{m^2+|\mathbf{p}|^2} \tag{2.39}$$

and $V = \int d^3\mathbf{x}$ is the volume of 3-space. This is easily recognised as the vacuum energy of free Dirac Q.F.T. before normal ordering. Thus, let $\hat{A} : \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$ be any operator for which $\hat{A}|vac_f\rangle$ should physically be zero, and define:

$$\hat{A}_{phys} \equiv \hat{A} - \langle vac_f|\hat{A}|vac_f\rangle\hat{1}$$

This new operator $\hat{A}_{phys}$ will then have the properties desired of $\hat{A}$. This process is easily seen to correspond to normal ordering of the free field operators in conventional Q.F.T.
Now, let us extend this concept to an electromagnetic background. Given an operator $\hat{A}(t) : \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$ (which may or may not be time dependent) corresponding to an observable which should be zero in the vacuum, define $\hat{A}(t)_{phys}$ as:

$$\hat{A}(t)_{phys} = \hat{A}(t) - \langle vac_t|\hat{A}(t)|vac_t\rangle\hat{1} \tag{2.40}$$

Note that now we are subtracting a time-dependent c-number valued function to give $\hat{A}(t)_{phys}$. This represents a departure from conventional techniques. Itzykson and Zuber[41] (Section 4-3) for example, simply normal order the free Hamiltonian, and leave the interaction Hamiltonian untouched. This corresponds to removing $\langle vac_f|\hat{H}_0|vac_f\rangle$, which places undue significance on the 'free states', and 'free fields', which are inherent in the definitions of $\hat{H}_0$ and $|vac_f\rangle$, and are, after all, just the zeroth order terms in a perturbation expansion. The method used here has the advantage of removing all reference to 'free' fields, and 'free' states, so that calculations can potentially be made in a truly non-perturbative way. Other normal ordering schemes are considered in Birrell and Davies[22], DeWitt[24] etc, and are discussed in more detail in Section 4.3.

I now consider some examples of this Hermitian extension/vacuum subtraction process.

1. I have already described the effect of vacuum subtraction on the Hamiltonian. However, it is also worth pointing out here, that the choice of $|vac_t\rangle$ made above, along with its vacuum subtraction, is *the only possible choice that guarantees the positive definiteness of* $\hat{H}_{phys}$. It is clearly $\hat{H}_{phys}$ which represents total energy (rather than the vacuum subtracted Hermitian extension of $\hat{H}_{ev,1}$ for instance) since:
   - As mentioned on page 6, it is the expectation value of $\hat{H}_1$ in the state ψ which which gives the '00-component' of the energy-momentum tensor.
   - The operator $\hat{H}_{ev,1}$ is clearly dependent on gauge, so could not possibly represent a physical observable.

2. Consider the unit operator on $\mathcal{H}$. The unitary extension of this is clearly still the unit operator, $\hat{1}$, while the Hermitian extension gives the grade of the state. The physical operator corresponding to this Hermitian extension satisfies:

$$\hat{1}_{phys}|F\rangle = \{grade(|F\rangle) - grade(|vac_t\rangle)\}|F\rangle = (grade(F_t^+) - grade(F_t^-))|F\rangle \tag{2.41}$$

   when acting on states in standard form (as introduced in (2.31)). This operator clearly represents charge; that is, $\hat{Q} = e\hat{1}_{phys}$. We saw in Section 2.1.4 that evolution is trivially grade preserving. This can now be seen to represent charge conservation. That the Hermitian extension/vacuum subtraction process should turn the unit operator into the physical charge is clearly appropriate, since the norm $\langle\psi|\hat{1}_1|\psi\rangle$ of a state $\psi \in \mathcal{H}$ is well known to be the conserved 'charge' of the Dirac Lagrangian conjugate to changes in phase.

3. The number operator is the main example of an operator that depends explicitly on the split of $\mathcal{H}$ into $\mathcal{H}^+(t)$ and $\mathcal{H}^-(t)$. To build this up in the same way as we have for energy and charge, start with the operator $\hat{N}_1(t) : \mathcal{H} \to \mathcal{H}$ defined by $\hat{N}_1(t_0) = \hat{P}^+(t_0) - \hat{P}^-(t_0)$, where $\hat{P}^+(t_0)\overrightarrow{\psi}$ is the projection of $\overrightarrow{\psi}$ onto $\mathcal{H}^+(t_0)$, and $\hat{P}^-(t_0)\overrightarrow{\psi}$ is the projection of $\overrightarrow{\psi}$ onto $\mathcal{H}^-(t_0)$. it's easy to see that this is Hermitian, but does not in general commute with time evolution. The operator $\hat{N}_{phys}$ obtained from $\hat{N}_1$ by the above procedure will of course inherit both of these properties. This means that the number operator represents a well-defined physical observable, but is not conserved. When acting on states in standard form, it gives:

$$\hat{N}_{phys,t}|F\rangle = (grade(F_t^+) + grade(F_t^-))|F\rangle \tag{2.42}$$

so that $\hat{N}_{phys,t}$ is positive definite, and has integer eigenvalues. To investigate this in more detail, let $\{\overrightarrow{u}_{\alpha,t}; \alpha \in I\}, \{\overrightarrow{v}_{\alpha,t}; \alpha \in I\}$ represent orthogonal bases for $\mathcal{H}^+(t)$ and $\mathcal{H}^-(t)$ respectively (I some index set) and assume for convenience that $\langle u_{\alpha,t}|u_{\beta,t}\rangle = f(\alpha)\delta(\alpha-\beta) = \langle v_{\alpha,t}|v_{\beta,t}\rangle$ for some (real) function $f(\alpha)$. Then it is straightforward to show that:

$$\hat{N}_{phys,t} = \int_I \frac{d\mu(\alpha)}{f(\alpha)}(\hat{N}_{u_{\alpha,t}} + \hat{N}_{v_{\alpha,t}}) \tag{2.43}$$

where

$$\hat{N}_{u_{\alpha,t}} = \overrightarrow{u}_{\alpha,t} \wedge (i_{\overrightarrow{u}_{\alpha,t}} \quad \text{and} \quad \hat{N}_{v_{\alpha,t}} = i_{\overrightarrow{v}_{\alpha,t}}(\overrightarrow{v}_{\alpha,t} \wedge \tag{2.44}$$

which agrees with conventional methods by comparison with equations (2.32) and (2.33). It is easy to see that each $\hat{N}_{u_{\alpha,t}}, \hat{N}_{v_{\alpha,t}}$ is Hermitian, but not conserved. Hence we can see that although particle numbers are not conserved, a particle interpretation is, at least in principle, a physically measurable concept. This disagrees with the claims of many recent authors (see [33] and [38] for instance). Some of the limitations of this concept are discussed in Section 2.2.4.

Before proceeding to the next Section, it is worth making some general comments regarding the construction just presented

1. It is easy to see now that evolution under $\hat{U}(t,t_0)$ corresponds to evolution w.r.t. $\hat{H}_{ev}(t)$, the Hermitian extension of $\hat{H}_{1,ev}(t)$. That is, if $F_{t_0}(t) = \hat{U}(t,t_0)F_{t_0}$ is a solution of the full multiparticle Dirac equation, then

$$i\frac{dF_{t_0}(t)}{dt} = \hat{H}_{ev}(t)F_{t_0}(t)$$

We could, if we desired, modify this so that physical states evolved under the vacuum subtracted version of $\hat{H}_{ev}(t)$, denoted $\hat{H}_{ev,phys}(t)$. This is achieved by defining the *physical state* $F_{phys,t_0}(t)$ by

$$F_{phys,t_0}(t) = F_{t_0}(t)e^{-i\int_{t_i}^t H_{ev,vac}(t')dt'}$$

where $H_{ev,vac}(t) \equiv \langle vac_t|\hat{H}_{ev}(t)|vac_t\rangle$and t_i is some initial time, chosen for convenience. This will satisfy

$$i\frac{dF_{phys,t_0}(t)}{dt} = \hat{H}_{ev,phys}(t)F_{phys,t_0}(t)$$

However, since it only differs from $F_{t_0}(t)$ by a change in phase, then they will both be equivalent for the calculation of transition probabilities and expectation values, so in practice we can simply use $F_{t_0}(t)$, safe in the knowledge that it is equivalent to using $F_{phys,t_0}(t)$.

2. I will demonstrate in Chapter 4 that the particle definition presented here, and generalised in the next Chapter, is equivalent to applying the 'Hamiltonian Diagonalisation' procedure (described by Grib and Mamaev[42, 43] and Grib, Mamayev and Mostepanenko[44]) to an appropriate Hamiltonian, albeit to a Hamiltonian (as generalised in the next Chapter) that has never before been considered. A thorough comparison of my method and the method of Hamiltonian diagonalisation is given in Section 4.4 (along with a discussion of Fulling's[45] criticism's of Hamiltonian diagonalisation). One point worth noting here is simply that, to the best of my knowledge, Hamiltonian diagonalisation has never been applied to fermions in electromagnetic backgrounds.

   A similar particle definition, based on a procedure more similar to my own, is that used by Fradkin, Gitman and Shvartsman[17]. Although still working in the Heisenberg picture, and using 'field operators' $\hat{\psi}(x)$ to describe evolution, their book uses eigenstates of the 'first quantized' Hamiltonian at times t_{in} and t_{out} to define 'in' and 'out' states, but immediately comments that the "initial, t_{in}, and final, t_{out} time instants are to be understood in the final expressions as shifted into the infinitely remote past and future respectively" (however they do consider, in Section 5 of their book, the possibility of removing these requirements).

   Another difference between the particle interpretation used here, and that used in Fradkin et. al.[17] is that [17] uses the spectrum of $\hat{H}_{ev}(t)$ rather than $\hat{H}_1(t)$ in order to specify particle and antiparticle states. This has the unfortunate consequence of making their formulation gauge dependent! To demonstrate this explicitly, one need only consider free field theory, in the gauge $A(x) = (A, 0, 0, 0)$, where A is some constant, for which $eA \gg m$. A more concerning example arises in the case of a constant electric field, where if the gauge $A = (Ez, 0, 0, 0)$ was chosen, the conclusion would be that no pair creation occurred in their approach! This gauge dependence in the choice of particle interpretation is a problem that has troubled all previous 'Bogoliubov coefficient' approaches to background quantum field theory, as has been succinctly pointed out by Sriramkumar and Padmanabhan[34]. This problem is usually 'patched up' by appealing to the tunnelling interpretation. That is, if the gauge $A = (Ez, 0, 0, 0)$ is chosen, then particle creation can be described in terms of particles 'tunnelling through the barrier that separates particle/antiparticle states' (see [66] or [65] for a description of these methods). This leaves us with two theories, each of which fails in precisely those cases where the other succeeds. Here however, the two approaches arise naturally as parts of the same formalism. By using eigenstates of the gauge-covariant Hamiltonian $\hat{H}_1 \neq \hat{H}_{ev}$ to define particle states, we see that particle creation in the background $A^\mu(x)$ requires either: that $A^0(x) \neq 0$, so that $A^0(x)$ can act as a potential, w.r.t. which tunnelling may occur, or; $\boldsymbol{A}(x) = (A^1(x), A^2(x), A^3(x))$ is time-dependent, so that eigenstates of $\hat{H}_1(t)$ will mix during evolution. Both of these effects contribute equally to the calculation of 'Bogoliubov coefficients', as is demonstrated explicitly in Section 5.1. An immediate consequence of this discussion is of course that if a gauge can be chosen such that $A^0 = 0$ and $\boldsymbol{A}$ is time-independent, then particle creation will not occur. That is, *time-independent magnetic fields cannot create particles.* This result is in agreement with Schwinger's calculation[28] in terms of an effective Lagrangian, and successfully fixes all of the inconsistency problems raised by Sriramkumar and Padmanabhan[34].

2.2.4 Non-locality, and Limitations of the Particle Concept

The particle definition presented in this Section, along with the definition of vacuum that accompanies it, are clearly non-local in the sense that they depend on the choice of spacelike Cauchy surface. We can easily see that this must be the case when we consider that declaring a system to be 'vacuum at time t' involves asserting that "at time t there were no particles anywhere". This statement clearly requires knowledge of a whole spacelike hypersurface defining space at time t. We may have hoped however to be able to define a local 'particle density' $N(x)$ with respect to which the 'vacuum on Σ' could be defined by the requirement that $N(x) = 0$ for all $x \in \Sigma$. However, this is also not possible, since no local particle density could possibly be consistent with the Unruh effect, which states that accelerating observers will observe particles to be present in states which inertial observers would deem to be vacuum. Another reason why we would not expect to be able to define a local particle density is because, as we know from elementary quantum mechanics, a particle is not a localised object, but rather is represented by a spread of probability, generally spread over length scales of the order of the Compton wavelength $\lambda_c = \frac{2\pi}{m}$ ($\hbar = c = 1$ here) of the particle concerned. Hence we could expect that a reasonable definition of particle would be non-local over length scales of the order of the Compton wavelength, but will become 'effectively local' at larger scales.

To investigate this, consider for simplicity the number operator at time t, in the case of free field theory. In this case we can write[46]:

$$\hat{N}_1(t)\psi(\mathbf{x},t) \equiv \hat{P}^+(t)\psi(\mathbf{x},t) - \hat{P}^-(t)\psi(\mathbf{x},t) \tag{2.45}$$

$$= \int d^3\mathbf{y} G(\mathbf{x}-\mathbf{y})\hat{H}_1(\mathbf{y},t)\psi(\mathbf{y},t) \tag{2.46}$$

$$\text{where } G(\mathbf{x}-\mathbf{y}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{\sqrt{|\mathbf{p}|^2+m^2}} \tag{2.47}$$

To derive this, simply expand an arbitrary state $\psi(\mathbf{x})$ in terms of positive and negative energy plane wave states, and substitute into (2.46). We know that $\hat{H}_1(\mathbf{y},t)$ is a local differential operator[3] so the function $G(\mathbf{x}-\mathbf{y})$ encodes the non-locality of the operator

[3]If we wish we can replace it by $i\frac{\partial}{\partial t}$ and require that $\psi(\mathbf{x},t)$ be a solution of the Dirac equation.

$\hat{N}_1$. We can evaluate this function, and we get:

$$G(\mathbf{x}-\mathbf{y}) = \frac{m}{\pi^2|\mathbf{x}-\mathbf{y}|}K_1(m|\mathbf{x}-\mathbf{y}|)$$

where $K_1(mr)$ is a Bessel function. It falls off exponentially for large $mr = \frac{2\pi r}{\lambda_c}$. Hence, as suspected, the particle definition is non-local on small scales, with the non-local contribution becoming negligible on scales larger than λ_c.

Consider now the effect this non-locality would have on measurements made in some finite volume V at time t_0. To measure the quantity A in the volume V, we should consider the operator $\hat{A}_{V,phys}$ obtained by Hermitian extension/vacuum subtraction from the operator:

$$\hat{A}_{V,1} \equiv \tfrac{1}{2}(\hat{\theta}_V\hat{A}_1 + \hat{A}_1\hat{\theta}_V) \tag{2.48}$$

where the operator $\hat{\theta}_V : \mathcal{H} \to \mathcal{H}$ is defined by:

$$\hat{\theta}_V\psi(\mathbf{x}) = \begin{cases} \psi(\mathbf{x}) & \mathbf{x} \in V \\ 0 & \mathbf{x} \notin V \end{cases} \tag{2.49}$$

which is Hermitian, but does not commute with most differential operators, since $\langle\phi|\hat{\theta}_V\hat{A}_1|\psi\rangle$ and $\langle\phi|\hat{A}_1\hat{\theta}_V|\psi\rangle$ will generally differ by terms on the surface of V. So, consider $\hat{A}_1 = \hat{N}_1$ for the moment, and consider the expectation value of $\hat{N}_{V,1}$ in some state $\psi(\mathbf{x},t_0) \in \mathcal{H}$.

$$\langle\psi(\mathbf{x},t_0)|\hat{N}_{V,1}|\psi(\mathbf{x},t_0)\rangle = \int_V \tfrac{1}{2}\{\psi^\dagger(\mathbf{x},t_0)(\hat{N}_1\psi(\mathbf{x},t_0)) + (\hat{N}_1\psi^\dagger(\mathbf{x},t_0))\psi(\mathbf{x},t_0)\}d^3\mathbf{x} \tag{2.50}$$

$$= \int_V N_{t_0}(\mathbf{x},t_0)d^3\mathbf{x} \tag{2.51}$$

$$\text{where } N_{t_0}(\mathbf{x},t_0) \equiv \tfrac{1}{2}\{\psi^\dagger(\mathbf{x},t_0)(\hat{N}_1\psi(\mathbf{x},t_0)) + (\hat{N}_1\psi^\dagger(\mathbf{x},t_0))\psi(\mathbf{x},t_0)\} \tag{2.52}$$

$$= \int d^3\mathbf{y}G(\mathbf{x}-\mathbf{y})Re[\psi^\dagger(\mathbf{x},t_0)\hat{H}_1(\mathbf{y},t_0)\psi(\mathbf{y},t_0)] \tag{2.53}$$

So, having chosen a spacelike Cauchy surface (in this case the surface $t = t_0$) containing the volume V, we can define a particle density $N_{t_0}(\mathbf{x},t_0)$ appropriate to this Cauchy surface. However, this density depends on the choice of Cauchy surface, and calculating it exactly (even at a point) requires integrating over the entire Cauchy surface. In practice, we would not have information about the state on the whole Cauchy surface, but would only have information about the state on some finite volume containing (and possibly equal to) V. Hence, in practice we would have to truncate the integral defining $N_{t_0}(\mathbf{x},t_0)$ to this finite volume. If this volume not only contained V, but also contained all points within a few Compton wavelengths of V, then the error due to this truncation will be exponentially small. However, there are some cases where this truncation might introduce significant errors:

However, by keeping it in the form above then not only does it apply regardless of the dynamics of ψ, but it also applies to any spatially uniform electric field upon replacing $\mathbf{p}$ with $\mathbf{p} - e\boldsymbol{A}(t)$.

- If the finite volume was taken to be equal to V. Then we would expect significant errors in the measured particle density near the boundary of V, corresponding essentially to wave-packets that are 'part outside' and 'part inside' V.

- If the particle detector was small in comparison to the Compton wavelength of the particle concerned. In this case it is unlikely that the particle detector could make reliable measurements of particle density.

- If the background present varied significantly on scales of the order of the Compton wavelength. In this case the theoretical predictions still remain precise, and the errors due to truncation are still small. However, since the predictions do not take into account the effect that the background might have on the detailed working of the specific detector chosen, then there is no guarantee that the results of an actual measurement will still agree with the theoretical predictions. Whether or not these effects are significant is a difficult question of measurement theory, which ultimately can only be answered by experiment. Unfortunately, such experiments appear to be infeasible with current technology.

Finally, it is worth emphasising that the equation governing the evolution of states is entirely local. The non-locality discussed in this section relates only to the calculation of expectation values in these states, and does not in any way effect their evolution.

2.3 Bogoliubov Coefficients and Their Application

In this Section I will define *time-dependent Bogoliubov coefficients* and use them to evaluate various physical quantities, such as the rate of pair creation in the 'evolved vacuum', the physical properties of the 'evolved vacuum', and the general S-Matrix element of the theory. A possible explanation of quantum anomalies is also presented, which explains the physical mechanism responsible for the anomalies, without requiring any reference to the UV-behaviour of the system. For simplicity I restrict my attention to the model example mentioned in Section 2.2.1, of a system with N positive energy, and N negative energy degrees of freedom. I need not specify what equation governs this system, except to assume that it is fermionic, linear, and that the system possesses an inner-product that is preserved under evolution. At the end of the Section I describe how the results can be extended to the more useful case, where the number of degrees of freedom is uncountably infinite, by taking the continuum limit of the finite dimensional case. I also describe briefly how this procedure can be generalised to cases where the inner product is not conserved under evolution, so that the S-Matrix is not unitary. This is potentially relevant to black hole physics, where conservation of the inner product relies on vanishing 'surface terms' not just at spatial infinity, but also at the singularity, and it is known that these latter surface terms will not vanish in general[47].

2.3.1 Some Definitions

Let $\{u_{i,t_0}; i = 1, \ldots N\}$ be an orthonormal basis for $\mathcal{H}^+(t_0)$, and let $\{v_{i,t_0}; i = 1 \ldots N\}$ be an orthonormal basis for $\mathcal{H}^-(t_0)$. Then $\{u_{i,t_0}, v_{i,t_0}; i = 1, \ldots N\}$ will be an orthonormal basis for $\mathcal{H}$ (since $\mathcal{H}^+(t_0)$ and $\mathcal{H}^-(t_0)$ are mutually orthogonal spaces). Similarly, let $\{u_{i,t}; i = 1,\dots N\}$, $\{v_{i,t}; i = 1,\dots N\}$ be orthonormal bases for $\mathcal{H}^+(t)$ and $\mathcal{H}^-(t)$ respectively, for some time $t > t_0$. Also, for each u_{i,t_0} let $u_{i,t_0}(t)$ denote the state obtained from u_{i,t_0} under evolution to time t, and define $v_{i,t_0}(t)$ similarly.

We can expand each of the vectors $u_{i,t_0}(t)$, $v_{i,t_0}(t)$ in terms of $u_{i,t}$ and $v_{i,t}$ as:

$$u_{i,t_0}(t) = \sum_j \{\alpha_{ji}(t,t_0)u_{j,t} + \gamma_{ji}(t,t_0)v_{j,t}\} \tag{2.54}$$

$$v_{i,t_0}(t) = \sum_j \{\beta_{ji}(t,t_0)u_{j,t} + \epsilon_{ji}(t,t_0)v_{j,t}\} \tag{2.55}$$

This defines the *Time-dependent Bogoliubov Coefficients*:

$$\alpha_{ij}(t,t_0) = \langle u_{i,t}|u_{j,t_0}(t)\rangle \quad \gamma_{ij}(t,t_0) = \langle v_{i,t}|u_{j,t_0}(t)\rangle \tag{2.56}$$

$$\beta_{ij}(t,t_0) = \langle u_{i,t}|v_{j,t_0}(t)\rangle \quad \epsilon_{ij}(t,t_0) = \langle v_{i,t}|v_{j,t_0}(t)\rangle \tag{2.57}$$

Comparing this with the evolution matrix on $\mathcal{H}$ (which I denote $\boldsymbol{S}_1(t,t_0)$ here), written in block matrix form:

$$S_{1,ij}(t,t_0) = \begin{bmatrix} \langle u_{i,t}|u_{j,t_0}(t)\rangle & \langle u_{i,t}|v_{j,t_0}(t)\rangle \\ \langle v_{i,t}|u_{j,t_0}(t)\rangle & \langle v_{i,t}|v_{j,t_0}(t)\rangle \end{bmatrix} \tag{2.58}$$

we see that:

$$\boldsymbol{S}_1(t,t_0) = \begin{bmatrix} \boldsymbol{\alpha}(t,t_0) & \boldsymbol{\beta}(t,t_0) \\ \boldsymbol{\gamma}(t,t_0) & \boldsymbol{\epsilon}(t,t_0) \end{bmatrix} \tag{2.59}$$

Since the inner product on $\mathcal{H}$ is conserved under evolution then the $\{u_{i,t_0}(t), v_{i,t_0}(t)\}$ form an orthonormal basis for $\mathcal{H}$, so that $\boldsymbol{S}_1^\dagger(t,t_0)\boldsymbol{S}_1(t,t_0) = \boldsymbol{I} = \boldsymbol{S}_1(t,t_0)\boldsymbol{S}_1^\dagger(t,t_0)$. That is, $\hat{S}_1(t,t_0)$ is unitary for all t, t_0. Since the evolution operator on $\mathcal{F}_\wedge(\mathcal{H})$ is the unitary extension of $\hat{S}_1(t,t_0)$ then it is also unitary. This shows how unitarity of the full evolution operator follows directly from conservation of the inner product under evolution (a result which is significantly more obscure in conventional treatments). When expressed in terms of the Bogoliubov coefficients, this gives the *Bogoliubov conditions*:

$$\begin{bmatrix} \boldsymbol{\alpha}^\dagger\boldsymbol{\alpha} + \boldsymbol{\gamma}^\dagger\boldsymbol{\gamma} & \boldsymbol{\alpha}^\dagger\boldsymbol{\beta} + \boldsymbol{\gamma}^\dagger\boldsymbol{\epsilon} \\ \boldsymbol{\beta}^\dagger\boldsymbol{\alpha} + \boldsymbol{\epsilon}^\dagger\boldsymbol{\gamma} & \boldsymbol{\beta}^\dagger\boldsymbol{\beta} + \boldsymbol{\epsilon}^\dagger\boldsymbol{\epsilon} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger + \boldsymbol{\beta}\boldsymbol{\beta}^\dagger & \boldsymbol{\alpha}\boldsymbol{\gamma}^\dagger + \boldsymbol{\beta}\boldsymbol{\epsilon}^\dagger \\ \boldsymbol{\gamma}\boldsymbol{\alpha}^\dagger + \boldsymbol{\epsilon}\boldsymbol{\beta}^\dagger & \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger + \boldsymbol{\epsilon}\boldsymbol{\epsilon}^\dagger \end{bmatrix} \tag{2.60}$$

It is also common to define matrices $\boldsymbol{U}, \boldsymbol{V}$ by:

$$\boldsymbol{U}(t,t_0) = \begin{bmatrix} \boldsymbol{\alpha}(t,t_0) & 0 \\ 0 & \boldsymbol{\epsilon}(t,t_0) \end{bmatrix} \text{ and } \boldsymbol{V}(t,t_0) = \begin{bmatrix} 0 & \boldsymbol{\beta}(t,t_0) \\ \boldsymbol{\gamma}(t,t_0) & 0 \end{bmatrix} \tag{2.61}$$

Then (2.60) becomes

$$\begin{aligned} \boldsymbol{U}^\dagger\boldsymbol{U} + \boldsymbol{V}^\dagger\boldsymbol{V} &= \boldsymbol{I} = \boldsymbol{U}\boldsymbol{U}^\dagger + \boldsymbol{V}\boldsymbol{V}^\dagger \\ \boldsymbol{U}^\dagger\boldsymbol{V} + \boldsymbol{V}^\dagger\boldsymbol{U} &= 0 = \boldsymbol{U}\boldsymbol{V}^\dagger + \boldsymbol{V}\boldsymbol{U}^\dagger \end{aligned} \tag{2.62}$$

in agreement with standard treatments (see Ottesen[35] or Thaller[37] for instance).

We can use the unitarity of $\boldsymbol{S}_1$ to invert (2.54) and (2.55), yielding:

$$u_{i,t} = \sum_j \{\alpha^*_{ij}(t,t_0)u_{j,t_0}(t) + \beta^*_{ij}(t,t_0)v_{j,t_0}(t)\} \tag{2.63}$$

$$v_{i,t} = \sum_j \{\gamma^*_{ij}(t,t_0)u_{j,t_0}(t) + \epsilon^*_{ij}(t,t_0)v_{j,t_0}(t)\} \tag{2.64}$$

where * denotes complex conjugation. From this we get for the inner derivative operators:

$$i_{u_{i,t}} = \sum_j \{\alpha_{ij}(t,t_0)i_{u_{j,t_0}(t)} + \beta_{ij}(t,t_0)i_{v_{j,t_0}(t)}\} \tag{2.65}$$

$$i_{v_{i,t}} = \sum_j \{\gamma_{ij}(t,t_0)i_{u_{j,t_0}(t)} + \epsilon_{ij}(t,t_0)i_{v_{j,t_0}(t)}\} \tag{2.66}$$

I now introduce the symbol $|\binom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} in_{t_0}\rangle$ to denote an 'in' state, of m particles (in states $u_{i_1} \ldots u_{i_m}$), and n antiparticles (corresponding to the absence of states $v_{j_1} \ldots v_{j_n}$), prepared at time t_0. This state is given by:

$$\begin{aligned} |\tbinom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} in_{t_0}\rangle &\equiv a^\dagger_{i_1,t_0} \ldots a^\dagger_{i_m,t_0} b^\dagger_{j_n,t_0} \ldots b^\dagger_{j_1,t_0} |vac_{t_0}\rangle \\ &= F^+_{m,t_0} \wedge (i_{G^-_{n,t_0}} |vac_{t_0}\rangle) & (2.67) \\ &= (-)^J u_{i_1,t_0} \wedge \ldots u_{i_m,t_0} \wedge v_{1,t_0} \wedge \ldots \check{v}_{j_1,t_0} \cdots \wedge \check{v}_{j_n,t_0} \cdots \wedge v_{N,t_0} & (2.68) \end{aligned}$$

where

$$1 \le i_1 < i_2 < \ldots i_m \le N \qquad 1 \le j_1 < j_2 < \ldots j_n \le N \qquad J = \sum_{k=1}^n (j_k + k)$$

$$F^+_{m,t_0} = u_{i_1,t_0} \wedge u_{i_2,t_0} \cdots \wedge u_{i_m,t_0}$$

$$G^-_{n,t_0} = v_{j_1,t_0} \wedge v_{j_2,t_0} \cdots \wedge v_{j_n,t_0}$$

and the check over v_{j,t_0} represents the fact that this degree of freedom is missing from the state. This state will evolve into

$$|\tbinom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} in_{t_0}(t)\rangle = (-)^J u_{i_1,t_0}(t) \wedge \cdots \wedge u_{i_m,t_0}(t) \wedge v_{1,t_0}(t) \wedge \check{v}_{j_1,t_0}(t) \cdots \wedge \check{v}_{j_n,t_0}(t) \cdots \wedge v_{N,t_0}(t)$$

which we can loosely write as $\equiv a^\dagger_{i_1,t_0}(t) \ldots a^\dagger_{i_m,t_0}(t) b^\dagger_{j_n,t_0}(t) \ldots b^\dagger_{j_1,t_0}(t)|vac_{t_0}\rangle$ where $a^\dagger_{i,t_0}(t)$ is shorthand for $u_{i,t_0}(t)\wedge$ and $b^\dagger_{j,t_0}(t)$ is shorthand for $i_{v_{j,t_0}(t)}$. It should however be stressed that $a^\dagger_{i,t_0}(t)$ **does not** represent the creation of a particle at time t, since $u_{i,t_0}(t)$ is not in $\mathcal{H}^+(t)$. Similarly, $b^\dagger_{j,t_0}(t)$ does not represent the creation of an antiparticle. Using this same loose notation we can write (2.64) and (2.65), in the more familiar form:

$$a_{i,t} = \sum_j \{\alpha_{ij}(t,t_0)a_{j,t_0}(t) + \beta_{ij}(t,t_0)b^\dagger_{j,t_0}(t)\} \tag{2.69}$$

$$b_{i,t} = \sum_j \{\gamma^*_{ij}(t,t_0)a^\dagger_{j,t_0}(t) + \epsilon^*_{ij}(t,t_0)b_{j,t_0}(t)\}$$

It is worth noting here that the labelling of the indices in (2.55) differs from that used in Manogue[48] for instance. Manogue's definitions correspond to setting $u_{i,t} = \sum_j \{\alpha_{ij} u_{j,t_0}(t) + \gamma_{ij} v_{j,t_0}(t)\}$ and $v_{i,t} = \sum_j \{\beta_{ij} u_{j,t_0}(t) + \epsilon_{ij} v_{j,t_0}(t)\}$, so that Manogue's Bogoliubov coefficients are the complex conjugates of those used here. Since the choice made here leads directly to equation (2.58), it is more convenient for comparison with Green's functions, evolution operators, etc. This is investigated further in Section 4.3.

2.3.2 Properties of the Evolved Vacuum

As a simple application of the methods just described, consider evolution of the vacuum. We have:

$$|vac_{t_0}\rangle = v_{1,t_0} \wedge v_{2,t_0} \wedge \cdots \wedge v_{N,t_0} \tag{2.70}$$

$$|vac_t\rangle = v_{1,t} \wedge v_{2,t} \wedge \cdots \wedge v_{N,t} \tag{2.71}$$

$$\text{and } |vac_{t_0}(t)\rangle = v_{1,t_0}(t) \wedge v_{2,t_0}(t) \wedge \cdots \wedge v_{N,t_0}(t) \tag{2.72}$$

We are interested in the particle content of $|vac_{t_0}(t)\rangle$ w.r.t. the Fock Space built on $|vac_t\rangle$. We could obtain this by substituting (2.54) and (2.55) into (2.72) and expanding, although this would be rather messy. We can learn a number of things about $|vac_{t_0}(t)\rangle$ without resorting to this. Consider for instance, the *expected number of particles of a certain type, in* $|vac_{t_0}(t)\rangle$. This is given by $\langle vac_{t_0}(t)|\hat{N}_{u_{i,t}}|vac_{t_0}(t)\rangle$ for the particle state $u_{i,t}$, and $\langle vac_{t_0}(t)|\hat{N}_{v_{i,t}}|vac_{t_0}(t)\rangle$ for the particle state $v_{i,t}$.

By substituting (2.63) - (2.66) into (2.44) $\hat{N}_{u_{i,t}}$, $\hat{N}_{v_{i,t}}$ can be expanded as:

$$\hat{N}_{u_{i,t}} = \sum_{j,k} \beta^*_{ik}\beta_{ij} v_{k,t_0}(t) \wedge (i_{v_{j,t_0}(t)} + \text{ other terms, involving } u \wedge (i_v \text{ etc} \tag{2.73}$$

$$\hat{N}_{v_{i,t}} = \sum_{jk} \gamma_{ik}\gamma^*_{ij} i_{u_{k,t_0}(t)}(u_{j,t_0}(t) \wedge + \text{ other terms} \tag{2.74}$$

When calculating $\langle vac_{t_0}(t)|\hat{N}_{u_{i,t}}|vac_{t_0}(t)\rangle$ it is clearly only terms of the form $v_{j,t_0}(t) \wedge (i_{v_{j,t_0}(t)}|vac_{t_0}(t)\rangle)$ (which $= |vac_{t_0}(t)\rangle$) which contribute, giving:

$$\langle vac_{t_0}(t)|\hat{N}_{u_{i,t}}|vac_{t_0}(t)\rangle = \sum_j \beta^*_{ij}(t,t_0)\beta_{ij}(t,t_0) \qquad = (\boldsymbol{\beta}\boldsymbol{\beta}^\dagger)_{ii} \tag{2.75}$$

$$\langle vac_{t_0}(t)|\hat{N}_{v_{i,t}}|vac_{t_0}(t)\rangle = \sum_j \gamma_{ij}(t,t_0)\gamma^*_{ij}(t,t_0) \qquad = (\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)_{ii} \tag{2.76}$$

This is the natural time-dependent generalisation of conventional methods (see also [17] for an alternative derivation of a similar result). The total number of particles (not including antiparticles) in the 'out vacuum' is thus given by:

$$\langle \hat{N}_+(t,t_0)\rangle = \sum_i \langle vac_{t_0}(t)|\hat{N}_{u_{i,t}}|vac_{t_0}(t)\rangle = Trace(\boldsymbol{\beta}(t,t_0)\boldsymbol{\beta}^\dagger(t,t_0)) \tag{2.77}$$

and similarly for the total number of antiparticles. From (2.60) (using $Trace(\boldsymbol{AB}) = Trace(\boldsymbol{BA})$) we see that $Trace(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger) = Trace(\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)$ so that:

$$\langle \hat{N}_+\rangle = \langle \hat{N}_-\rangle \tag{2.78}$$

in agreement with charge conservation. The total number of particles (including anti-particles) in the 'out vacuum' is just $\langle \hat{N}(t,t_0)\rangle = 2Trace(\boldsymbol{\gamma}(t,t_0)\boldsymbol{\gamma}^\dagger(t,t_0))$. Since it has been calculated for all times, then we can use it to calculate the 'rate of production' of particles $\frac{d}{dt}\langle \hat{N}(t,t_0)\rangle$ for instance, something which could not be done with any previous treatments. This represents the number of particles produced per unit time, throughout all space.

General Expectation Values of The Evolved Vacuum

In this subsection I will present an alternative proof of the previous result, which will also allow us to deduce $\langle vac_{t_0}(t)|\hat{A}_{phys}(t)|vac_{t_0}(t)\rangle$ for any operator, $\hat{A}_{phys}(t)$ which is obtained from some operator $\hat{A}_1(t)$ via the Hermitian extension/vacuum subtraction process. First, use property 4 of Section 2.1.3, along with (2.71) and (2.72) to write:

$$\langle vac_{t_0}(t)|\hat{A}_{phys}(t)|vac_{t_0}(t)\rangle = \sum_{i=1}^{N}\langle v_{i,t_0}(t)|\hat{A}_1(t)|v_{i,t_0}(t)\rangle - \sum_{i=1}^{N}\langle v_{i,t}|\hat{A}_1(t)|v_{i,t}\rangle \tag{2.79}$$

Then substitute (2.55) and rearrange to get:

$$\langle vac_{t_0}(t)|\hat{A}_{phys}(t)|vac_{t_0}(t)\rangle = Trace(\boldsymbol{\beta}^\dagger\boldsymbol{A}^{++}\boldsymbol{\beta} + \boldsymbol{\beta}^\dagger\boldsymbol{A}^{+-}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}^\dagger\boldsymbol{A}^{-+}\boldsymbol{\beta} + \boldsymbol{\epsilon}^\dagger\boldsymbol{A}^{--}\boldsymbol{\epsilon}) - Trace(\boldsymbol{A}^{--}) \tag{2.80}$$

$$= Trace(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger\boldsymbol{A}^{++} - \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger\boldsymbol{A}^{--} + \boldsymbol{\epsilon}\boldsymbol{\beta}^\dagger\boldsymbol{A}^{+-} + \boldsymbol{\beta}\boldsymbol{\epsilon}^\dagger\boldsymbol{A}^{-+}) \tag{2.81}$$

where I have defined:

$$\begin{aligned} \boldsymbol{A}^{++}_{jk} &\equiv \langle u_{j,t}|\hat{A}_1(t)|u_{k,t}\rangle \\ \boldsymbol{A}^{--}_{jk} &\equiv \langle v_{j,t}|\hat{A}_1(t)|v_{k,t}\rangle \\ \boldsymbol{A}^{+-}_{jk} &\equiv \langle u_{j,t}|\hat{A}_1(t)|v_{k,t}\rangle \\ \text{and } \boldsymbol{A}^{-+}_{jk} &\equiv \langle v_{j,t}|\hat{A}_1(t)|u_{k,t}\rangle \\ &= \overline{\boldsymbol{A}^{+-}_{kj}} \text{ if } \hat{A}_1 \text{ is Hermitian} \end{aligned} \tag{2.82}$$

We can rederive $\langle \hat{N}(t,t_0)\rangle$ from (2.81) by using $\hat{N}_1(t) = \hat{P}^+(t) - \hat{P}^-(t)$ (so that $\boldsymbol{N}^{++}_{jk} = \delta_{jk} = -\boldsymbol{N}^{--}_{jk}$ and $\boldsymbol{N}^{+-}_{jk} = 0 = \boldsymbol{N}^{-+}_{jk}$). Conservation of charge follows simply from considering $\hat{A}_1 = \hat{1}$. Another interesting example worth considering briefly here, is the expected number of particles present in a finite volume V (at time t). In this case the operator $\hat{A}_1 : \mathcal{H} \to \mathcal{H}$ is

$$\hat{N}_{V,1}(t) = \tfrac{1}{2}\{(\hat{P}^+(t) - \hat{P}^-(t))\hat{\theta}_V + \hat{\theta}_V(\hat{P}^+(t) - \hat{P}^-(t))\} \tag{2.83}$$

$$= \hat{P}^+(t)\hat{\theta}_V\hat{P}^+(t) - \hat{P}^-(t)\hat{\theta}_V\hat{P}^-(t) \tag{2.84}$$

So that $\boldsymbol{N}^{+-}_{V;jk} = 0 = \boldsymbol{N}^{-+}_{V;jk}$, while $\boldsymbol{N}^{++}_{V;jk} = \langle u_{j,t}|u_{k,t}\rangle_V$ and $\boldsymbol{N}^{--}_{V;jk} = -\langle v_{j,t}|v_{k,t}\rangle_V$, where $\langle|\rangle_V$ denotes that the spatial integral is restricted to the volume V.

Now, the derivation of $\langle F_{t_0}(t)|\hat{A}_{phys}(t)|F_{t_0}(t)\rangle$ for an arbitrary state F_{t_0} is identical to the derivation of (2.81), and results in:

$$\langle \binom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} in_{t_0}(t)|\hat{A}_{phys}(t)|\binom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} in_{t_0}(t)\rangle$$
$$= \sum_{k=1}^{m} \langle u_{i_k,t_0}(t)|\hat{A}_1(t)|u_{i_k,t_0}(t)\rangle - \sum_{k=1}^{n} \langle v_{j_k,t_0}(t)|\hat{A}_1(t)|v_{j_k,t_0}(t)\rangle + \langle vac_{t_0}(t)|\hat{A}_{phys}(t)|vac_{t_0}(t)\rangle \tag{2.85}$$

$$= \sum_{k=1}^{m} (\boldsymbol{\alpha}^\dagger \boldsymbol{A}^{++} \boldsymbol{\alpha} + \boldsymbol{\alpha}^\dagger \boldsymbol{A}^{+-} \boldsymbol{\gamma} + \boldsymbol{\gamma}^\dagger \boldsymbol{A}^{-+} \boldsymbol{\alpha} + \boldsymbol{\gamma}^\dagger \boldsymbol{A}^{--} \boldsymbol{\gamma})_{i_k i_k}$$
$$- \sum_{k=1}^{n} (\boldsymbol{\beta}^\dagger \boldsymbol{A}^{++} \boldsymbol{\beta} + \boldsymbol{\beta}^\dagger \boldsymbol{A}^{+-} \boldsymbol{\epsilon} + \boldsymbol{\epsilon}^\dagger \boldsymbol{A}^{-+} \boldsymbol{\beta} + \boldsymbol{\epsilon}^\dagger \boldsymbol{A}^{--} \boldsymbol{\epsilon})_{j_k j_k} + \langle vac_{t_0}(t)|\hat{A}_{phys}(t)|vac_{t_0}(t)\rangle \tag{2.86}$$

Equations (2.81) and (2.86) will be used extensively in all future applications. It is worth noting here that the subtraction $(\boldsymbol{\epsilon}\boldsymbol{\epsilon}^\dagger)_{kj} - \delta_{ij} = -(\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)_{kj}$ used to get (2.81) from (2.80) relies on the fact that we are vacuum subtracting with respect to $|vac_t\rangle$, rather than $|vac_{t_0}\rangle$ or $|vac_{-\infty}\rangle$ for instance.

It is interesting to consider here the case where $\hat{A}_1$ represents a quantity that is conserved at the level of the Dirac equation. In this case we can write $\langle v_{i,t_0}(t)|\hat{A}_1|v_{i,t_0}(t)\rangle = \langle v_{i,t_0}|\hat{A}_1|v_{i,t_0}\rangle$ so that (2.81) becomes:

$$\langle vac_{t_0}(t)|\hat{A}_{phys}|vac_{t_0}(t)\rangle = Trace(\boldsymbol{A}^{--}(t) - \boldsymbol{A}^{--}(t_0)) \tag{2.87}$$

We see that due to the varying particle interpretation this can be non-zero even when $\hat{A}_1$ is independent of time. This provides an elegant physical description of quantum anomalies. That is, although the total value of $\hat{A}$ in any given state does not change with time, the portion of this that is to be attributed to the vacuum may change as the definition of vacuum changes. Since the amount that would be measured in experiment is the amount 'in excess of the vacuum' (the vacuum subtracted amount) then this can change with time. For example, a nice treatment of the axial anomaly in an external electromagnetic background, which appeals to a similar physical mechanism as above but which uses conventional calculational techniques, can be found in [49]. The subtraction in (2.87) will generally be of the form $(\infty - \infty)$, although it is likely that writing it in the form of (2.81) will fix this. This will be investigated further in future applications. Notice also that, if (2.87) does indeed represent quantum anomalies, then for quantities (such as the current density for instance) that are not anomolous, the vacuum subtraction will be independent of the hypersurface on which the subtraction takes place. This is important when considering back-reaction for instance, since it ensures that $J_{vac,t_0}(\mathbf{x},t)$ is a well-defined covariant quantity. I demonstrate in Chapter 7 that, atleast for spatially uniform electric fields, the vacuum subtraction of the current density is indeed independent of the time at which the subtraction is calculated.

Fluctuations

Equations (2.85) and (2.86) allow us to calculate the expectation value, in any given state, of any operator $\hat{A}_{phys}$ that can be obtained from $\hat{A}_1$ by Hermitian extension and vacuum subtraction. However, not every physical operator will be of this form. For instance, the operator $(\hat{A}_{phys})^2$ will in general not be the Hermitian extension of any operator on $\mathcal{H}$. So, consider the expectation value of $(\hat{A}_{phys})^2$ in the state $\psi_1 \wedge \cdots \wedge \psi_n$ where $\psi_1 \ldots \psi_n$ are assumed for convenience to be orthonormal (although need not be in $\mathcal{H}^+(t)$ or $\mathcal{H}^-(t)$ for any t). By writing $(\hat{A}_{phys})^2 = (\hat{A}_H)^2 - 2\langle vac_t|\hat{A}_H|vac_t\rangle \hat{A}_H + (\langle vac_t|\hat{A}_H|vac_t\rangle)^2\hat{1}$ we can obtain the desired expectation value by noting that:

$$\begin{aligned}\langle\psi_1 \wedge \cdots \wedge \psi_n|(\hat{A}_H)^2|\psi_1 \wedge \cdots \wedge \psi_n\rangle = \langle\psi_1 \wedge \cdots \wedge \psi_n|(\hat{A}_1^2)_H|\psi_1 \wedge \cdots \wedge \psi_n\rangle + \\ 2\sum_{i<j}(\langle\psi_i|\hat{A}_1|\psi_i\rangle\langle\psi_j|\hat{A}_1|\psi_j\rangle - \langle\psi_i|\hat{A}_1|\psi_j\rangle\langle\psi_j|\hat{A}_1|\psi_i\rangle)\end{aligned}$$

This allows us to calculate the fluctuations of $\hat{A}_{phys}$ in the state $\psi_1 \wedge \cdots \wedge \psi_n$:

$$\begin{aligned}\langle\psi_1 \wedge \cdots \wedge \psi_n|(\hat{A}_{phys})^2|\psi_1 \wedge \cdots \wedge \psi_n\rangle - (\langle\psi_1 \wedge \cdots \wedge \psi_n|\hat{A}_{phys}|\psi_1 \wedge \cdots \wedge \psi_n\rangle)^2 \\ = \langle\psi_1 \wedge \cdots \wedge \psi_n|(\hat{A}_H)^2|\psi_1 \wedge \cdots \wedge \psi_n\rangle - (\langle\psi_1 \wedge \cdots \wedge \psi_n|\hat{A}_H|\psi_1 \wedge \cdots \wedge \psi_n\rangle)^2 \quad (2.88) \\ = \sum_i\langle\psi_i|\hat{A}_1^2|\psi_i\rangle - \sum_{i,j}|\langle\psi_i|\hat{A}_1|\psi_j\rangle|^2 \quad (2.89)\end{aligned}$$

where I have used the assumed hermiticity of $\hat{A}_1$. This is an interesting result, which is worth investigating further. One straightforward and interesting application of this result is to calculate the vacuum fluctuations in the number of particles present in a finite volume V. That is $\langle vac_t|(\hat{N}_{V,phys}(t))^2|vac_t\rangle$. (Note that $\langle vac_t|\hat{N}_{V,phys}(t)|vac_t\rangle$ is trivially zero, since $\hat{N}_{V,phys}(t)$ is obtained from Hermitian extension and vacuum subtraction.) To calculate this, use equation (2.84) to deduce:

$$(\hat{N}_{V,1}(t))^2 = \hat{P}^+(t)\hat{\theta}_V\hat{P}^+(t)\hat{\theta}_V\hat{P}^+(t) + \hat{P}^-(t)\hat{\theta}_V\hat{P}^-(t)\hat{\theta}_V\hat{P}^-(t) \quad (2.90)$$

So:

$$\begin{aligned}\langle v_{i,t}|(\hat{N}_{V,1})^2|v_{i,t}\rangle &= \langle\hat{\theta}_V v_{i,t}|\hat{P}^-(t)|\hat{\theta}_V v_{i,t}\rangle \quad (2.91) \\ &= \sum_j\langle\hat{\theta}_V v_{i,t}|v_{j,t}\rangle\langle v_{j,t}|\hat{\theta}_V v_{i,t}\rangle \quad (2.92) \\ &= \sum_j|\langle v_{i,t}|v_{j,t}\rangle_V|^2 \quad (2.93)\end{aligned}$$

Substituting into (2.89) gives:

$$\begin{aligned}\langle vac_t|(\hat{N}_{V,phys}(t))^2|vac_t\rangle &= \sum_i\sum_j|\langle v_{i,t}|v_{j,t}\rangle_V|^2 - \sum_{ij}|\langle v_{i,t}|v_{j,t}\rangle_V|^2 \quad (2.94) \\ &= 0 \quad (2.95)\end{aligned}$$

That is:

There are no fluctuations in the number of particles present in a finite volume in the physical vacuum.

I should stress here that I am referring to the actual vacuum $|vac_t\rangle$ at the time of interest, not to any 'evolved vacuum state' $|vac_{t_0}(t)\rangle$ (the evolved vacuum will of course have fluctuations in the number of particles present). Also, it is likely that if we truncated the integral defining $\hat{N}_1$, as described in Section 2.2.4, then this result would no longer be zero, and would depend on the form of the truncation. I have not considered this effect here.

To consider the fluctuations of some quantity $\hat{A}_{phys}(t)$ in the 'evolved vacuum' $|vac_{t_0}(t)\rangle$, define $\hat{P}^-_{t_0}(t) \equiv \sum_i |v_{i,t_0}(t)\rangle\langle v_{i,t_0}(t)| = \hat{U}_1(t,t_0)\hat{P}^-(t_0)\hat{U}_1^\dagger(t,t_0)$ and $\hat{P}^+_{t_0}(t) \equiv \sum_i |u_{i,t_0}(t)\rangle\langle u_{i,t_0}(t)| = \hat{U}_1(t,t_0)\hat{P}^+(t_0)\hat{U}_1^\dagger(t,t_0)$. Then we can write:

$$\langle vac_{t_0}(t)|(\hat{A}_{phys})^2|vac_{t_0}(t)\rangle - (\langle vac_{t_0}(t)|\hat{A}_{phys}|vac_{t_0}(t)\rangle^2 \tag{2.96}$$
$$= Trace(\hat{P}^-_{t_0}(t)\hat{A}_1(t)^2) - Trace(\hat{P}^-_{t_0}(t)\hat{A}_1(t)\hat{P}^-_{t_0}(t)\hat{A}_1(t)) \tag{2.97}$$
$$= Trace(\hat{P}^-_{t_0}(t)\hat{A}_1(t)\hat{P}^+_{t_0}(t)\hat{A}_1(t)) \tag{2.98}$$

So we see that the size of the fluctuations in the evolved vacuum are determined by the commutation between $\hat{A}_1(t)$ and $\hat{P}^-_{t_0}(t)$. If we consider again fluctuations in the *physical vacuum at time t*, $|vac_t\rangle$ (by putting $t_0 = t$ above), we see that any operator that commutes with the Hamiltonian (and hence with $\hat{P}^+(t)$) will have zero fluctuations in the physical vacuum. This is just a reflection of the fact that, since the physical vacuum is constructed from the spectrum of the Hamiltonian, then it will be an eigenstate of any operator that commutes with the Hamiltonian.

The Vacuum Component of $|vac_{t_0}(t)\rangle$

Just as in Section 2.2.1, we can substitute (2.71) and (2.72) into (2.15), and we immediately deduce that the vacuum - vacuum transition amplitude is:

$$\langle vac_t|vac_{t_0}(t)\rangle = \det(\boldsymbol{\epsilon}(t,t_0)) \tag{2.99}$$

Alternatively, the same conclusion can be reached by substituting (2.55) into (2.72) and extracting the component proportional to $|vac_t\rangle$. The reader is invited to compare the ease of these derivations with the analogous result presented by Fradkin et. al.[17], or (under simplifying assumptions) by Schwinger[50], or with the bosonic result, presented by DeWitt[24]. It is common to see this result quoted in terms of the Feynman propagator (see [17] pg 37, [22] or [50] for instance) as:

$$\langle vac, out|vac, in\rangle = \frac{\det S_F^0}{\det S_F} \tag{2.100}$$

where $S_F(x,y)$ is the Feynman propagator in the relevant background, and $S_F^0(x,y)$ is the free Feynman propagator. The relation of this result to (2.99) is investigated in Section

4.5. Now, the probability that the vacuum at time t_0 will again be the vacuum at time t is:

$$\mathcal{P}_{|vac_{t_0}\rangle\to|vac_t\rangle} = |\det(\boldsymbol{\epsilon}(t,t_0))|^2 = \det(\boldsymbol{\epsilon}(t,t_0)\boldsymbol{\epsilon}^\dagger(t,t_0)) \tag{2.101}$$

$$= \exp(Trace(\log(1-\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)) = \exp(-\sum_{n=1}^{\infty}\frac{1}{n}Trace((\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)^n)) \tag{2.102}$$

In the case of an electromagnetic background we will see that $\boldsymbol{\gamma}$ is first order in the coupling constant e, so that $\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger$ is second order. Then (2.102) allows us to write:

$$\mathcal{P}_{|vac_{t_0}\rangle\to|vac_t\rangle} = \exp(-\tfrac{1}{2}\langle\hat{N}(t,t_0)\rangle) + O(e^4) \tag{2.103}$$

This fact is used in Itzykson and Zuber[41] as a method for finding $\langle\hat{N}(\infty,-\infty)\rangle$ (to this order in e) without using Bogoliubuv coefficients. However, in Itzykson and Zuber[41] the result above is simply justified on physical grounds, it is not proved.

2.3.3 S-Matrix Elements in General

Consider the in-state $|\binom{i_1 i_2\ldots i_m}{j_1 j_2\ldots j_n} in_{t_0}\rangle \equiv a^\dagger_{i_1,t_0}\ldots a^\dagger_{i_m,t_0} b^\dagger_{j_n,t_0}\ldots b^\dagger_{j_1,t_0}|vac_{t_0}\rangle$ as defined in (2.68), and the out state $|\binom{i'_1 i'_2\ldots i'_{m'}}{j'_1 j'_2\ldots j'_{n'}} out_t\rangle = a^\dagger_{i'_1,t}\ldots a^\dagger_{i'_{m'},t} b^\dagger_{j'_{n'},t}\ldots b^\dagger_{j'_1,t}|vac_t\rangle$. We can use (2.15) to immediately calculate the S-matrix element:

$$S_{out,in}(t,t_0) \equiv \langle\binom{i'_1 i'_2\ldots i'_{m'}}{j'_1 j'_2\ldots j'_{n'}} out_t|\binom{i_1 i_2\ldots i_m}{j_1 j_2\ldots j_n} in_{t_0}(t)\rangle \tag{2.104}$$

$$= (-)^{J-J'}\det\begin{bmatrix} \begin{bmatrix} \alpha_{i'_1 i_1} & \cdots & \alpha_{i'_1 i_m} \\ \vdots & & \vdots \\ \alpha_{i'_{m'} i_1} & \cdots & \alpha_{i'_{m'} i_m} \end{bmatrix} & \begin{bmatrix} \beta_{i'_1 1} & \binom{j_1\ldots j_n}{missing} & \beta_{i'_1 N} \\ \vdots & & \vdots \\ \beta_{i'_{m'} 1} & \binom{j_1\ldots j_n}{missing} & \beta_{i'_{m'} N} \end{bmatrix} \\ \begin{bmatrix} \gamma_{1 i_1} & \cdots & \gamma_{1 i_m} \\ \binom{j'_1\ldots j'_{n'}}{missing} & & \binom{j'_1\ldots j'_{n'}}{missing} \\ \gamma_{N i_1} & \cdots & \gamma_{N i_m} \end{bmatrix} & \begin{bmatrix} \epsilon_{11} & \binom{j_1\ldots j_n}{missing} & \epsilon_{1N} \\ \binom{j'_1\ldots j'_{n'}}{missing} & & \binom{j'_1\ldots j'_{n'}}{missing} \\ \epsilon_{N1} & \binom{j_1\ldots j_n}{missing} & \epsilon_{NN} \end{bmatrix} \end{bmatrix} \tag{2.105}$$

if $m-n = m'-n'$ (charge conservation) and zero otherwise. This is a completely general formula for an arbitrary S-Matrix element, and it is quite remarkable that using Slater determinants makes the derivation of this result so trivial. I will show in Section 4.2 how this is derived without using Slater determinants, and the reader will hopefully agree that the derivation in Section 4.2 is rather more cumbersome!

It is often useful in practice to factor out $\det(\boldsymbol{\epsilon})$ from this expression, leaving an expression for $\frac{S_{out,in}(t,t_0)}{\langle vac_t|vac_{t_0}(t)\rangle}$. I do this now, first for the cases when $|in_{t_0}\rangle = |vac_{t_0}\rangle$ or $|out_t\rangle = |vac_t\rangle$ respectively, and then for the general case.

Case 1// $\langle\binom{i'_1 i'_2\ldots i'_{m'}}{j'_1 j'_2\ldots j'_{n'}} out_t|vac_{t_0}(t)\rangle$**:** In this case $m' = n'$ and $m = n = 0$, so (2.105) reduces to:

$$S_{out,in}(t,t_0) = (-)^{J'} \det \begin{bmatrix} \overrightarrow{\beta}_{i'_1} \\ \vdots \\ \overrightarrow{\beta}_{i'_{n'}} \\ \overrightarrow{\epsilon}_1 \\ \binom{j'_1 \ldots j'_{n'}}{missing} \\ \overrightarrow{\epsilon}_N \end{bmatrix}$$

where $\overrightarrow{\beta}_i$ represents the i^{th} row of $\boldsymbol{\beta}$, and similarly for $\overrightarrow{\epsilon}_i$. Multiplying this on the right by the matrix $\begin{bmatrix} \overrightarrow{\epsilon}^{-1}_{j'_1} & \cdots & \overrightarrow{\epsilon}^{-1}_{j'_{n'}} & \overrightarrow{\epsilon}^{-1}_1 & \binom{j'_1 \ldots j'_{n'}}{missing} & \overrightarrow{\epsilon}^{-1}_N \end{bmatrix}$ (the determinant of which is $\frac{(-1)^{\Sigma_1^{n'}(j'_k-1)}}{\det(\boldsymbol{\epsilon})}$) gives:

$$\begin{aligned} S_{out,vac}(t,t_0) &= \langle \binom{i'_1 i'_2 \ldots i'_{m'}}{j'_1 j'_2 \ldots j'_{n'}} out_t | vac_{t_0}(t) \rangle \\ &= (-)^{\frac{n'}{2}(n'-1)} \det(\boldsymbol{\epsilon}) \det \begin{bmatrix} V_{i'_1 j'_1} & \cdots & V_{i'_1 j'_{n'}} \\ \vdots & & \vdots \\ V_{i'_{n'} j'_1} & \cdots & V_{i'_{n'} j'_{n'}} \end{bmatrix} \\ &= (-)^{\frac{n'}{2}(n'-1)} \det(\boldsymbol{\epsilon}) \det(\boldsymbol{V}_{out}) \qquad \text{say} \end{aligned} \tag{2.106}$$

where $\boldsymbol{V} \equiv \boldsymbol{\beta}\boldsymbol{\epsilon}^{-1}$ and $\boldsymbol{V}_{out}$ is constructed from $\boldsymbol{V}$ using only those matrix entries relevant to the desired out state. This is the finite-time generalisation of equation (124) of [50] (the extra factor of $(-)^{\frac{n'}{2}(n'-1)}$ is simply due to the different order of the $b^\dagger_{j,t_0}$ operators in (2.68)). I invite the reader to compare the ease of derivation afforded by this formalism with that of Schwinger's calculation[50]. For the infinite time Bosonic equivalent of this result, the reader is referred to [24] or [22]. I will present the finite time Bosonic equivalent of this in Section 8.2.

Case 2// $\langle vac_t | \binom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} in_{t_0}(t) \rangle$ **:** We have $m' = n' = 0$ and $m = n$ now. The method here is just as before, and the result is:

$$\begin{aligned} S_{vac,in}(t,t_0) &= \langle vac_t | \binom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} in_{t_0}(t) \rangle \\ &= (-)^{\frac{n}{2}(n-1)} \det(\boldsymbol{\epsilon}) \det \begin{bmatrix} \Lambda_{j_1 i_1} & \cdots & \Lambda_{j_1 i_n} \\ \vdots & & \vdots \\ \Lambda_{j_n i_1} & \cdots & \Lambda_{j_n i_n} \end{bmatrix} \\ &= (-)^{\frac{n}{2}(n-1)} \det(\boldsymbol{\epsilon}) \det(\boldsymbol{\Lambda}_{in}) \end{aligned} \tag{2.107}$$

where $\boldsymbol{\Lambda} \equiv \boldsymbol{\epsilon}^{-1}\boldsymbol{\gamma}$.

The General Case: The most obvious way to calculate this is to write:

$$S_{out,in}(t,t_0) = \langle vac_t | b_{j'_1,t} \ldots b_{j'_{n'},t} a_{i'_{m'},t} \ldots a_{i'_1,t} a^\dagger_{i_1,t_0}(t) \ldots a^\dagger_{i_m,t_0}(t) b^\dagger_{j_n,t_0}(t) \ldots b^\dagger_{j_1,t_0}(t) | vac_{t_0}(t) \rangle$$

and use equation (2.69) to convert $S_{out,in}(t,t_0)$ into a series of S-matrix elements of the form given in Case 2. I will not attempt the general case here, but will simply derive, as an example, the S-Matrix element between two 1-electron states[4]

$$\begin{aligned} S_{in,out} &= \langle ({}^{i'})\, out_t | ({}^{i})\, in_{t_0}(t) \rangle \\ &= \det(\boldsymbol{\epsilon})(\alpha_{i'i} - \sum_k \beta_{i'k}\Lambda_{ki}) \qquad (2.108) \\ &= \det(\boldsymbol{\epsilon})(\boldsymbol{\alpha} - \boldsymbol{\beta}\boldsymbol{\epsilon}^{-1}\boldsymbol{\gamma})_{i'i} = \overline{\alpha^{-1}_{ii'}} \det(\boldsymbol{\epsilon}) \qquad (2.109) \end{aligned}$$

where I have used $\boldsymbol{\alpha} - \boldsymbol{\beta}\boldsymbol{\epsilon}^{-1}\boldsymbol{\gamma} = (\boldsymbol{\alpha}^\dagger)^{-1}$, which follows from the Bogoliubov conditions. The two terms in (2.108) represent two different contributions to the 'evolved 1-particle state' $u_{i,t_0}(t) \wedge |vac_{t_0}(t)\rangle$. The first contribution corresponds to the positive energy component of $\overrightarrow{u}_{1,t_0}(t)$ acting on the vacuum component of $|vac_{t_0}(t)\rangle$, while the second contribution comes from the negative energy component of $\overrightarrow{u}_{1,t_0}(t)$ destroying the antiparticle from one of the two-particle components of $|vac_{t_0}(t)\rangle$. We see that the 'naive transition amplitude' $\alpha_{i'i} = \langle u_{i',t} | u_{i,t_0}(t) \rangle$ has been changed to $\overline{\alpha^{-1}_{ii'}} \det(\boldsymbol{\epsilon})$ by considering the effects of vacuum instability (that is, pair creation in the 'evolved vacuum'). This simple example demonstrates how vacuum instability has a significant effect not just on vacuum evolution, but on the evolution of all states. Similarly, the S-matrix element between two antiparticle states is:

$$\langle ({}_{j'})\, out_t | ({}_{j})\, in_{t_0}(t) \rangle = (\epsilon^{-1})_{jj'} \det(\boldsymbol{\epsilon}) \qquad (2.110)$$

Some general comments are now in order:

1. We can use the Bogoliubov conditions to deduce:

$$\boldsymbol{\Lambda}^\dagger = -\boldsymbol{\alpha}^{-1}\boldsymbol{\beta} \qquad\qquad \boldsymbol{V}^\dagger = -\boldsymbol{\gamma}\boldsymbol{\alpha}^{-1} \qquad (2.111)$$

$$\boldsymbol{\epsilon}^\dagger\boldsymbol{\epsilon} = (1 + \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\dagger)^{-1} \qquad\qquad \boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger = (1 + \boldsymbol{V}\boldsymbol{V}^\dagger)^{-1} \qquad (2.112)$$

   Noting that $\boldsymbol{\alpha}^{-1}$ and $\boldsymbol{\epsilon}^{-1}$ represent the relative particle/particle and antiparticle/antiparticle transition amplitudes respectively, then (2.112), along with $\boldsymbol{V}^\dagger\boldsymbol{\alpha} = -\boldsymbol{\epsilon}\boldsymbol{\Lambda}$ and $\boldsymbol{\alpha}\boldsymbol{\Lambda}^\dagger = -\boldsymbol{V}\boldsymbol{\epsilon}$ (which follow immediately from (2.111) and the definitions of $\boldsymbol{V}$, $\boldsymbol{\Lambda}$) constitute the fermionic equivalent of DeWitt's[24, 51] relativistic generalisation of the Optical Theorem (see [17] also).

2. It is straightforward to generalise the procedure above to the case where the inner product is not conserved under evolution. We can still define the Bogoliubov coefficients by equations (2.54) and (2.55), and these still give (2.56) and (2.57). The main difference is that since $\boldsymbol{S}(t,t_0)$ is no longer unitary, we have to use $\boldsymbol{S}^{-1}$ rather than $\boldsymbol{S}^\dagger$ in (2.63) - (2.66). Equation (2.105) is still exactly the same, and although most other equations will change, the general procedure is just as before. This is in stark contrast to conventional quantum field theory, where the entire construction

---

[4]The notation $|({}^{i_1 i_2 \dots i_m}) in_{t_0}\rangle$ is used to denote a state consisting of m particles, and no antiparticles, while the state $|({}_{j_1 j_2 \dots j_n}) in_{t_0}\rangle$ consists of n antiparticles, and no particles.

relies on unitarity of the S-Matrix from the outset. This is potentially relevant to black hole physics, where the termination of geodesics on the singularity will cause decay of the wave packet of an in-falling particle. If back-reaction is properly taken into account then the charge, mass etc lost by the quantum system, will be donated to the classical background, while the loss of the quantum 'state-information' will add to the entropy of the black hole. This will be investigated fully in future work.

3. Consider now the generalisation of this procedure to cases where the dimension of $\mathcal{H}$ is uncountable. For instance, consider again the case of Dirac field theory, and suppose we have a basis $\{u_{\mathbf{p},\lambda;t}(\mathbf{x}); \mathbf{p} \in \boldsymbol{R}^3, \lambda = 1,2\}$ for $\mathcal{H}^+(t)$ and a basis $\{v_{\mathbf{p},\lambda;t}(\mathbf{x}); \mathbf{p} \in \boldsymbol{R}^3, \lambda = 1,2\}$ for $\mathcal{H}^-(t)$ satisfying:

$$\langle u_{\mathbf{p},\lambda;t}(\mathbf{x}) | u_{\mathbf{q},\sigma;t}(\mathbf{x})\rangle = f(\mathbf{p})\delta_{\lambda\sigma}\delta(\mathbf{p}-\mathbf{q}) = \langle v_{\mathbf{p},\lambda;t}(\mathbf{x}) | v_{\mathbf{q},\sigma;t}(\mathbf{x})\rangle$$

We can define:

$$\begin{aligned} \alpha_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t,t_0) &= \langle u_{\mathbf{p},\lambda;t}(\mathbf{x}) | u_{\mathbf{q},\sigma;t_0}(\mathbf{x},t)\rangle \\ \gamma_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t,t_0) &= \langle v_{\mathbf{p},\lambda;t}(\mathbf{x}) | u_{\mathbf{q},\sigma;t_0}(\mathbf{x},t)\rangle \qquad (2.113) \\ \beta_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t,t_0) &= \langle u_{\mathbf{p},\lambda;t}(\mathbf{x}) | v_{\mathbf{q},\sigma;t_0}(\mathbf{x},t)\rangle \\ \epsilon_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t,t_0) &= \langle v_{\mathbf{p},\lambda;t}(\mathbf{x}) | v_{\mathbf{q},\sigma;t_0}(\mathbf{x},t)\rangle \qquad (2.114) \end{aligned}$$

just as in (2.56) and (2.57). All the previous results can be carried over, except that the $\sum_j$ which appears in matrix multiplications must be replaced by $\int \frac{d^3\mathbf{p}}{f(\mathbf{p})}\sum_\lambda$, so that the 'identity matrix' has entries $f(\mathbf{p})\delta_{\lambda\sigma}\delta(\mathbf{p}-\mathbf{q})$, and the S-Matrix, as defined in (2.59) is unitary w.r.t. this form of multiplication. $\langle\hat{N}(t,t_0)\rangle$ now becomes:

$$\langle\hat{N}(t,t_0)\rangle = 2\int\frac{d^3\mathbf{p}}{f(\mathbf{p})}\sum_\lambda\int\frac{d^3\mathbf{q}}{f(\mathbf{q})}\sum_\sigma|\gamma_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t,t_0)|^2 \qquad (2.115)$$

which is seen to be dimensionless, despite the fact that $\gamma_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t,t_0)$ is not (since $\delta(\mathbf{p}-\mathbf{q})$ has dimensions of $(mass)^{-3}$, while the dimensions of $f(\mathbf{p})$ have been left arbitrary). (2.77) however becomes:

$$N^-_{\mathbf{p},\lambda}(t,t_0) = (\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)_{\mathbf{p},\lambda;\mathbf{p},\lambda} = \int\frac{d^3\mathbf{q}}{f(\mathbf{q})}\sum_\sigma|\gamma_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t,t_0)|^2$$

which is clearly not dimensionless. This is simply because $N^-_{\mathbf{p},\lambda}(t,t_0)$ is a *number density*. That is, the expected number of positrons in the evolved vacuum with spin λ and momenta in the range $d^3\mathbf{p}$ around $\mathbf{p}$ is given by $N^-_{\mathbf{p},\lambda}(t,t_0)\frac{d^3\mathbf{p}}{f(\mathbf{p})}$ which is dimensionless, as required. To give meaning to $\det(\boldsymbol{\epsilon}\boldsymbol{\epsilon}^\dagger)$ we can use (2.102) to change the determinant into a trace, and hence into an integral.

2.4 Back-Reaction and the Mean Field Approximation

So far in this dissertation I have only considered the evolution of a fermionic system in a classical electromagnetic background - I have not considered the effect of the system on the background. In this Section I describe how the current of the system of electrons can be coupled back into the equation for the background electromagnetic field, hence formulating the 'back reaction' problem of spinor QED, at the level of the mean field approximation. It has already been demonstrated (Cooper et. al.[52]) that this semiclassical theory provides a useful leading-order approximation to full QED.

For this purpose introduce orthonormal bases $\{u_{i,t_0}(\mathbf{x}); i \in I\}$, $\{v_{i,t_0}(\mathbf{x}); i \in I\}$ for $\mathcal{H}^{\pm}(t_0)$ where I have supposed for convenience that I is countable (we could achieve this by 'putting the system in a box' for instance). Let $|\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} in_{t_0}\rangle$ be some chosen initial state. In the previous Sections (ignoring back-reaction) we would obtain $|\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} in_{t_0}(t)\rangle$ by solving:

$$\begin{aligned} i\nabla_0 u_{i_k,t_0}(\mathbf{x},t) &= \hat{H}_1^{A(\mathbf{x},t)} u_{i_k,t_0}(\mathbf{x},t) \qquad & u_{i_k,t_0}(\mathbf{x},t_0) &= u_{i_k,t_0}(\mathbf{x}) \qquad & k &= 1\dots m \\ i\nabla_0 v_{i,t_0}(\mathbf{x},t) &= \hat{H}_1^{A(\mathbf{x},t)} v_{i,t_0}(\mathbf{x},t) \qquad & v_{i,t_0}(\mathbf{x},t_0) &= v_{i,t_0}(\mathbf{x}) \qquad & i &\in I - \{j_1 \dots j_n\} \\ A^\mu(\mathbf{x},t) &= A^\mu_{ext}(\mathbf{x},t) \end{aligned}$$

and taking the Slater determinant of these solutions. I have used the notation $\hat{H}_1^{A(\mathbf{x},t)}$ rather than $\hat{H}_1(\mathbf{x},t)$ here to emphasise the dependence of $\hat{H}_1$ on the background $A^\mu(\mathbf{x},t)$. This background field $A^\mu_{ext}(\mathbf{x},t)$ will satisfy $\partial^2 A^\mu_{ext} - \partial^\mu \partial \cdot A_{ext} = J^\mu_{ext}$ for some external (classical) current density $J^\mu_{ext}(\mathbf{x},t)$ (which may be zero). We can calculate the current density induced in the state $|\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} in_{t_0}(t)\rangle$ as the expectation value of the Hermitian extension of the operator[5] $\hat{J}^\mu_1 \delta(\mathbf{x}-\mathbf{y})$, where $\hat{J}^\mu = e\gamma^0\gamma^\mu$, and the $\delta-$function is included to cancel the $\int d^3\mathbf{x}$ in the inner product, so that we get a current density rather than a total current. From equation (2.85) this gives:

$$J^\mu_{in,t_0}(\mathbf{x},t) = \sum_{k=1}^{m} J^\mu(u_{i_k,t_0}(\mathbf{x},t)) - \sum_{k=1}^{n} J^\mu(v_{j_k,t_0}(\mathbf{x},t)) + J^\mu_{vac,t_0}(\mathbf{x},t) \tag{2.116}$$

where $J^\mu(\psi) = e\bar{\psi}\gamma^\mu\psi$ is the current of the 'first quantized' state $\psi(\mathbf{x},t)$ and $J^\mu_{vac,t_0}(\mathbf{x},t)$ is the 'vacuum current density':

$$\begin{aligned} J^\mu_{vac,t_0}(\mathbf{x},t) &\equiv J^\mu(|vac_{t_0}(t)\rangle) \\ &= \sum_i \{J^\mu(v_{i,t_0}(\mathbf{x},t)) - J^\mu(v_{i,t}(\mathbf{x}))\} \qquad \text{from (2.79)} \qquad (2.117) \\ &= Trace(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger \boldsymbol{J}^{++}(\mathbf{x},t) - \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger \boldsymbol{J}^{--}(\mathbf{x},t) + \boldsymbol{\epsilon}\boldsymbol{\beta}^\dagger \boldsymbol{J}^{+-}(\mathbf{x},t) + \boldsymbol{\beta}\boldsymbol{\epsilon}^\dagger \boldsymbol{J}^{-+}(\mathbf{x},t)) \qquad (2.118) \end{aligned}$$

where $J^{\mu++}_{ij}(\mathbf{x},t) = e\bar{u}_{i,t}(\mathbf{x})\gamma^\mu u_{j,t}(\mathbf{x})$, $J^{\mu+-}_{ij}(\mathbf{x},t) = e\bar{u}_{i,t}(\mathbf{x})\gamma^\mu v_{j,t}(\mathbf{x})$ etc, and we can ignore the imaginary parts, since they will cancel due to the hermiticity of $\hat{J}^\mu$.

The full back-reaction problem can now be presented simply by including this current in the equation for $A(\mathbf{x},t)$. That is, $|\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} in_{t_0}(t)_{BR}\rangle$ is given by solving:

[5] This procedure is not entirely satisfactory, since this operator will generally have infinite fluctuations when evaluated at a point. This is one of a number of concerns regarding the back-reaction problem, a general discussion of which is left until the final Chapter.

$$
\begin{aligned}
i\nabla_0 u_{i_k,t_0}(\mathbf{x},t) &= \hat{H}_1^{A(\mathbf{x},t)} u_{i_k,t_0}(\mathbf{x},t) \qquad u_{i_k,t_0}(\mathbf{x},t_0) = u_{i_k,t_0}(\mathbf{x}) \qquad k = 1\ldots m \\
i\nabla_0 v_{i,t_0}(\mathbf{x},t) &= \hat{H}_1^{A(\mathbf{x},t)} v_{i,t_0}(\mathbf{x},t) \qquad v_{i,t_0}(\mathbf{x},t_0) = v_{i,t_0}(\mathbf{x}) \qquad i \in I - \{j_1\ldots j_n\} \\
\partial^2 A^\mu - \partial^\mu \partial \cdot A &= J^\mu_{ext}(\mathbf{x},t) + J^\mu_{in,t_0}(\mathbf{x},t)
\end{aligned}
\tag{2.119}
$$

with suitable initial conditions on $A^\mu(\mathbf{x},t)$. This is a well-defined set of coupled differential equations. It has the same basic form as that presented in Kluger et. al.[18]. However, the calculation of $J^\mu_{vac,t_0}(\mathbf{x},t))$ used here is different to that used in [18]. In [18] the vacuum current density is:

$$
J^\mu_{vac,in}(\mathbf{x},t) = \frac{e}{2}\langle 0|[\hat{\bar{\psi}}(\mathbf{x},t), \gamma^\mu \hat{\psi}(\mathbf{x},t)]|0\rangle \tag{2.120}
$$

$$
= \tfrac{1}{2}\sum_i \{J(v_{i,in}(\mathbf{x},t)) - J(u_{i,in}(\mathbf{x},t))\} \tag{2.121}
$$

It is straightforward to show, atleast in the case of spatially uniform electric fields (see Chapter 7) that $J(u_{i,in}(\mathbf{x},t)) = -J(v_{i,in}(\mathbf{x},t))$, so that (2.121) becomes equivalent to the non-vacuum subtracted current, given by the first term in (2.117). Then, having calculated this, they obtain the vacuum subtracted version simply by 'discarding terms in the integrand which are odd in $\mathbf{p}$'. This achieves the desired result, since the current in the physical vacuum (not the 'evolved vacuum') at any given time is an odd function of momentum. However, since this odd function is divergent (see Chapter 7 for the details of this), then it is not entirely satisfactory to simply 'discard it because it is odd'.

Finally I should point out that in general the integral over modes which defines the vacuum current will, even after vacuum subtracting or 'discarding odd terms', diverge logarithmically. This logarithmic divergence (which will be proportional to $\partial_\mu F^{\mu\nu}$) must be isolated, and absorbed into a renormalisation of the coupling constant, before the last equation in (2.119) becomes physically reasonable. This is discussed in more detail in [18] (among other places) and will not be considered further here.

Chapter 3

Gravitational Backgrounds

In this chapter I extend the formalism of Chapter 2 to include gravitational backgrounds. I assume throughout this chapter not only that the gravitational background admits a spin structure, but also that it is globally hyperbolic, so that it admits a family of spacelike Cauchy surfaces. For convenience I also assume that there is no electromagnetic background present here, although it is straightforward to generalise this to the situation involving both electromagnetic and gravitational backgrounds.

The convention I will be using is the "Landau, Lifshitz timelike convention", referred to as $-++$ in Misner, Thorne and Wheeler's[53] table of sign conventions. Specifically this means that the metric has signature $(+---)$, while the curvature, connection etc are given by:

$$R^{\alpha}{}_{\mu\nu\sigma} = \partial_\nu \Gamma^{\alpha}_{\mu\sigma} - \partial_\sigma \Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\alpha}_{\lambda\sigma}\Gamma^{\lambda}_{\mu\nu} \tag{3.1}$$

$$\Gamma^{\mu}_{\nu\sigma} = \tfrac{1}{2} g^{\mu\alpha}(\partial_\sigma g_{\alpha\nu} + \partial_\nu g_{\alpha\sigma} - \partial_\alpha g_{\nu\sigma}) \tag{3.2}$$

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \qquad\qquad R = R^{\mu}{}_{\mu} \tag{3.3}$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.4}$$

3.1 Dirac Equation in a Gravitational Background

The Lagrangian density for the Dirac equation in a gravitational background[54] is[1] :

$$\mathcal{L} = \Re[e(x)\bar{\psi}(x)(i\gamma^\mu(x)\nabla_\mu - m)\psi(x)] \tag{3.5}$$

where $\gamma^\mu(x) = e^\mu_a(x)\bar{\gamma}^a$ satisfies:

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x) \tag{3.6}$$

[1] I should point out that Kaku[54] does not take the real part as I have here, and so mistakenly presents a complex valued Lagrangian. A more accurate description of this is given by Hehl et. al.[55] (and is also described in [22]). There the Lagrangian agrees with that given here, although they write it (up to a change in signature) as $\mathcal{L} = \frac{ie(x)}{2}[\bar{\psi}(x)\gamma^\mu(x)\nabla_\mu\psi(x) - (\nabla_\mu\bar{\psi}(x))\gamma^\mu(x)\psi(x)] - e(x)m\bar{\psi}(x)\psi(x)$

$\bar{\gamma}^a$ are a representation of the normal, flat space Dirac equations, $\bar{\psi} \equiv \psi^\dagger \bar{\gamma}^0$, $e^\mu_a(x)$ is the *Vierbein* (defined by equation (3.6)), and

$$e(x) \equiv \det(e^a_\mu(x)) = \sqrt{-\det(g_{\mu\nu}(x))} \tag{3.7}$$

The vierbein provides a map between the chosen coordinate system, and the flat tangent space at the point x. The tangent space indices are denoted by the Latin letters $a, b, c, \ldots$, and can be raised and lowered using $\eta_{ab} = diag(+ - --) = \eta^{ab}$, while the general 'coordinate indices' are denoted by Greek letters, and are raised and lowered using the metric tensor. The covariant derivative ∇_μ acts on spinors as:

$$\nabla_\mu \psi(x) = (\partial_\mu - \frac{i}{4}\omega^{ab}_\mu \sigma_{ab})\psi(x) \tag{3.8}$$

where $\sigma_{ab} = \frac{i}{2}[\bar{\gamma}_a, \bar{\gamma}_b]$ and the connection field $\omega^{ab}_\mu(x)$ is given (in the absence of spin-torsion interactions) by:

$$0 = \nabla_\mu e^a_\nu(x) = \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + \omega^{ab}_\mu e^b_\nu \tag{3.9}$$

That is:

$$\omega^{ab}_\mu = -e^{\nu b}\partial_\mu e^a_\nu + \Gamma^\lambda_{\mu\nu} e^{\nu b} e^a_\lambda \tag{3.10}$$

The Lagrangian of (3.5) leads to the governing equation:

$$(i\gamma^\mu(x)\nabla_\mu - m)\psi(x) = 0 \tag{3.11}$$

The inner product $\langle\psi|\phi\rangle_\Sigma$ on the spacelike Cauchy surface Σ is:

$$\langle\psi(x)|\phi(x)\rangle_\Sigma = \int e(x)\bar{\psi}(x)\gamma^\mu(x)\phi(x)d\Sigma_\mu \tag{3.12}$$

and is independent of Σ by virtue of (3.11).

The energy-momentum tensor $T_{\mu\nu}$ can be written as:

$$T_{\mu\nu}(\psi) = \Re[i\bar{\psi}\gamma_{(\mu}\nabla_{\nu)}\psi] - \frac{g_{\mu\nu}}{e(x)}\mathcal{L} \tag{3.13}$$

It is customary to omit the $\mathcal{L}$ term from this expression, since it is zero whenever ψ satisfies (3.11). However, I have included it here so that $H_{\tau_0}(\psi)$ in (3.17) depends only on $\psi(x|_{\Sigma_{\tau_0}})$, and so that it leads directly to the result (3.18).

3.2 An Observer and Their Particle Interpretation

We desire a number operator $\hat{N}(\tau) : \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$ whose expectation value represents the (expected) total number of particles present 'at time τ'. This will require:

- Introducing a hypersurface Σ_τ generalising our concept of 'at time τ',

- Categorising states as having positive/negative energy 'at time τ'. Hence, we will need a Hamiltonian 'at time τ'.

Since we expect the particle definition to depend on the motion of an observer (Unruh Effect), then (as mentioned in the last Chapter) we expect the definition to be non-local.

So, consider an observer travelling on path $\gamma : x^\mu = x^\mu_{(\gamma)}(\tau)$ with proper time τ. Define:

$$\begin{aligned}
\tau^+(x) &\equiv \text{(earliest possible) proper time at which a photon leaving} \\
&\qquad\qquad \text{point } x \text{ could intercept } \gamma. \\
\tau^-(x) &\equiv \text{(latest possible) proper time at which a photon could} \\
&\qquad\qquad \text{leave } \gamma\text{, and still reach point } x. \\
\tau(x) &\equiv \tfrac{1}{2}(\tau^+(x) + \tau^-(x)) \qquad = \text{ `radar time'.} \\
\rho(x) &\equiv \tfrac{1}{2}(\tau^+(x) - \tau^-(x)) \qquad = \text{ `radar distance'.} \\
\Sigma_{\tau_0} &\equiv \{x : \tau(x) = \tau_0\} = \text{ observers `hypersurface of simultaneity at time } \tau_0\text{'.}
\end{aligned}$$

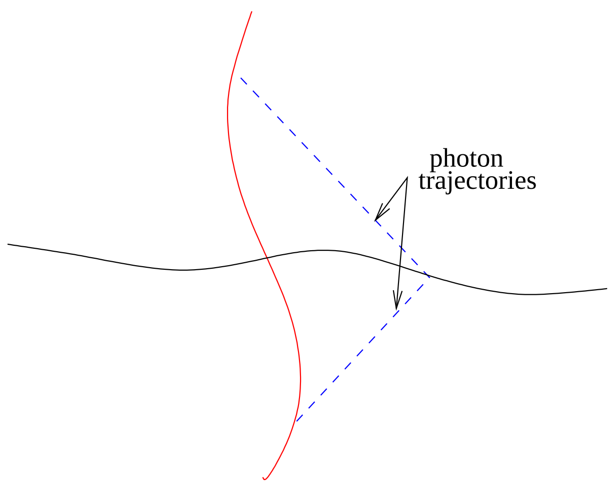

Figure 3.1: Schematic of the definition of 'radar time' $\tau(x)$.

This is a generalisation of the definitions made popular by Bondi in his work on special relativity and k-calculus[56, 57, 58]. However (to the best of my knowledge) the idea of defining a hypersurface of simultaneity in terms of 'radar time' has never before been applied to non-inertial observers and non-flat spacetimes. This might be due in part to Bondi's own belief that "how a clock reacts to acceleration [or equivalently, to gravitational fields] is utterly dependent on how the clock is constructed" (see [56] pg 49). This belief prompted him to claim that radar time and radar distance could only be defined by inertial observers in flat space. However, not only has this belief been refuted by experiment (see [59], or [60] pgs 144,145, and the references therein), but the assumption that 'suitable clocks' will behave identically under acceleration (or under gravitational fields) is a basic premise of general relativity, without which proper time would cease to have physical meaning. There seems therefore to be no reason for not using 'radar time' for accelerating observers in curved spacetimes. (Another point in favour of using 'radar time' is that it is the only possible construction which both agrees with proper time on the observers path, and is invariant under 'time-reversal' - that is, under reversing the sign of the observers proper time.) Some examples of this construction are given in the following subsections.

It is intuitively obvious from these definitions that Σ_{τ_1} lies to the future of Σ_{τ_0} (for $\tau_1 > \tau_0$) except at the observers particle horizon (when one exists) at which point the various Σ_τ will converge. Also notice that the domain of $\tau(x)$ is not necessarily all of spacetime - it is only that part of spacetime with which the observer can communicate (send and receive signals).

Also, define

$$k_\mu(x) \equiv \frac{\frac{\partial \tau}{\partial x^\mu}}{g^{\sigma\nu}\frac{\partial \tau}{\partial x^\sigma}\frac{\partial \tau}{\partial x^\nu}}$$

This represents the perpendicular distance between neighbouring hypersurfaces of simultaneity. That is: it is normal to these hypersurfaces, and satisfies $k^\mu(x)\frac{\partial \tau}{\partial x^\mu} = 1$.

Now, use the identity:

$$i\not{k}\gamma^\mu\nabla_\mu = ik^\mu\nabla_\mu + \sigma^{ab}e_a^\mu e_b^\nu k_\mu\nabla_\nu \tag{3.14}$$

to write (3.11) as:

$$ik^\mu\nabla_\mu\psi = -\sigma^{ab}e_a^\mu e_b^\nu k_\mu\nabla_\nu\psi + m\not{k}\psi \tag{3.15}$$

from which we can define the 'Hamiltonian on Σ' $\hat{H}_{nh}(\tau)$ by:

$$\hat{H}_{nh}(\tau) : \psi \to -\sigma^{ab}e_a^\mu e_b^\nu k_\mu\nabla_\nu\psi + m\not{k}\psi \tag{3.16}$$

An interesting point that should be noted here is that $\hat{H}_{nh}(\tau)$ is *not Hermitian!* (hence the subscript nh here). At first sight this seems to disagree with the conservation of (3.12). However, these are consistent since the inner product (3.12) contains explicit time dependence, so that the standard proof that 'unitary evolution ⇔ Hermitian Hamiltonian' no longer applies[2]. This is a significant and interesting point, which has received little attention in the literature.

To investigate the relationship between $\hat{H}_{nh}(\tau_0)$ and the energy momentum tensor, define:

$$H_{\tau_0}(\psi) \equiv \int_{\Sigma_{\tau_0}} e(x)T_{\mu\nu}(\psi(x))k^\mu d\Sigma^\nu \tag{3.17}$$

Substituting (3.13) we see that:

$$H_{\tau_0}(\psi) = \Re[\langle\psi|\hat{H}_{nh}(\tau_0)|\psi\rangle_\Sigma] = \langle\psi|\hat{H}_1(\tau_0)|\psi\rangle_\Sigma \tag{3.18}$$

$$\text{where } \hat{H}_1(\tau_0) \equiv \tfrac{1}{2}\{\hat{H}_{nh}(\tau_0) + \hat{H}_{nh}^\dagger(\tau_0)\} \tag{3.19}$$

[2]Another reason why the standard proof no longer applies is that the Hermitian form is no longer simply $i\frac{d}{dt}|\psi(t)\rangle = \hat{H}_1(t)|\psi(t)\rangle$.

So, define the projection operators $\hat{P}^{\pm}(\tau_0)$ and the spaces $\mathcal{H}^{\pm}(\tau_0)$ by the requirement that:

$$H_{\tau_0}(\hat{P}^{+}(\tau_0)\psi) \geq H_{\tau_0}(\psi) \geq H_{\tau_0}(\hat{P}^{-}(\tau_0)\psi) \tag{3.20}$$

for all ψ. This definition clearly depends only on the choice of observer, and is equivalent to defining:

$\mathcal{H}^{+}(\tau_0)$ is the span of the positive spectrum of $\hat{H}_1(\tau_0)$

$\mathcal{H}^{-}(\tau_0)$ is the span of the negative spectrum of $\hat{H}_1(\tau_0)$

Note that we cannot refer in general to the spectrum of $\hat{H}_{nh}(\tau_0)$ since the eigenvectors corresponding to different eigenvalues need not be orthogonal. However, if $\hat{H}_{nh}(\tau_0)$ and $\hat{H}^{\dagger}_{nh}(\tau_0)$ commute then we can show that eigenvectors of different eigenvalues are orthogonal, and in that case we find that $\mathcal{H}^{+}(\tau_0)$ is the span of that part of the spectrum of $\hat{H}_{nh}(\tau_0)$ for which the real part of the eigenvalue is positive, while $\mathcal{H}^{-}(\tau_0)$ is the span of that part of the spectrum of $\hat{H}_{nh}(\tau_0)$ for which the real part of the eigenvalue is negative.

Also, as I will demonstrate in the next Chapter, in a conventional Approach to Background Q.F.T. this definition would correspond to Hamiltonian diagonalisation of the second quantized Hamiltonian obtained by substituting the field operator $\hat{\psi}(x)$ into $H_{\tau_0}(\psi)$ above. This will be discussed in more detail in Section 4.4.

I now present some examples of observers and their hypersurfaces of simultaneity.

3.2.1 The Travelling Twin.

Consider the hypersurfaces of simultaneity of the 'travelling twin' in the well-known 'twin paradox'. I have shown these in Figure 3.2, for the case when the travelling twin turns around instantaneously (left) and gradually (right). The dotted vertical line represents the twin who 'stays at home'.

In the left hand example we have:

$$x^{\mu}_{(\gamma)}(\tau) = \begin{cases} \frac{(\tau,0,0,-v\tau)}{\sqrt{1-v^2}} & \tau < 0 \\ \frac{(\tau,0,0,v\tau)}{\sqrt{1-v^2}} & \tau > 0 \end{cases}$$

A simple calculation reveals that Σ_{τ_0} is given by:

$$t_{\tau_0}(z) = \begin{cases} \tau_0\sqrt{\frac{1+v}{1-v}} & \text{region I} \\ \tau_0\sqrt{\frac{1-v}{1+v}} & \text{region II} \\ \tau_0\sqrt{1-v^2} - vz & \text{region F} \\ \tau_0\sqrt{1-v^2} + vz & \text{region P} \end{cases}$$

as shown. It might seem reasonable to think that the twin paradox has long been understood, and that no confusion remains on this topic. There is certainly no disagreement

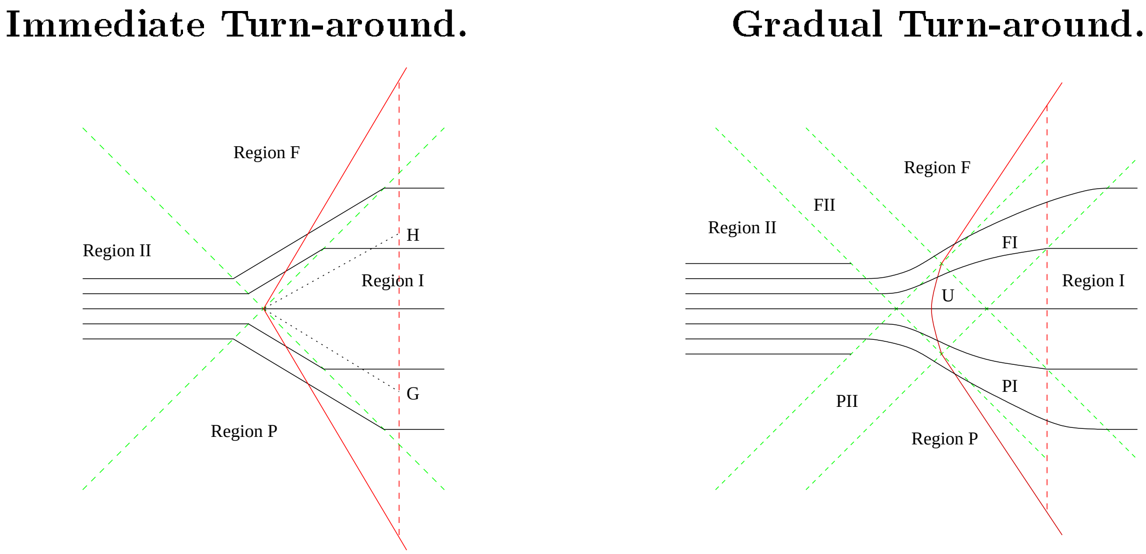

Figure 3.2: Hypersurfaces of simultaneity of the 'travelling twin' in the well-known 'twin paradox' (schematic).

about the relative aging of the two twins, so there is no longer any 'paradox'. There is also no confusion regarding 'when events are seen' by the two twins, or regarding the description, by the stay-at-home twin, of 'when events happened'. However, the description of 'when events happened' according to the travelling twin, seems to have never quite been settled. Papers discussing this issue have been published as recently as 1995[61]. Even Unruh[62] has proposed his own definition for this purpose, based in his case on parallax distance. (Interestingly, not only does Unruh's definition have the unfortunate quality that the travelling twin describes his brother has having travelled backwards in time during part of his journey, but also, rather surprisingly, when applied to a uniformly accelerating observer, it fails to give the Rindler time coordinate which is of paramount importance in derivations of the 'Unruh Effect'!) As for the textbook treatments of this issue, region I is consistently misrepresented (or ignored) in the literature. Both Bohm[57] and D'Inverno[58] for instance devote a whole chapter to k-calculus, and derive the hypersurfaces of simultaneity of inertial observers using a definition of time just like that used above. Then, both go on to claim that in region I the hypersurfaces of simultaneity continue to have the same gradient as in regions F and P! Bohm even goes on to claim that points G and H appear to the travelling twin as simultaneous, and that points in between will be described as "jumping backwards in time by an amount GH at the instant of turnaround"! Other authors either make no attempt to describe how the travelling twin would assign a time to any given event, or claim either that "a complete explanation of the problem can only be given within the framework of general relativity" (see Pauli[63] for instance) or that the solution of the 'G and H appear simultaneous' problem lies in recognising that the travelling twin must in fact take some time (at a finite acceleration) to turn around. This is precisely the case considered in the right hand example above. Here we have:

$$x^{\mu}_{(\gamma)}(\tau) = \begin{cases} (az_c(\tau-\tau_c)+t_c, 0, 0, at_c(\tau-\tau_c)+z_c) & \tau \geq \tau_c \\ (\frac{\sinh(a\tau)}{a}, 0, 0, \frac{\cosh(a\tau)}{a}) & |\tau| \leq \tau_c \\ (az_c(\tau+\tau_c)-t_c, 0, 0, -at_c(\tau+\tau_c)+z_c) & \tau \leq -\tau_c \end{cases}$$

where $z_c = \frac{\cosh(a\tau_c)}{a}$, $t_c = \frac{\sinh(a\tau_c)}{a}$ and (for comparison with the previous case) $v = \tanh(a\tau_c)$, so that $\sqrt{\frac{1+v}{1-v}} = e^{a\tau_c}$ and $(1-v^2)^{-\frac{1}{2}} = \cosh a\tau_c$. This gives hypersurfaces of simultaneity of the form:

$$t_{\tau_0}(z) = \begin{cases} \tau_0 e^{\pm a\tau_c} & \text{regions I and II respectively} \\ \frac{\tau_0 \pm \tau_c}{\cosh(a\tau_c)} \mp z\tanh(a\tau_c) & \text{regions P and F respectively} \\ z\tanh(a\tau_0) & \text{region U} \end{cases}$$

while in the other 4 regions they are given implicitly by:

$$\log(a|z+t_{\tau_0}(z)|+a(t_{\tau_0}(z)-z)e^{\pm a\tau_c} = 2a\tau_0 \mp a\tau_c - 1 \quad \text{regions FII and PI respectively}$$
$$\log(a|z-t_{\tau_0}(z)|-a(t_{\tau_0}(z)+z)e^{\pm a\tau_c} = -2a\tau_0 \mp a\tau_c - 1 \quad \text{regions FI and PII respectively}$$

These are shown in the diagram above. We see that, as we would hope, there is very little difference (for small τ_c) between the 'immediate turn-around' and 'gradual turn-around' cases. It is interesting now to consider the limit as $\tau_c \to \infty$, corresponding to a uniformly accelerating observer.

3.2.2 Uniformly Accelerating Observer.

The case of a uniformly accelerating observer is shown here.

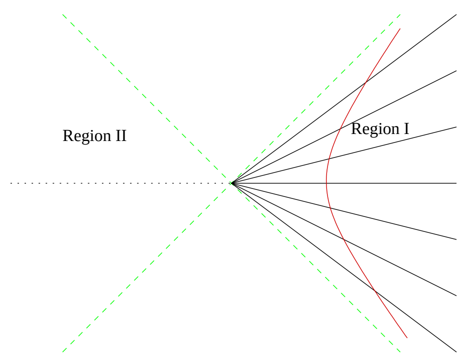

Figure 3.3: Hypersurfaces of simultaneity of a uniformly accelerating observer.

Now (I use 1 spatial coordinate for convenience) we have:

$$x^{\mu}_{(\gamma)}(\tau) = (\frac{\sinh(a\tau)}{a}, \frac{\cosh(a\tau)}{a})$$
$$\text{so } \tau(x) = \frac{1}{2a}\log(\frac{z+t}{z-t}) \quad \text{and} \quad \rho(x) = \frac{1}{2a}\log(a^2(z^2-t^2)) \tag{3.21}$$
$$\text{so that } t_{\tau_0}(z) = z\tanh(a\tau_0) \tag{3.22}$$

as expected. We see that ρ and τ are simply Rindler coordinates, and cover only region I. The metric in these coordinates is $ds^2 = e^{2a\rho}(d\tau^2 - d\rho^2)$. It is also common to define $u = \frac{e^{a\rho}}{a}$, in terms of which the metric takes the familiar form $ds^2 = u^2 d\tau^2 - du^2$. Also notice that the vector field k^μ is given by:

$$k = z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z} \qquad \text{or } = \frac{\partial}{\partial \tau} \text{ in Rindler coordinates.}$$

This is the Killing vector field that is used to define positive/negative frequency modes in conventional derivations of the Unruh effect. This is just as we would have hoped. It shows that my general particle definition does reduce to accepted definitions in conventional situations, so that it can be viewed as a generalisation of (rather than an alternative to) conventional definitions. If we considered more than one spatial coordinate, then τ, and hence k_μ would take the same form (and would still be Killing). Only the 'radar distance' ρ would become more complicated, and this plays no part in the particle definition. Finally, the dotted line in Region II requires some explanation. This does not appear if we simply consider a uniformly accelerating observer. However, if we assume that a uniformly accelerating observer should be represented as the infinite time limit of the 'Gradual Turn-around' case above, then we find that this dotted line forms part of the hypersurface of simultaneity for all time, and that $k_\mu = 0$ on this hypersurface. This leads to the requirement that the observer's state space only be constructed from those modes which are zero in region II. This fact is essential to many conventional derivations of the Unruh Effect (see [64, 21, 22, 65] for instance), but is usually included as an assumption, with little or no justification.

3.2.3 DeSitter Space.

Consider the metric (I again consider only one spatial dimension):

$$ds^2 = dt^2 - a(t)^2 dr^2 \qquad\qquad a(t) \equiv e^{\lambda t}$$

and consider an observer stationary at the origin. In this space, photon trajectories will be of the form $r(t) = \frac{\pm e^{-\lambda t}}{\lambda} + C$. The trajectory $r_\infty(t) = \frac{e^{-\lambda t}}{\lambda}$ is a particle horizon, separating events which can send signals to our observer, from events that can't. Notice also that it takes much less 'cosmic time' (the t above) to send a photon than it does to receive it back. A consequence of this is that the 'radar time' of an event (t, r) is generally larger than the cosmic time t. Indeed, we find:

$$\tau(x) = \frac{-1}{2\lambda}\log(e^{-2\lambda t} - \lambda^2 r^2) \qquad \rho(x) = \frac{-1}{2\lambda}\log(\frac{e^{-\lambda t} + \lambda r}{e^{-\lambda t} - \lambda r}) \tag{3.23}$$

as shown below.

The metric in these coordinates is $ds^2 = \cosh^{-2}(\lambda\rho)(d\tau^2 - d\rho^2)$. Meanwhile, the vector field k^μ is given by:

$$k = \frac{\partial}{\partial t} - \lambda r\frac{\partial}{\partial r} \qquad \text{or } = \frac{\partial}{\partial \tau} \text{ in } (\tau, \rho) \text{ coordinates}$$

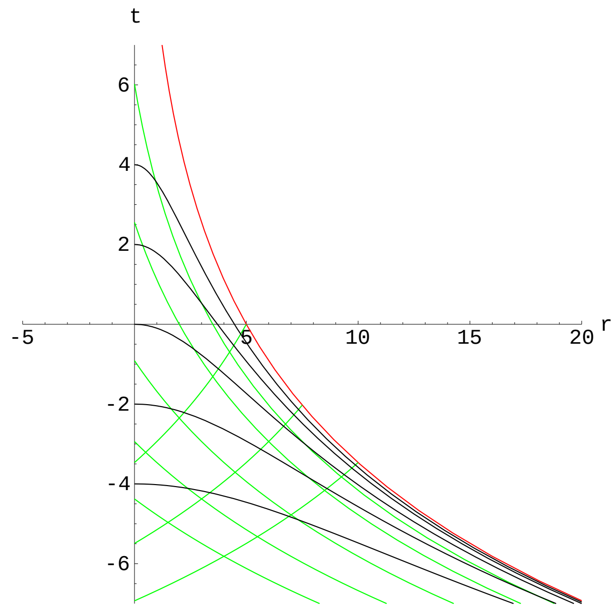

Figure 3.4: DeSitter space in (t, r) coordinates. The particle horizon $r_\infty(t) = \frac{e^{-\lambda t}}{\lambda}$ is shown in mid grey. The light grey lines represent ingoing and outgoing photon trajectories, while the black lines are the observers hypersurfaces of simultaneity for various τ_0.

Again this is a timelike Killing vector field on the domain of $\tau(x)$, becoming spacelike outside this region. To relate this to the coordinates used by Gibbons and Hawking[26] in 1977, put $u = \frac{\tanh(\lambda\rho)}{\lambda}$. In these coordinates we have:

$$ds^2 = (1 - \lambda^2 u^2) d\tau^2 - \frac{du^2}{1 - \lambda^2 u^2}$$

as desired. This allows us to see that in the case of a comoving observer in DeSitter space, my general particle definition reduces to that used by Gibbons and Hawking[26], so that we would expect it to lead (after a suitable choice of initial conditions) to the prediction of thermal particle creation at the Gibbons-Hawking temperature.

3.3 Bogoliubov Coefficients, S-Matrix Elements Etc.

As in the previous Chapter, let $\{u_{i,\tau_{in}}(x|_{\Sigma_{\tau_{in}}})\}$ be an orthonormal basis for $\mathcal{H}^+(\tau_{in})$, and let $\{u_{i,\tau_{in}}(x)\}$ be the corresponding solutions of (3.11). I have used a discrete index i here for convenience, although in practice this will be a continuous index. Similarly, let $\{v_{i,\tau_{in}}(x|_{\Sigma_{\tau_{in}}})\}$ be an orthonormal basis for $\mathcal{H}^-(\tau_{in})$, and let $\{v_{i,\tau_{in}}(x)\}$ be the corresponding solutions of (3.11). Let $\{u_{i,\tau_{out}}(x)\}$, $\{v_{i,\tau_{out}}(x)\}$ be similarly formed. Then we may define the Bogoliubov Coefficients:

$$\begin{aligned} \alpha_{ij}(\tau_{out}, \tau_{in}) &= \langle u_{i,\tau_{out}}(x) | u_{j,\tau_{in}}(x) \rangle_\Sigma \quad & \gamma_{ij}(\tau_{out}, \tau_{in}) &= \langle v_{i,\tau_{out}}(x) | u_{j,\tau_{in}}(x) \rangle_\Sigma \\ \beta_{ij}(\tau_{out}, \tau_{in}) &= \langle u_{i,\tau_{out}}(x) | v_{j,\tau_{in}}(x) \rangle_\Sigma \quad & \epsilon_{ij}(\tau_{out}, \tau_{in}) &= \langle v_{i,\tau_{out}}(x) | v_{j,\tau_{in}}(x) \rangle_\Sigma \end{aligned} \tag{3.24}$$

which are independent of Σ by virtue of (3.11). We can use these, along with the formalism of Chapter 2, to calculate expectation values and S-Matrix elements just as in

Section 2.3. For instance, the total energy 'at time τ' of a state that was vacuum 'at time τ_{in}' may be written (using equations (2.79) and (2.81)) as:

$$\langle vac_{\tau_{in}}(\tau)|\hat{H}_{phys}(\tau)|vac_{\tau_{in}}(\tau)\rangle = \sum_i H_\tau(v_{i,\tau_{in}}(x)) - \sum_i H_\tau(v_{i,\tau}(x)) \quad (3.25)$$

$$= Trace(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger \boldsymbol{H}^{++} - \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger \boldsymbol{H}^{--}) \quad (3.26)$$

where $\boldsymbol{H}^{++}_{jk}(\tau) = \langle u_{j,\tau}(x)|\hat{H}_1(\tau)|u_{k,\tau}(x)\rangle$ and $\boldsymbol{H}^{--}_{jk}(\tau) = \langle v_{j,\tau}(x)|\hat{H}_1(\tau)|v_{k,\tau}(x)\rangle$. We could choose to require that $u_{i,\tau}(x|_\tau)$ be an eigenstate of $\hat{H}_1(\tau)$ with eigenvalue $E^+_i(\tau) > 0$, and that $v_{i,\tau}(x|_\tau)$ be an eigenstate of $\hat{H}_1(\tau)$ with eigenvalue $E^-_i(\tau) < 0$. Then (3.26) can be written as:

$$\langle vac_{\tau_{in}}(\tau)|\hat{H}_{phys}(\tau)|vac_{\tau_{in}}(\tau)\rangle = \sum_{i,j}\{|\beta_{ij}(\tau,\tau_{in})|^2 E^+_i(\tau) - |\gamma_{ij}(\tau,\tau_{in})|^2 E^-_i(\tau)\} \quad (3.27)$$

Unfortunately, (3.27) will diverge in general. That is, vacuum subtraction on its own is not sufficient to render $\langle vac_{\tau_{in}}(\tau)|\hat{H}_{phys}(\tau)|vac_{\tau_{in}}(\tau)\rangle$ finite, and further renormalisation is required. Some techniques appropriate for the renormalisation of quantities 'after vacuum subtraction' are presented in Grib et. al.[44] for instance. Alternatively, we may find it convenient to simply calculate the 'pre-normal ordered' quantity $\langle vac_{\tau_{in}}(\tau)|\hat{H}_{1,H}(\tau)|vac_{\tau_{in}}(\tau)\rangle = \sum_i H_\tau(v_{i,\tau_{in}}(x))$, which is independent of the particle interpretation on τ, and then use the standard covariant renormalisation techniques (See [22] or [23] for instance) to render this finite. I should emphasise however that the results of doing this would still depend on the choice of particle interpretation 'at time τ_{in}', since it is only through this interpretation that we can choose which of the possible initial conditions on $\Sigma_{\tau_{in}}$ can be identified with the physical vacuum there.

Some other points worth noting are:

1. We see from equation (3.15) that, in direct analogy with the comments at the end of Section 2.2, particle creation can be caused either by the 'time'-dependence of $\hat{H}_1(\tau)$, or by the presence of a non-zero 'time-component' of the connection field, $k^\mu\omega^{ab}_\mu(x) \neq 0$.

2. I should point out that in proving that (3.12) is independent of the choice of hypersurface Σ I rely on using the divergence theorem to convert the difference between the two surface integrals into a volume integral. This in turn relies on the assumption that any other surface integrals bounding this volume, such as at spatial infinity, will vanish. However, if there are any singularities present then these will introduce extra surface terms, which may not vanish in general. It is commented in Lasenby et. al.[47] that this causes the inner product to decay with time, resulting in non-unitarity of the evolution operator. In the case of the Unruh effect, the choice of $k^\mu(x)$ leads to boundary conditions on allowed states at the particle horizon, which appear to 'shield' the states from any singularities. More work has to be done on this however, to investigate whether or not this is a general propoerty of such systems. It is an interesting area for future research, and is of importance to the 'information loss problem' in black hole physics.

Chapter 4

Relationship to Conventional Methods

In order to compare the formalism of the previous Chapter with more conventional treatments[24, 22, 23] this Chapter is devoted to describing the conventional approach to background quantum field theory, and to describing how my particle definition and vacuum subtraction scheme can be incorporated into this approach. My presentation will be analogous to that used (for real scalar fields) in Section 6 of DeWitt[24]. Although the scalar field results presented in [24] have been studied and applied extensively since their publication, the work presented here represents the first thorough treatment of fermionic systems along these lines. Some of these results have been obtained in special cases by previous authors[16, 18, 66, 48, 31] and they were obtained by very different methods in Schwinger[50].

In Section 4.1 I introduce 'in' and 'out' basis solutions, the field operator, Bogoliubov coefficients etc, and describe how the particle interpretation introduced in the previous Chapter allows these to be generalised to arbitrary backgrounds, arbitrary observers, and arbitrary 'in' and 'out' times. In Section 4.2 I show how the arbitrary S-Matrix elements of the theory can be calculated without requiring a concrete representation of the states involved, and compare the difficulty of these derivations with the equivalent derivations from the previous Chapter. In Section 4.3 I look at the calculation of vacuum expectation values, and show how the vacuum subtraction scheme differs from normal ordering, while in Section 4.4 I describe the relationship between my particle definition and the method of Hamiltonian diagonalisation. In Section 4.5 I compare the various Greens functions of the theory, defined in terms of vacuum expectation values of field operators, with the evolution operator $\hat{U}_1(t, t_0)$ on $\mathcal{H}$, and with the Bogoliubov matrices, and describe the link between equations (2.66) and (2.67) for the vacuum - vacuum transition amplitude.

4.1 Preliminaries

Let $\{u_{i,in}(x), v_{i,in}(x); i \in I\}$ be an orthonormal basis of solutions of the 'first quantized' Dirac equation, where I is some index set, which is assumed for convenience to be countable. Let these solutions be such that the $u_{i,in}(x)$ can be interpreted as representing 'in' particles, and the $v_{i,in}(x)$ as representing 'in' antiparticles. This can be achieved here via whatever particle interpretation is considered appropriate (I will introduce my particle definition soon). Let $\{u_{i,out}(\mathbf{x},t),\, v_{i,out}(\mathbf{x},t); i \in I\}$ be an orthonormal basis of solutions representing 'out' particles and 'out' antiparticles respectively. We can expand the 'in' basis in terms of the 'out' basis as:

$$u_{i,in}(x) = \sum_j \{\alpha_{ji} u_{j,out}(x) + \gamma_{ji} v_{j,out}(x)\} \tag{4.1}$$

$$v_{i,in}(x) = \sum_j \{\beta_{ji} u_{j,out}(x) + \epsilon_{ji} v_{j,out}(x)\} \tag{4.2}$$

just as in (2.54) and (2.55). This defines the Bogoliubov coefficients, which are given by expressions analogous to (2.56) and (2.57), and allows us to deduce the Bogoliubov conditions, and to invert (4.1) and (4.2) just as in Section 2.4.1.

The field operator $\hat{\psi}(x)$ can be written in terms of these as:

$$\hat{\psi}(x) = \sum_i \{u_{i,in}(x) a_{i,in,h} + v_{i,in}(x) b^{\dagger}_{i,in,h}\} \tag{4.3}$$

where $a_{i,in,h}$ is the annihilation operator of the *Heisenberg picture state* (hence the subscript h) representing the solution $u_{i,in}(x)$, and similarly for $b_{i,in,h}$. By requiring that $\hat{\psi}(x)$ satisfy the (equal time[1]) Canonical Anticommutation Relations, we can deduce that $a_{i,in,h}$, $b_{i,in,h}$ must satisfy:

$$\{a_{i,in,h}, a^{\dagger}_{j,in,h}\} = \delta_{ij} = \{b_{i,in,h}, b^{\dagger}_{j,in,h}\} \tag{4.4}$$

with all other anticommutators zero. Similarly, we can write $\hat{\psi}(\mathbf{x},t)$ as:

$$\hat{\psi}(x) = \sum_i \{u_{i,out}(x) a_{i,out,h} + v_{i,out}(x) b^{\dagger}_{i,out,h}\} \tag{4.5}$$

By requiring that these two expressions for $\hat{\psi}(x)$ be equivalent, we can deduce that

$$a_{i,out,h} = \sum_j \{\alpha_{ij} a_{j,in,h} + \beta_{ij} b^{\dagger}_{j,in,h}\} \tag{4.6}$$

$$b_{i,out,h} = \sum_j \{\gamma^*_{ij} a^{\dagger}_{j,in,h} + \epsilon^*_{ij} b_{j,in,h}\}$$

and that:

$$a_{i,in,h} = \sum_j \{\alpha^*_{ji} a_{j,out,h} + \gamma^*_{ji} b^{\dagger}_{j,out,h}\} \tag{4.7}$$

$$b_{i,in,h} = \sum_j \{\beta_{ji} a^{\dagger}_{j,out,h} + \epsilon_{ji} b_{j,out,h}\}$$

[1]By 'equal time' here we can read 'on any spacelike hypersurface'.

as in Section 2.4.1. The (Heisenberg picture) 'in' vacuum $|vac_{in}, h\rangle$ is defined by the requirement that

$$a_{i,in,h}|vac_{in}, h\rangle = 0 = b_{i,in,h}|vac_{in}, h\rangle \qquad \langle vac_{in}, h|vac_{in}, h\rangle = 1 \tag{4.8}$$

and the 'in Fock space' is defined by acting on $|vac_{in}, h\rangle$ with all possible combinations of creation operators, and taking the span of this. The 'out' vacuum and the 'out Fock space' are defined similarly. If $\langle vac_{out}, h|vac_{in}, h\rangle \neq 0$ then the 'in' and 'out' Fock spaces may be seen to be simply different parametrisations of the same space, whereas if $\langle vac_{out}, h|vac_{in}, h\rangle = 0$ then the two spaces are mutually orthogonal spaces!

Recall that the state spaces are Heisenberg picture spaces, and so do not evolve with time. This means that the Heisenberg picture state $|vac_{in}, h\rangle$ represents not just $|vac_{-\infty}\rangle$ but rather represents $|vac_{-\infty}(\tau)\rangle$ for all τ, so that for instance $\langle vac_{out}|vac_{in}\rangle \leftrightarrow \langle vac_{\infty}(\tau)|vac_{-\infty}(\tau)\rangle$ which is independent of τ by virtue of conservation of the inner product (and hence $= \langle vac_{\infty}|vac_{-\infty}(\infty)\rangle$ in agreement with the Schrödinger picture version). At first sight the Heisenberg picture might seem to be inconsistent with having different 'in' and 'out' parametrisations of state space. The reason that this is consistent is of course that the operators that describe the properties of these states evolve with time in the Heisenberg picture, so that the states with represent particles/antiparticles at early times need not represent the same thing at late times. Notice also that no concrete representation of state space, or of the creation and annihilation operators was necessary here, since properties (4.4) - (4.8), are sufficient to completely describe state space. I will show in the next Section that these properties are indeed enough for us to deduce the general S-Matrix element of the theory, but that these derivations are more difficult, and conceptually more obscure than those given in Section 2.4.3.

Having described the conventional Canonical construction of the 'in' and 'out' spaces, and Bogoliubov coefficients, we are now in a position to introduce into this framework the particle definition defined in the previous Chapter. Clearly that definition allows us to replace $\{u_{i,in}(x), v_{i,in}(x)\}$ and $\{u_{i,out}(x), v_{i,out}(x)\}$ with $\{u_{i,\tau_0}(x), v_{i,\tau_0}(x)\}$ and $\{u_{i,\tau_1}(x), v_{i,\tau_1}(x)\}$ where τ_0 is some arbitrary 'in time' and $\tau_1 > \tau_0$ is some arbitrary 'out' time, according to some chosen observer (and we need no longer place any 'asymptotic niceness' restrictions on the background that is present). We can then define the finite time Bogoliubov coefficients $\alpha_{ij}(\tau_1, \tau_0), \beta_{ij}(\tau_1, \tau_0)$ etc, just as in (4.1) and (4.2) (which agree with equations (3.24) now). The field operator $\hat{\psi}(x)$ can then be written as

$$\hat{\psi}(x) = \sum_i \{u_{i,\tau_0}(x) a_{i,\tau_0,h} + v_{i,\tau_0}(x) b^{\dagger}_{i,\tau_0,h}\} \tag{4.9}$$

and is completely independent of τ_0. The Heisenberg picture vacuum $|vac_{\tau_0}, h\rangle$ can be defined by the requirement that

$$a_{i,\tau_0,h}|vac_{\tau_0}, h\rangle = 0 = b_{i,\tau_0,h}|vac_{\tau_0}, h\rangle \tag{4.10}$$

and the 'τ_0 Fock space' can be defined just as before. One note worth making here is that, if we used the explicit representation of state space introduced in the previous Chapter (or at least the Heisenberg picture equivalent of it) then we could substitute (2.32) and (2.33) into (4.9) to get:

$$\hat{\psi}(x) = \sum_i \psi_i(x) i_{\psi_h}$$

where the $\{\psi_i(x)\}$ form an arbitrary orthonormal basis for the solution space. In this form we see that $\hat{\psi}(x)$ is trivially independent of τ_0, and similarly, that it does not depend on the split of solution space into '$+ve/-ve$ energy solutions'.

4.2 S-Matrix Elements

Define the arbitrary 'in' and 'out' states:

$$|\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} in_{\tau_0}, h\rangle \equiv a^\dagger_{i_1,\tau_0,h} \dots a^\dagger_{i_m,\tau_0,h} b^\dagger_{j_n,\tau_0,h} \dots b^\dagger_{j_1,\tau_0,h} |vac_{\tau_0}, h\rangle$$

$$\text{and } |\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} out_{\tau_1}, h\rangle \equiv a^\dagger_{i_1,\tau_1,h} \dots a^\dagger_{i_m,\tau_1,h} b^\dagger_{j_n,\tau_1,h} \dots b^\dagger_{j_1,\tau_1,h} |vac_{\tau_1}, h\rangle$$

We wish to calculate $\langle \binom{i'_1 i'_2 \dots i'_{m'}}{j'_1 j'_2 \dots j'_{n'}} out_{\tau_1}, h | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} in_{\tau_0}, h\rangle$ using only the formalism of Section 4.1. To do this, start by defining:

$$c_v \equiv \langle vac_{\tau_1}, h | vac_{\tau_0}, h\rangle \tag{4.11}$$

$$V\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} \equiv \frac{1}{c_v} \langle \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} out_{\tau_1}, h | vac_{\tau_0}, h\rangle \tag{4.12}$$

$$\Lambda\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} \equiv \frac{1}{c_v} \langle vac_{\tau_1}, h | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} in_{\tau_0}, h\rangle \tag{4.13}$$

Once we have calculated $c_v, V\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}$ and $\Lambda\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}$, we can calculate any arbitrary S-Matrix element by using (4.6) or (4.7) to convert it into a linear combination of $V\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}$'s or $\Lambda\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}$'s.

From the definitions of V and Λ we have:

$$|vac_{\tau_0}, h\rangle = c_v \sum_{n,m=0}^{\infty} \sum_{\substack{i_1 < i_2 \dots < i_m \\ j_1 < j_2 \dots < j_n}} V\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} |\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} out_{\tau_1}, h\rangle \tag{4.14}$$

$$|vac_{\tau_1}, h\rangle = c_v \sum_{n,m=0}^{\infty} \sum_{\substack{i_1 < i_2 \dots < i_m \\ j_1 < j_2 \dots < j_n}} \Lambda^*\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} |\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} in_{\tau_0}, h\rangle \tag{4.15}$$

Acting on (4.15) with $a_{i,\tau_1,h}$ and using (4.6) gives:

$$0 = \sum_{n,m=0}^{\infty} \sum_{\substack{i_1 < i_2 \dots < i_m \\ j_1 < j_2 \dots < j_n}} \{ \sum_{r=1}^{m} (-)^{r-1} \alpha_{i i_r} \Lambda^*\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} |\binom{i_1 \dots \check{i}_r \dots i_m}{j_1 j_2 \dots j_n} in_{\tau_0}, h\rangle$$

$$+ \sum_j (-)^m \beta_{ij} \Lambda^*\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} |\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n j} in_{\tau_0}, h\rangle \} \tag{4.16}$$

true for all i. Consider now the coefficient of $|\binom{}{j_1} in_{\tau_0}, h\rangle$ in (4.16). This gives:

$$0 = \sum_{i_1} \alpha_{ii_1} \Lambda^* \binom{i_1}{j_1} + \beta_{ij_1} \Lambda^*()$$

where $\Lambda() = \frac{c_v}{c_v} = 1$. So, if we define the matrix $\boldsymbol{\Lambda} \equiv \boldsymbol{\epsilon}^{-1}\boldsymbol{\gamma}$, so that $\boldsymbol{\Lambda}^\dagger = -\boldsymbol{\alpha}^{-1}\boldsymbol{\beta}$ (from (2.111)) then we have $\Lambda\binom{i_1}{j_1} = \Lambda_{j_1 i_1}$ in agreement with (2.107). Looking at the coefficient of $|\binom{i_1 i_2 \ldots i_{m-1}}{j_1 j_2 \ldots j_n} in_{\tau_0}, h\rangle$ in (4.16) gives (after some work)

$$\sum_l \alpha_{il} \Lambda^* \binom{i_1 \ldots i_{m-1} l}{j_1 j_2 \ldots j_n} = (-)^{n+m} \sum_{r=1}^{n} (-)^r \beta_{ij_r} \Lambda^* \binom{i_1 i_2 \ldots i_{m-1}}{j_1 \ldots \check{j}_r \ldots j_n}$$

Multiplying through by $\alpha^{-1}_{i_m i}$, summing over i and conjugating gives:

$$\Lambda\binom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} = (-)^{n+m} \sum_{r=1}^{n} (-)^{r+1} \Lambda_{j_r i_m} \Lambda \binom{i_1 i_2 \ldots i_{m-1}}{j_1 \ldots \check{j}_r \ldots j_n} \tag{4.17}$$

For $n = m$ this gives (by induction)

$$\Lambda\binom{i_1 i_2 \ldots i_n}{j_1 j_2 \ldots j_n} = (-)^{\frac{n}{2}(n-1)} \det \begin{bmatrix} \Lambda_{j_1 i_1} & \cdots & \Lambda_{j_1 i_n} \\ \vdots & & \vdots \\ \Lambda_{j_n i_1} & \cdots & \Lambda_{j_n i_n} \end{bmatrix} \tag{4.18}$$

in agreement with (2.107). Looking at the coefficient of $|\binom{i_1 \ldots i_{m-1}}{} in_{\tau_0}, h\rangle$ in (4.16) gives $\Lambda\binom{i_1 i_2 \ldots i_m}{} = 0$ whenever $m > 0$, while $\Lambda\binom{}{j_1 j_2 \ldots j_n} = 0$ can be deduced by acting on (4.15) with $b_{i,t_1,h}$ and looking at the coefficient of $|\binom{}{j_1 j_2 \ldots j_{n-1}} in_{\tau_0}, h\rangle$. Combined with (4.17) this gives

$$\Lambda\binom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} = 0 \text{ for } m \neq n$$

as required. Similarly, by acting on (4.14) with $a_{i,\tau_0,h}$ and $b_{i,\tau_0,h}$ and extracting components we can deduce:

$$V\binom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} = \delta_{mn} (-)^{\frac{n}{2}(n-1)} \det \begin{bmatrix} V_{i_1 j_1} & \cdots & V_{i_1 j_n} \\ \vdots & & \vdots \\ V_{i_n j_1} & \cdots & V_{i_n j_n} \end{bmatrix} \tag{4.19}$$

We now know enough to calculate any arbitrary S-Matrix element up to a factor of the vacuum - vacuum transition amplitude c_v. In order to calculate c_v we can use

$$1 = \sum_{n,m=0}^{\infty} \sum_{\substack{i_1 < i_2 \cdots < i_m \\ j_1 < j_2 \cdots < j_n}} |\binom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} out_{\tau_1}, h\rangle\langle\binom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} out_{\tau_1}, h| \tag{4.20}$$

to get

$$1 = \sum_{n=0}^{\infty} \sum_{\substack{i_1<i_2\cdots<i_n \\ j_1<j_2\cdots<j_n}} |\langle \left({}^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n} \right) out_{\tau_1}, h | vac_{\tau_0}, h\rangle|^2$$

$$= |c_v|^2 (1 + \sum_{n=1}^{\infty} \sum_{\substack{i_1<i_2\cdots<i_n \\ j_1<j_2\cdots<j_n}} |\det \begin{bmatrix} V_{i_1 j_1} & \cdots & V_{i_1 j_n} \\ \vdots & & \vdots \\ V_{i_n j_1} & \cdots & V_{i_n j_n} \end{bmatrix} |^2) \tag{4.21}$$

$$= |c_v|^2 \det(1 + \boldsymbol{V}\boldsymbol{V}^\dagger) \tag{4.22}$$

where we have used the rather obscure matrix identity[2] :

$$\det(1 + \boldsymbol{V}\boldsymbol{V}^\dagger) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{i_1<i_2\cdots<i_n \\ j_1<j_2\cdots<j_n}} |\det \begin{bmatrix} V_{i_1 j_1} & \cdots & V_{i_1 j_n} \\ \vdots & & \vdots \\ V_{i_n j_1} & \cdots & V_{i_n j_n} \end{bmatrix} |^2$$

Hence

$$|c_v|^2 = \det(1 + \boldsymbol{V}\boldsymbol{V}^\dagger)^{-1} \tag{4.23}$$

$$= \det(\boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger) \qquad \text{from (2.112)} \tag{4.24}$$

$$= |\det(\boldsymbol{\alpha}(t_1, t_0))|^2 = |\det(\boldsymbol{\epsilon}(t_1, t_0))|^2 \tag{4.25}$$

where we have used the Bogoliubov conditions again in the last line. Notice that we cannot calculate the phase of c_v by this method, only its magnitude. This is of no importance to us, since the phase has no effect on transition probabilities, or expectation values (calculated by the methods of the previous Chapter, or of the next Section). However, if we wish to work with the functional integral $Z = \langle vac_{out} | vac_{in} \rangle$ (in the presence of external sources), then the phase of this S-Matrix element becomes vitally important - it is the real part of the effective action W (defined by $Z[0] = e^{iW}$). This is discussed in DeWitt[24] and in Birrell and Davies[22], but will not be discussed further here. However, in Section 4.5 I will justify equation (2.100), which is the equation generally used for evaluating W.

4.3 Expectation Values and Vacuum Subtraction

The first thing I wish to consider here is the relationship between Hermitian extension and conventional 'second quantization'. To do this, suppose we have chosen an observer (and so have chosen a split of spacetime into 'space' and 'time') and consider a 'first quantized' operator $\hat{A}_1(\tau) : \mathcal{H} \to \mathcal{H}$ (this will generally be a differential operator on $x|_{\Sigma_\tau}$). By inserting factors of $1 = \sum_i |u_{i,\tau_0}(x|_{\Sigma_\tau})\rangle\langle u_{i,\tau_0}(x|_{\Sigma_\tau})| + |v_{i,\tau_0}(x|_{\Sigma_\tau})\rangle\langle v_{i,\tau_0}(x|_{\Sigma_\tau})|$ either side of

[2]This identity follows from the Cauchy-Binet formula, applied (in the notation of Horn and Johnson[67], pg 22) to $A = [V, I_N] = B^\dagger$ with $\alpha = \beta = \{1, \ldots, N\}$ and $r = N$

$\hat{A}_1(\tau)$ we can write:

$$\hat{A}_1(\tau) = \sum_{ij} \langle u_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|u_{j,\tau_0}(x|_{\Sigma_\tau})\rangle |u_{i,\tau_0}(x|_{\Sigma_\tau})\rangle\langle u_{j,\tau_0}(x|_{\Sigma_\tau})| +$$
$$\langle u_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{j,\tau_0}(x|_{\Sigma_\tau})\rangle |u_{i,\tau_0}(x|_{\Sigma_\tau})\rangle\langle v_{j,\tau_0}(x|_{\Sigma_\tau})| + \tag{4.26}$$
$$\langle v_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|u_{j,\tau_0}(x|_{\Sigma_\tau})\rangle |v_{i,\tau_0}(x|_{\Sigma_\tau})\rangle\langle u_{j,\tau_0}(x|_{\Sigma_\tau})| +$$
$$\langle v_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{j,\tau_0}(x|_{\Sigma_\tau})\rangle |v_{i,\tau_0}(x|_{\Sigma_\tau})\rangle\langle v_{j,\tau_0}(x|_{\Sigma_\tau})|$$

By noticing that under Hermitian extension $|u_{i,\tau_0}(x|_{\Sigma_\tau})\rangle\langle u_{j,\tau_0}(x|_{\Sigma_\tau})| \to a^\dagger_{i,\tau_0}(\tau)a_{j,\tau_0}(\tau)$, $|u_{i,\tau_0}(x|_{\Sigma_\tau})\rangle\langle v_{j,\tau_0}(x|_{\Sigma_\tau})| \to a^\dagger_{i,\tau_0}(\tau)b^\dagger_{j,\tau_0}(\tau)$ etc.. (in the notation introduced at the end of Section 2.3.1), we can now write $\hat{A}_H(\tau)$ as:

$$\hat{A}_H(\tau) = \sum_{ij} \{\langle u_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|u_{j,\tau_0}(x|_{\Sigma_\tau})\rangle a^\dagger_{i,\tau_0}(\tau)a_{j,\tau_0}(\tau) +$$
$$\langle u_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{j,\tau_0}(x|_{\Sigma_\tau})\rangle a^\dagger_{i,\tau_0}(\tau)b^\dagger_{j,\tau_0}(\tau) + \langle v_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|u_{j\tau_0}(x|_{\Sigma_\tau})\rangle b_{i,\tau_0}(\tau)a_{j,\tau_0}(\tau) +$$
$$\langle v_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{j,\tau_0}(x|_{\Sigma_\tau})\rangle b_{i,\tau_0}(\tau)b^\dagger_{j,\tau_0}(\tau)\} \tag{4.27}$$

Meanwhile, in the conventional approach to second quantization the pre-normal-ordered 'second quantized' operator $\hat{A}_{naive}(\tau)$ is obtained by substituting the field operator $\hat{\psi}(x)$ from (4.9) into the integral expression for the 'first quantized expectation value' $\langle\psi(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|\psi(x|_{\Sigma_\tau})\rangle$. This gives:

$$\hat{A}_{naive}(\tau) = \sum_{ij} \{\langle u_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|u_{j,\tau_0}(x|_{\Sigma_\tau})\rangle a^\dagger_{i,\tau_0,h}a_{j,\tau_0,h} +$$
$$\langle u_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{j,\tau_0}(x|_{\Sigma_\tau})\rangle a^\dagger_{i,\tau_0,h}b^\dagger_{j,\tau_0,h} + \langle v_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|u_{j\tau_0}(x|_{\Sigma_\tau})\rangle b_{i,\tau_0,h}a_{j,\tau_0,h} +$$
$$\langle v_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{j,\tau_0}(x|_{\Sigma_\tau})\rangle b_{i,\tau_0,h}b^\dagger_{j,\tau_0,h}\} \tag{4.28}$$

which is clearly the Heisenberg picture version of (4.27). Notice that by construction neither of these operators depends on the choice of τ_0. It is also worth noting that although (4.27) and (4.28) are equivalent, the expression (4.27) is seldom very useful in my formalism. It is much more convenient in most situations to work directly from the definition (2.17). The same comments apply to conventional multiparticle quantum mechanics (see [68, 69] for instance) where expressions like (4.27) appear (with only the $u_i(x)'s$ and a_i's), but are seldom found useful. Part of the reason for this is that writing down (4.27) requires having at your disposal a complete set of solutions to the (first quantized) governing equation, and the usefulness of (4.27) then relies on the matrix elements taking a convenient form in this set. Indeed, even writing down the field operator $\hat{\psi}(x)$ requires this complete set of solutions. This is in my opinion a significant drawback of the standard approach to QFT. The definition (2.17) on the other hand, which is precisely the definition used in multiparticle quantum mechanics, is well-defined regardless of whether or not we have such a complete set.

We can now calculate the expectation value of $\hat{A}_{naive}(\tau)$ in the state $|vac_{\tau_{in}}, h\rangle$ (representing the state that 'at time τ_{in}' was vacuum) for some time τ_{in}. To do this choose

$\tau_0 = \tau_{in}$ in (4.28), to get:

$$\langle vac_{\tau_{in}}, h|\hat{A}_{naive}(\tau)|vac_{\tau_{in}}, h\rangle = \sum_i \langle v_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{i\tau_0}(x|_{\Sigma_\tau})\rangle \tag{4.29}$$

which can be recognised as the first term in (2.79) (which followed directly from property 4 of Section 2.1.3). Now consider normal ordering. Although $\hat{A}_{naive}(\tau)$ is independent of the choice of τ_0, the normal ordering process is clearly dependent upon this choice. Hence, denote *normal ordering with respect to the* τ' *expansion of* $\hat{\psi}(x)$ by $:\quad :_{\tau'}$. Then:

$$\begin{aligned} : \hat{A}_{naive}(\tau) :_{\tau'} &= \sum_{ij} \langle u_{i,\tau'}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|u_{j,\tau'}(x|_{\Sigma_\tau})\rangle a^\dagger_{i,\tau',h} a_{j,\tau',h} \\ &+ \langle u_{i,\tau'}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{j,\tau'}(x|_{\Sigma_\tau})\rangle a^\dagger_{i,\tau',h} b^\dagger_{j,\tau',h} + \langle v_{i,\tau'}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|u_{j\tau'}(x|_{\Sigma_\tau})\rangle b_{i,\tau',h} a_{j,\tau',h} \\ &- \langle v_{i,\tau'}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{j,\tau'}(x|_{\Sigma_\tau})\rangle b^\dagger_{j,\tau',h} b_{i,\tau',h} & (4.30) \\ &= \hat{A}_{naive}(\tau) - \sum_i \langle v_{i,\tau'}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{i,\tau'}(x|_{\Sigma_\tau})\rangle \hat{1} & (4.31) \end{aligned}$$

Comparing the second expression (4.31) above with (2.79), or with (2.40) reveals that vacuum subtraction corresponds to the choice $\tau' = \tau$. That is, it corresponds to normal ordering *at the time of measurement*:

$$\hat{A}_{phys}(\tau) \;\; \Leftrightarrow \; : \hat{A}_{naive}(\tau) :_\tau \tag{4.32}$$

This is also the choice used in the 'Hamiltonian diagonalisation' procedure discussed in the next Section. I will assume that $\tau' = \tau$ from now on. Expression (4.30) is what we obtain if we normal order by 'moving the creation operators to the left, and changing appropriate signs', while expression (4.31) is the acknowledgement that this is the same as subtracting an appropriate multiple of the unit operator. Since (4.31) still allows us to express $\hat{A}_{naive}(\tau)$ in terms of an arbitrary τ_0, then it is generally more convenient than the first expression (4.30) above, which requires expressing $\hat{A}_{naive}(\tau)$ in terms of $\tau_0 = \tau'(= \tau$ now). To illustrate this, consider $\langle vac_{\tau_{in}}, h| : \hat{A}_{naive}(\tau) :_\tau |vac_{\tau_{in}}, h\rangle$ for some $\tau_{in} \neq \tau$. Using (4.31) allows us to write:

$$\begin{aligned} &\langle vac_{\tau_{in}}, h| : \hat{A}_{naive}(\tau) :_\tau |vac_{\tau_{in}}, h\rangle \\ &\quad = \sum_i \langle v_{i,\tau_0}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{i\tau_0}(x|_{\Sigma_\tau})\rangle - \sum_i \langle v_{i,\tau}(x|_{\Sigma_\tau})|\hat{A}_1(\tau)|v_{i\tau}(x|_{\Sigma_\tau})\rangle \end{aligned} \tag{4.33}$$

just as in (2.79). Then, if we wish (sometimes this will be convenient, but certainly not always), we can expand the $\{v_{i,\tau_0}(x|_{\Sigma_\tau})\}$ in terms of the $\{u_{i,\tau}(x|_{\Sigma_\tau}), v_{i,\tau}(x|_{\Sigma_\tau})\}$ just as in (2.55) or (4.2) to obtain:

$$\begin{aligned} &\langle vac_{\tau_{in}}, h| : \hat{A}_{naive}(\tau) :_\tau |vac_{\tau_{in}}, h\rangle \\ &\quad = Trace(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger \boldsymbol{A}^{++}(\tau) - \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger \boldsymbol{A}^{--}(\tau) + \boldsymbol{\epsilon}\boldsymbol{\beta}^\dagger \boldsymbol{A}^{+-}(\tau) + \boldsymbol{\beta}\boldsymbol{\epsilon}^\dagger \boldsymbol{A}^{-+}(\tau)) \end{aligned} \tag{4.34}$$

just as in (2.81).

However, if we had tried to use (4.30) to evaluate $\langle vac_{\tau_{in}},h| : \hat{A}_{naive}(\tau) :_\tau |vac_{\tau_{in}},h\rangle$ (as is common in the literature) then we would first obtain:

$$\langle vac_{\tau_{in}},h| : \hat{A}_{naive}(\tau) :_\tau vac_{\tau_{in}},h\rangle = \sum_{ij} \boldsymbol{A}^{++}_{ij}(\tau)\langle vac_{\tau_{in}},h|a^\dagger_{i,\tau,h}a_{j,\tau,h}|vac_{\tau_{in}},h\rangle$$
$$+\boldsymbol{A}^{+-}_{ij}(\tau)\langle vac_{\tau_{in}},h|a^\dagger_{i,\tau,h}b^\dagger_{j,\tau,h}|vac_{\tau_{in}},h\rangle + \boldsymbol{A}^{-+}_{ij}(\tau)\langle vac_{\tau_{in}},h|b_{i,\tau',h}a_{j,\tau',h}|vac_{\tau_{in}},h\rangle$$
$$-\boldsymbol{A}^{--}_{ij}(\tau)\langle vac_{\tau_{in}},h|b^\dagger_{j,\tau',h}|vac_{\tau_{in}},h\rangle b_{i,\tau',h}$$

into which we must substitute (4.6) in order to get (4.34) above. Although this still yields the right result, it is more cumbersome than using (4.31), and it suppresses the fact that (4.34) is simply (4.33)

Finally, it will be convenient for the next section to consider the operator $\hat{H}_{naive}(\tau)$ (obtained by substituting $\hat{\psi}(x)$ into (3.17)) expressed in terms of $\tau_0 = \tau$. This can be written as:

$$\hat{H}_{naive}(\tau) = \sum_{ij} \boldsymbol{H}^{++}_{ij}(\tau)a^\dagger_{i,\tau,h}a_{j,\tau,h} + \boldsymbol{H}^{+-}_{ij}(\tau)a^\dagger_{i,\tau,h}b^\dagger_{j,\tau,h}$$
$$+\boldsymbol{H}^{-+}_{ij}(\tau)b_{i,\tau,h}a_{j,\tau,h} + \boldsymbol{H}^{--}_{ij}(\tau)b_{i,\tau,h}b^\dagger_{j,\tau,h} \tag{4.35}$$

Since $\{u_{j,\tau}(x|_{\Sigma_\tau})\}$ and $\{v_{j,\tau}(x|_{\Sigma_\tau})\}$ are chosen to satisfy the particle definition introduced in the previous Chapters. (see pages 40 and 41 for instance) then we clearly have $\boldsymbol{H}^{-+}(\tau) = 0 = \boldsymbol{H}^{+-}(\tau)$, so that $\hat{H}_{naive}(\tau)$ contains no $a^\dagger_{i,\tau,h}b^\dagger_{j,\tau,h}$, or $b_{i,\tau,h}a_{j,\tau,h}$ terms. We also have that $\boldsymbol{H}^{++}(\tau)$ and $-\boldsymbol{H}^{--}(\tau)$ are positive definite. Conversely, it is straightforward to show that requiring $\boldsymbol{H}^{-+}(\tau) = 0 = \boldsymbol{H}^{+-}(\tau)$, (and that $\boldsymbol{H}^{++}(\tau)$ and $-\boldsymbol{H}^{--}(\tau)$ are positive definite) is enough to guarantee that $\{u_{j,\tau}(x|_{\Sigma_\tau})\}$ span $\mathcal{H}^+(\tau)$, and that $\{v_{j,\tau}(x|_{\Sigma_\tau})\}$ span $\mathcal{H}^-(\tau)$. This amounts to a proof of the fact that the particle definition given in Section 3.2 is equivalent to Hamiltonian diagonalisation of the Hamiltonian[3] obtained by substituting $\hat{\psi}(x)$ into (3.17). To put this result in context, we now turn to a discussion of Hamiltonian diagonalisation.

4.4 On Hamiltonian Diagonalisation

The idea of using 'Hamiltonian diagonalisation' as a prescription for defining particle/anti-particle states dates back to a paper by Imamura[70] in 1960, which was in turn based on an analogy with Bogoliubov's work on superfluidity[71] and superconductivity[72]. However, the idea was not developed in any detail until the papers by Grib and Mamaev[42, 43] and Grib, Mamayev and Mostepanenko[44] throughout the 1970's. The basic procedure is:

[3]An interesting note worth mentioning here is that the generalisation of the well-known formula $\frac{\partial\hat{\psi}}{\partial t} = i[\hat{H},\hat{\psi}]$ to gravitational backgrounds is not $k^\mu\nabla_\mu\hat{\psi}(x)|_{\Sigma_\tau} = i[\hat{H}_{naive}(\tau),\hat{\psi}(x|_{\Sigma_\tau})]$ as might have been expected, but rather $k^\mu\nabla_\mu\hat{\psi}(x)|_{\Sigma_\tau} = i[\hat{H}_{nh,naive}(\tau),\hat{\psi}(x|_{\Sigma_\tau})]$, where $\hat{H}_{nh,naive}(\tau)$ can be written in terms of $\hat{\psi}(x)$ by substituting $\hat{\psi}(x)$ into the expression (see Section 3.2) for $\langle\psi|\hat{H}_{nh}(\Sigma)|\psi\rangle_\Sigma$. Normal ordering of course makes no difference to these results.

1. Obtain a (pre-normal ordered) 'second quantised Hamiltonian' or 'energy operator' by whatever means necessary. For fermionic systems (about which comparatively little has been done) this involved appealing to the energy-momentum tensor, while in Bosonic systems it often involved some sort of canonical quantisation procedure.

2. Substitute the expansion of $\hat{\psi}(x)$ (in terms of creation/annihilation operators and a basis of solutions of the governing equation) into this Hamiltonian, to get an expression for $\hat{H}(t)$ in terms of creation/annihilation operators, with coefficients depending on the choice of basis solutions.

3. Choose the basis solutions that 'represent particle/antiparticle states at time t_0' by the requirement that $\hat{H}(t_0)$ be diagonal, in the sense that it does not contain any terms proportion to $b_{i,t_0}a_{j,t_0}$ or proportional to $a^{\dagger}_{i,t_0}b^{\dagger}_{j,t_0}$.

4. Applying this requirement at two different times leads to two different mode-expansions of $\hat{\psi}(x)$, which will be related by Bogoliubov coefficients, which can then be used to calculate S-matrix elements, vacuum expectation values etc.

We see then that if the 'second quantized Hamiltonian' is taken to be that obtained by substituting $\hat{\psi}(x)$ into (3.17), and we take as implicit to Hamiltonian diagonalisation the requirement that $\boldsymbol{H}^{++}(\tau)$ and $-\boldsymbol{H}^{--}(\tau)$ be positive definite (that is, that particles have positive '1st quantized' energy, and antiparticles have negative '1st quantized energy'), then my particle definition becomes equivalent to Hamiltonian diagonalisation. However, as is probably quite apparent to the reader, the Hamiltonian diagonalisation prescription is quite meaningless unless a Hamiltonian is specified, and the results of Hamiltonian diagonalisation will depend on precisely which Hamiltonian we choose. The Hamiltonian proposed in Section 3.2 has never before been studied, and it is the first Hamiltonian that is defined directly in terms of the motion of an observer, and that depends *only* on the choice of observer and the background present, not on the choice of coordinates or (in an electromagnetic background) the choice of gauge (a problem which plagued previous 'Bogoliubov coefficient' methods[34]). It is also the first definition that does not rely on the spacetime possessing any convenient symmetries.

Early attempts at Hamiltonian diagonalisation received much criticism, most notably by Fulling[45] in 1979. Fulling's main criticism was the ambiguity that existed in early attempts to define a Hamiltonian operator. (He even went as far as to point out that there exists a Hamiltonian for any desired purpose, and demonstrated this by deriving a Hamiltonian whose diagonalisation would yield the 'adiabatic expansion' prescription that he himself advocated.) In the early examples of Hamiltonian diagonalisation, this Hamiltonian came from a canonical quantisation procedure, with respect to an arbitrarily chosen time coordinate. This was hopelessly ambiguous, since the results were sensitively dependent on this arbitrarily chosen time coordinate. Another procedure was to simply take the Hamiltonian to be $\int d^3\mathbf{x}T_{oo}(\hat{\psi}(x))$, (or the even more ambiguous[73] $\int T_{o\mu}d\Sigma^{\mu}$ over an 'arbitrary spacelike hypersurface') which is again hopelessly dependent on the choice of time coordinate (and the choice of hypersurface). However, more recent work[44, 74] has gone some way towards remedying this situation, by using $\int T_{\nu\mu}k^{\nu}d\Sigma^{\mu}$

(as in (3.17)), where k^μ is taken to be a conformal timelike Killing vector field. At this point the Hamiltonian diagonalisation procedure has become very similar to the method proposed by Gibbons[75] in 1975, which amounts to diagonalising $\int T_{\nu\mu} k^\nu d\Sigma^\mu$ where k^μ is taken to be a Killing vector field there. Diagonalisation procedures based on these 'improved Hamiltonians' have been used in recent work on inflationary cosmology, such as in Chung[76] et. al.'s recent work on particle creation during an inflationary era, and the possible production of 'wimpzillas'.

However, even these 'improved' diagonalisation procedures are far from satisfactory. DeWitt[24] for instance, after outlining Gibbons[75]' prescription, comments that "this is just as in conventional particle physics. The only trouble with it is: *it's wrong.* It is not wrong in a technical mathematical sense. It simply provides a grossly inadequate foundation for the theory." Of the various situations DeWitt mentions for which Gibbon's construction is inadequate (see [24] for details) the most obvious and irrefutable one is the fact that most spacetimes (including our own) do not possess timelike Killing vector fields! Even for spacetimes that do admit Killing vector fields they might not be timelike everywhere, and what if there are more than one of them? We are expected to assume that if more than one timelike Killing vector field exists, then the different Killing vector fields are to be associated with different choices of observer, but no prescription is given to describe which Killing vector field is to be associated with which observer. Also, what about those observers among us us who occasionally stray from our Killing path - surely we still have the right to count particles once in a while?!

I have demonstrated in the previous Chapter that my particle definition is precisely the definition needed to fix these problems. By defining $k^\mu(x)$ directly in terms of the motion of an observer, I not only provide a definition that appropriately takes observer dependence into account, but I ensure that the definition depends *only* on the motion of the observer - not on the choice of coordinates, the choice of gauge (in electromagnetic backgrounds) or the existence of special symmetries. This also means that my definition is well-defined even in an arbitrary electromagnetic or gravitational background, and for an arbitrary observer. Another advantage of the prescription presented in Chapters 2 and 3 is that I have not just provided a 'Hamiltonian diagonalisation condition' (equation (3.20)) which $\mathcal{H}^\pm(\tau_0)$ must satisfy. I have also provided a clear procedure for constructing $\mathcal{H}^\pm(\tau_0)$ (in terms of eigenstates of the 1st quantized Hamiltonian $\hat{H}_1(\tau)$) which allows us to tackle arbitrarily complicated situations. With the conventional Hamiltonian diagonalisation prescription however, unless the spacetime allows separation of variables no clue is given as to how to actually satisfy the 'Hamiltonian diagonalisation condition'. It is not surprising then, that in all the examples of Hamiltonian diagonalisation so far considered, this separation of variables was indeed possible.

This need to be able to separate variables is also a problem which plagues the 'adiabatic approximation' techniques advocated by Fulling[23, 45] and others. Having chosen such a separation of variables (when this is possible) the adiabatic approximation technique does provide an unambiguous prescription for which parametrisation of time ($\tau(t)$ say) is the appropriate one. However, as we saw in Sections 3.2.2 and 3.2.3, there is often more than one way of separating variables, and each such separation will lead to a different adiabatic prescription.

I would also like to return here to the fact, as pointed out in Section 2.2.4, that any physically reasonable definition of particle must be non-local! No local definition could possibly be consistent with the Unruh effect. I have demonstrated with a simple example that, as we would hope, this non-locality is only significant on scales of the order of the Compton wavelength of the particle concerned. Despite this, this non-locality has caused significant concern among quantum field theorists working in gravitational backgrounds, causing many in the field to abandon the concept of particle[33]. I find this level of concern quite surprising, especially considering the fact (as also pointed out in Section 2.2.4) that this non-locality was already present in free quantum field theory in flat Minkowski space! Even more relevant is the fact that, while the early quantum field theorists were trying to formulate a completely local theory, Bell and others were setting up experiments to check the validity of some of the non-local claims of quantum mechanics (see Wick[77] for an excellent historical overview of these experiments). They found, as we all know, that those non-local claims were right! These experiments have now been carried out over distances as large as 10 kilometres, and the result is still the same - not only is quantum mechanics non-local, but if it wasn't non-local, it would be wrong! Roger Penrose[78] for instance not only points out that "Experiments of the EPR type give clear observational substance to this puzzling but essential fact of quantum physics: it is *non-local.*" but goes on to point out that "The difficulty of a consistent relativistic interpretation of the quantum jumps occurring with [state-vector reduction] that the EPR-type experiments leave us with, is not even touched by quantum field theory". These non-local aspects of quantum mechanics form the basis of a whole new field of research (that of quantum computation), where researchers are even considering the possibility of harnessing this non-locality in the next generation of computers. With these points in mind, I would claim that not only must we accept the appearance of the observer and their hypersurface of simultaneity at the heart of QFT, but we should embrace this fact. An ability to consider non-local issues such as entanglement and state-vector reduction within a relativistic context is not only essential, but also long overdue! Collapse of the wave-function, whether viewed as a fundamental postulate, or as a convenient approximation to the detailed interactions involved in any measurement (as in the decoherence viewpoint) seems to be here to stay. If we don't make a place for it in relativistic quantum physics, then it will be at our peril. Doing so clearly requires (just as I have done) that a choice of hypersurface be associated with any measurement, while consistency requires that this hypersurface be specified in terms of the motion of an observer.

It is also worth noting here, that state-vector reduction is significantly easier to consider in the Schrödinger picture, where we work directly in terms of the states of the system. Although it may well be possible to describe state-vector reduction in terms of the field operator $\hat{\psi}(x)$, doing so would be both cumbersome and conceptually awkward. I should also stress here that the non-locality that is demonstrated to be present in quantum physics does not lead to any violation of causality. This is because, as mentioned in Section 2.2.4, the governing equation is completely local. So, although measurements may reveal correlations between spacelike separated events, there is no way of sending signals at superluminal velocities, and hence no possible violation of causality.

The need to consider states as being defined on an entire Cauchy surface, (not just locally definable) is of course not restricted to EPR related issues. Even in a problem as straightforward as the hydrogen atom, deducing the allowed energy levels requires not just solving the eigenvalue equation locally (which can be done for any eigenvalue), but also extending these solutions out to spatial infinity, so as to deduce which states converge there.

Finally, I would like to address Fullings[45] comment that "With the field itself thereby put in its rightful place at the centre of the physical interpretation of the theory, the particle concept is reduced to a somewhat arbitrary technical aid". This is directly analogous to saying that the wave function of a quantum system is most fundamental to its physical interpretation, and that its decomposition in terms of the eigenstates of a Complete Set of Commuting Operators should be considered to be merely a 'technical aid'. Quite clearly, knowing the wave function of a particular hydrogen atom for instance, even knowing the expected energy density associated with this wave function, tells us next to nothing about the properties of that atom. Such properties as the allowed energy levels, and the probability that an electron will occupy a certain allowed energy level, will involve the expectation values of projection operators onto portions of the spectrum of the Hamiltonian, all of which are non-local in precisely the same sense as is my particle definition (which is constructed entirely from projection operators onto portions of the spectrum of the Hamiltonian). If we are to believe that such projection operators are mere 'technical aids', or as Davies[33] puts it, that "particles do not exist", must we now convince the chemists and the astronomers that the atomic spectral lines they measure everyday are simply 'technical aids' and do not really exist!

4.5 Greens Functions

In this Section I wish to compare the Greens functions, defined in terms of vacuum expectation values of field operators, with the evolution operator $\hat{U}_1(t, t_0)$ on $\mathcal{H}$, and with the Bogoliubov matrices. For simplicity, I restrict my attention to inertial observers in electromagnetic backgrounds (and I work in the inertial observers rest frame) so that I may refer to $(\mathbf{x}, t)$ just as in Chapter 2. However, most of this Section is easily generalised to arbitrary observers and arbitrary spacetimes. It will often be convenient in this Section to write the solutions $u_{i,t_0}(\mathbf{x}, t)$ as $\langle \mathbf{x} | u_{i,t_0}(t) \rangle$, denoting the fact that $u_{i,t_0}(\mathbf{x}, t)$ is the 'coordinate-space' representation of the state vector $|u_{i,t_0}(t)\rangle$. $\langle u_{i,t_0}(t) | \mathbf{x} \rangle$ then represents the 'conjugate transpose spinor' $u^\dagger_{i,t_0}(\mathbf{x}, t)$. The equation $\hat{U}_1(t, t')|\psi_{t'}\rangle = |\psi_{t'}(t)\rangle$ can now be written in coordinate space as:

$$\begin{aligned} \langle \mathbf{x} | \psi_{t'}(t) \rangle &= \int d^3\mathbf{x}' \langle \mathbf{x} | \hat{U}_1(t, t') | \mathbf{x}' \rangle \langle \mathbf{x}' | \psi_{t'} \rangle \\ &= \int d^3\mathbf{x}' i S(x, x') \gamma_0 \psi_{t'}(\mathbf{x}') \end{aligned} \tag{4.36}$$

where I have defined the *full propagator* $S(x, x')$, by $iS(x, x')\gamma_0 = \langle \mathbf{x} | \hat{U}_1(t, t') | \mathbf{x}' \rangle$.

$\hat{U}_1(t,t')$ can be written as:

$$\hat{U}_1(t,t') = \sum_i |\psi_i(t)\rangle\langle\psi_i(t')| \tag{4.37}$$

where $\{|\psi_i(t)\rangle\}$ is any orthonormal basis of solutions to the Dirac equation. From this it is clear that $S(x,x')$ satisfies:

$$(i\gamma^\mu \frac{\partial}{\partial x^\mu} - e\gamma^\mu A_\mu(x) - m)S(x,x') = 0 \tag{4.38}$$

and satisfies the conjugate Dirac equation in x'.

It is often convenient to also define the *forward evolution operator* $\hat{U}_1^+(t,t')$ by $\hat{U}_1^+(t,t') = \theta(t-t')\hat{U}_1(t,t')$. That is $\hat{U}_1^+(t,t') = \hat{U}_1(t,t')$ for $t > t'$ and $\hat{U}_1^+(t,t') = 0$ for $t < t'$, so that $\hat{U}_1^+(t,t')$ only propagates forward in time. This is written in integral form, as:

$$\psi_{t'}(\mathbf{x},t) = -i\int d^3\mathbf{x}' S_R(x,x')\gamma_0\psi_{t'}(\mathbf{x}') \tag{4.39}$$

where $-iS_R(x,x')\gamma_0 = \langle\mathbf{x}|\hat{U}_1^+(t,t')|\mathbf{x}'\rangle$ is the *retarded Greens function*, and satisfies

$$(i\gamma^\mu \frac{\partial}{\partial x^\mu} - e\gamma^\mu A_\mu(x) - m)S_R(x,x') = i\delta(x-x') \tag{4.40}$$

with $S_R(x,x') = 0$ for $t < t'$. In fact, it can be shown that $S_R(x,x')$ is zero except when x' is in the causal past of x, as required for causal propagation. Any function satisfying equation (4.40) is called a Greens function. There are of course many possible Greens functions, corresponding to different choices of boundary conditions. The commonly used Greens functions can be defined in terms of the field operator $\hat{\psi}(x)$ as:

$$\begin{aligned}
S^+(x,x')\gamma_0 &\equiv \langle 0|\hat{\psi}(x)\hat{\psi}^\dagger(x')|0\rangle \\
S^-(x,x')\gamma_0 &\equiv \langle 0|\hat{\psi}^\dagger(x')\hat{\psi}(x)|0\rangle \\
iS_F(x,x')\gamma_0 &\equiv \langle 0|T\hat{\psi}(x)\hat{\psi}^\dagger(x')|0\rangle \\
&= \theta(t-t')S^+(x,x')\gamma_0 - \theta(t'-t)S^-(x,x')\gamma_0 \\
S^{(1)}(x,x')\gamma_0 &\equiv \langle 0|[\hat{\psi}(x),\hat{\psi}^\dagger(x')]|0\rangle \\
&= S^+(x,x')\gamma_0 - S^-(x,x')\gamma_0 \\
iS(x,x')\gamma_0 &\equiv \{\hat{\psi}(x),\hat{\psi}^\dagger(x')\} \\
&= S^+(x,x')\gamma_0 + S^-(x,x')\gamma_0 \\
S_R(x,x') &\equiv -\theta(t-t')S(x,x') \\
S_A(x,x') &\equiv \theta(t'-t)S(x,x')
\end{aligned} \tag{4.41}$$

where $S^\pm(x,x')$ are the *Wightman functions*, $iS_F(x,x')$ is the *Feynman propagator*, $S^{(1)}(x,x')$ is the *Hadamard function*, $iS(x,x')$ is the full propagator, and $S_R(x,x')$, $S_A(x,x')$ are the *retarded* and *advanced* Greens functions respectively. Notice that I have included

$S^{\pm}(x,x')$, $S^{(1)}(x,x')$ and $S(x,x')$ in this list despite the fact that technically these are not Greens functions - they satisfy (4.38) rather than (4.40). I will show soon that the definitions of $S_R(x,x')$ and $S(x,x')$ given here agree with those given above.

Clearly these definitions are not yet complete, since I have not specified what state will constitute $\langle 0|$ and $|0\rangle$. Three common choices (see Gavrilov et. al.[79] or DeWitt[24] for instance) are $\langle vac_{out}| \quad |vac_{out}\rangle$, $\langle vac_{in}| \quad |vac_{in}\rangle$, and $\frac{\langle vac_{out}| \quad |vac_{in}\rangle}{\langle vac_{out}|vac_{in}\rangle}$, with the last being the most useful for the perturbative calculation of S-matrix elements (see Fradkin et. al.[17] or Birrell et. al.[80]).

The first two can now be generalised to $\langle vac_{t_0}, h| \quad |vac_{t_0}, h\rangle$ for arbitrary t_0. Substituting $\hat{\psi}(\mathbf{x},t)$ into (4.41) gives:

$$\begin{aligned} S^+_{t_0}(x,x')\gamma_0 &= \sum_i \langle \mathbf{x}|u_{i,t_0}(t)\rangle\langle u_{i,t_0}(t')|\mathbf{x}'\rangle \\ &= \langle \mathbf{x}|\hat{S}^+_{t_0}(t,t')\gamma_0|\mathbf{x}'\rangle \\ \text{where } \hat{S}^+_{t_0}(t,t')\gamma_0 &= \sum_i |u_{i,t_0}(t)\rangle\langle u_{i,t_0}(t')| \\ &= \hat{U}_1(t,t_0)\hat{P}^+(t_0)\hat{U}_1(t_0,t') \end{aligned} \tag{4.42}$$

From (4.42) we see that, $\hat{S}^+_{t_0}(t,t')$ evolves a state from time t' to time t, but only if the state was in $\mathcal{H}^+(t_0)$ at time t_0. Similarly we can write $S^-_{t_0}(x,x')\gamma_0 = \langle \mathbf{x}|\hat{S}^-_{t_0}(t,t')\gamma_0|\mathbf{x}'\rangle$ where

$$\begin{aligned} \hat{S}^-_{t_0}(t,t')\gamma_0 &= \sum_i |v_{i,t_0}(t)\rangle\langle v_{i,t_0}(t')| \\ &= \hat{U}_1(t,t_0)\hat{P}^-(t_0)\hat{U}_1(t_0,t') \end{aligned} \tag{4.43}$$

From these we can calculate all the other Greens functions. It is clear from (4.41) that $i\hat{S}(t,t')\gamma_0 = \hat{U}_1(t,t')$ and $-i\hat{S}_R(t,t')\gamma_0 = \hat{U}_1^+(t,t')$ as claimed. We also see that $\hat{S}(t,t')$, $\hat{S}_R(t,t')$ and $\hat{S}_A(t,t')$ are independent of t_0, while the others depend on t_0.

The separation of Greens functions into those that depend on t_0 and those that do not is equivalent to the split (see Fulling[23] for instance) into those Greens functions that do, or do not, depend on the decomposition of solutions into positive/negative frequency components.

Notice also that when defining the evolution operator in equation (2.19) I required that $\psi_{t'}(\mathbf{x}')$ represent our chosen initial conditions at time t'. That is, I required $\psi_{t'}(\mathbf{x}',t') = \psi_{t'}(\mathbf{x}')$ (or at least $\lim_{t\downarrow t'}\psi_{t'}(\mathbf{x}',t) = \psi_{t'}(\mathbf{x}')$) Although substituting any one of the Greens functions into equation (4.40) would generate a solution of the Dirac equation, it is only $S_R(x,x')$ (and $S(x,x')$, although this is not a Greens function) that satisfies this condition. That is, it is only the retarded Greens function that evolves a chosen initial state forward in time.

In conventional Q.F.T. the retarded Greens function is discarded, due to the claim that it would lead to a non-zero probability for a transition from a particle to an antiparticle state (see Feynman[81] for a succinct account of this). This of course does not happen in the formalism of the previous Chapter, since antiparticle states have different grade to particle states. Rather, as we have seen already, the development of a negative energy component under evolution manifests itself physically as pair creation. Conventional approaches, having discarded the retarded Greens function, replace it with the Feynman propagator $iS_F(x,x')$, which propagates positive energy particles forward in time, and negative energy particles backwards in time. However, due to the distinction between positive and negative energy states (dependence on t_0), and the consequence that $iS_F(x,x') \neq 0$ even for x, x' spacelike separated, the Feynman Propagator becomes troublesome whenever the Hamiltonian is time-dependent. Fulling[23] (p 79) comments that "there is no obvious, natural definition of [the Feynman propagator] if the dynamics is not independent of time", and goes on to point out that this is a "fundamental problem for the quantum theory of such systems". We have seen above that the reason why there is "no obvious, natural definition of **the** Feynman propagator", is simply that when the Hamiltonian is time dependent we have a whole family of possible Feynman propagators, namely $S_{F,t_0}(x,x')$ for each t_0. The fact that the formalism I introduced in Chapters 2 and 3 uses the causally well-behaved, uniquely defined Greens function $S_R(x,x_0)$ (or $S(x,x_0)$) rather than $S_F(x,x_0)$ thus represents a significant improvement on conventional methods.

Now, to compare the operators $\hat{S}^{\pm}_{t_0}(t,t')$, $\hat{S}_{F,t_0}(t,t')$ etc with the Bogoliubov coefficients, define the *Bogoliubov operators*:

$$\begin{aligned}
\hat{\alpha}_{t_1,t_0}(t,t') &= \hat{U}_1(t,t_1)\hat{P}^+_{t_1}\hat{U}_1(t_1,t_0)\hat{P}^+_{t_0}\hat{U}_1(t_0,t') \\
\hat{\beta}_{t_1,t_0}(t,t') &= \hat{U}_1(t,t_1)\hat{P}^+_{t_1}\hat{U}_1(t_1,t_0)\hat{P}^-_{t_0}\hat{U}_1(t_0,t') \\
\hat{\gamma}_{t_1,t_0}(t,t') &= \hat{U}_1(t,t_1)\hat{P}^-_{t_1}\hat{U}_1(t_1,t_0)\hat{P}^+_{t_0}\hat{U}_1(t_0,t') \\
\hat{\epsilon}_{t_1,t_0}(t,t') &= \hat{U}_1(t,t_1)\hat{P}^-_{t_1}\hat{U}_1(t_1,t_0)\hat{P}^-_{t_0}\hat{U}_1(t_0,t')
\end{aligned}$$

These are related to the Bogoliubov coefficients by:

$$\begin{aligned}
&\langle u_{i,t_1}(t)|\hat{\alpha}_{t_1,t_0}(t,t')|u_{j,t_0}(t')\rangle = \alpha_{ij}(t_1,t_0) \\
&\langle v_{i,t_1}(t)|\hat{\alpha}_{t_1,t_0}(t,t')|u_{j,t_0}(t')\rangle = \langle u_{i,t_1}(t)|\hat{\alpha}_{t_1,t_0}(t,t')|v_{j,t_0}(t')\rangle = \langle v_{i,t_1}(t)|\hat{\alpha}_{t_1,t_0}(t,t')|v_{j,t_0}(t')\rangle = 0
\end{aligned}$$

and similarly for $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\epsilon}$. In Section 2.4 I only needed to use $\hat{\alpha}_{t_1,t_0}(t_1,t_0)$, $\hat{\beta}_{t_1,t_0}(t_1,t_0)$ etc, but I may need the more general form of $\hat{\alpha}_{t_1,t_0}(t,t')$ when investigating interacting quantum field theories, which I hope to do in future work. There are a variety of ways in which the Bogoliubov operators can be related to the Greens functions. These are summarised in the formulae below:

$$\begin{aligned}
\hat{\alpha}_{t_1,t_0}(t,t') &= \hat{S}^+_{t_1}(t,\tau)\gamma_0\hat{S}^+_{t_0}(\tau,t')\gamma_0 \qquad & \hat{\beta}_{t_1,t_0}(t,t') &= \hat{S}^+_{t_1}(t,\tau)\gamma_0\hat{S}^-_{t_0}(\tau,t')\gamma_0 \\
\hat{\gamma}_{t_1,t_0}(t,t') &= \hat{S}^-_{t_1}(t,\tau)\gamma_0\hat{S}^+_{t_0}(\tau,t')\gamma_0 \qquad & \hat{\epsilon}_{t_1,t_0}(t,t') &= \hat{S}^-_{t_1}(t,\tau)\gamma_0\hat{S}^-_{t_0}(\tau,t')\gamma_0
\end{aligned}$$

where τ is completely arbitrary. Taking $t_0 = t_1$ gives:

$$\hat{\alpha}_{t_0,t_0}(t,t') = \hat{S}^+_{t_0}(t,t')\gamma_0 \qquad \hat{\epsilon}_{t_0,t_0}(t,t') = \hat{S}^-_{t_0}(t,t')\gamma_0$$
$$\hat{\beta}_{t_0,t_0}(t,t') = 0 = \hat{\gamma}_{t_0,t_0}(t,t')$$

Finally we can generate a Greens function by adding two Bogoliubov operators:

$$\begin{aligned}\hat{\alpha}_{t_1,t_0}(t,t') + \hat{\beta}_{t_1,t_0}(t,t') &= \hat{S}^{+}_{t_1}(t,t')\gamma_0\\ \hat{\alpha}_{t_1,t_0}(t,t') + \hat{\gamma}_{t_1,t_0}(t,t') &= \hat{S}^{+}_{t_0}(t,t')\gamma_0\\ \hat{\epsilon}_{t_1,t_0}(t,t') + \hat{\beta}_{t_1,t_0}(t,t') &= \hat{S}^{-}_{t_0}(t,t')\gamma_0\\ \hat{\epsilon}_{t_1,t_0}(t,t') + \hat{\gamma}_{t_1,t_0}(t,t') &= \hat{S}^{-}_{t_1}(t,t')\gamma_0\end{aligned}$$

The Bogoliubov operators can be written in a form analogous to (4.43) as:

$$\begin{aligned}\hat{\alpha}_{t_1,t_0}(t,t') &= \sum_{ij} |u_{i,t_1}(t)\rangle \alpha_{ij}(t_1,t_0) \langle u_{j,t_0}(t')|\\ \hat{\beta}_{t_1,t_0}(t,t') &= \sum_{ij} |u_{i,t_1}(t)\rangle \beta_{ij}(t_1,t_0) \langle v_{j,t_0}(t')|\\ \hat{\gamma}_{t_1,t_0}(t,t') &= \sum_{ij} |v_{i,t_1}(t)\rangle \gamma_{ij}(t_1,t_0) \langle u_{j,t_0}(t')|\\ \hat{\epsilon}_{t_1,t_0}(t,t') &= \sum_{ij} |v_{i,t_1}(t)\rangle \epsilon_{ij}(t_1,t_0) \langle v_{j,t_0}(t')|\end{aligned}$$

I now turn my attention to the third form of Greens functions, which can be generalised from $\frac{\langle vac_{out}| \quad |vac_{in}\rangle}{\langle vac_{out}|vac_{in}\rangle}$ to $\frac{\langle vac_{t_1},h| \quad |vac_{t_0},h\rangle}{\langle vac_{t_1},h|vac_{t_0},h\rangle}$, where t_1 is any chosen 'out time', and t_0 any chosen 'in time'. Substituting $\hat{\psi}(\mathbf{x},t)$ gives:

$$\begin{aligned}\hat{S}^{+}_{t_1,t_0}(t,t')\gamma_0 &= \sum_{ij} |u_{i,t_1}(t)\rangle (\alpha^{\dagger})^{-1}_{ij}(t_1,t_0)\langle u_{j,t_0}(t')|\\ \hat{S}^{-}_{t_1,t_0}(t,t')\gamma_0 &= \sum_{ij} |v_{i,t_0}(t)\rangle (\epsilon^{-1})_{ij}(t_1,t_0)\langle v_{j,t_1}(t')|\end{aligned}$$

where I have used (2.109) and (2.110) to write $\frac{\langle \binom{i}{} out_{t_1},h|\binom{j}{} in_{t_0},h\rangle}{\langle 0,t_1,h|0,t_0,h\rangle} = \overline{\alpha^{-1}_{ji}}$ and $\frac{\langle \binom{}{j} out_{t_1},h|\binom{}{i} in_{t_0},h\rangle}{\langle 0,t_1,h|0,t_0,h\rangle} = (\epsilon^{-1})_{ij}$. The other Greens functions follow from (4.41). For instance

$$\begin{aligned}i\hat{S}_{F;t_1,t_0}(t,t')\gamma_0 = \sum_{ij}\{&\theta(t-t')|u_{i,t_1}(t)\rangle(\alpha^{\dagger})^{-1}_{ij}(t_1,t_0)\langle u_{j,t_0}(t')|\\ &-\theta(t'-t)|v_{i,t_0}(t)\rangle(\epsilon)^{-1}_{ij}(t_1,t_0)\langle v_{j,t_1}(t')|\}\end{aligned}$$

So $i\hat{S}_{F;t_1,t_0}(t,t')\gamma_0$ evolves a state from t' to t, but if $t > t'$ it takes as input a state which had positive energy at time t_0 (a positive energy 'in state') and gives as output a state which has positive energy at time t_1 (a positive energy 'out state'), while if $t < t'$ it takes a negative energy 'out state' as input, and gives a negative energy 'in state' as output. It is interesting to see the matrices $(\boldsymbol{\alpha}^{\dagger})^{-1}$ and $\boldsymbol{\epsilon}^{-1}$ appearing here. Schematically we can identify $i\hat{S}_{F;t_1,t_0}(t,t')\gamma_0$ with $(\boldsymbol{\alpha}^{\dagger})^{-1}$ for $t > t'$ and with $\boldsymbol{\epsilon}^{-1}$ for $t < t'$, and so (ignoring factors of i and γ_0) we can identify

$$\det(\hat{S}_{F;t_1,t_0}(t,t')) \propto \begin{cases} \overline{\det(\boldsymbol{\alpha}(t_1,t_0))}^{-1} & t > t' \\ \det(\boldsymbol{\epsilon}(t_1,t_0))^{-1} & t < t' \end{cases}$$

Since these determinants differ only by a phase, then the result (2.100) has been justified. For a more concrete derivation of (2.100) the reader is referred to Schwinger[50].

4.6 Conclusion

In this Chapter I have shown how the particle definition and vacuum subtraction scheme introduced in the previous Chapter can be incorporated into a more conventional, field operator based formalism, and I have used this formalism to rederive many of the results of Section 2.3. I have also shown how the various Greens functions, defined in terms of vacuum expectation values of field operators, are related to the evolution operator $\hat{U}_1(t, t_0)$ on $\mathcal{H}$, and to the Bogoliubov matrices, and I have justified (although not proved) the conventional formula for the action functional.

It is now worth saying a few words regarding the formalism introduced in the previous Chapter. Firstly, comparing the immediate evaluation of (2.105) and the ease of derivation of (2.106) and (2.107) with the derivation of (4.18) and (4.19) the reader will hopefully be convinced that the use of Slater determinants gives a much simpler derivation of S-Matrix Elements than that used in this Chapter. The derivation of c_v in this Chapter also obscures somewhat the fact that it is simply the inner product of two Dirac seas. Being able to work in the Schrödinger picture, where S-Matrix elements are seen as the overlap between the evolved state and the desired out state, is a conceptual advantage over the Heisenberg picture description, where the S-Matrix elements are simply overlaps between two different choices of particle interpretation. This conceptual advantage is seen also in comparing the vacuum subtraction scheme with normal ordering, and the choice of τ'. Other advantages of the formalism presented in Chapters 2 and 3 include the ease with which unitarity of the S-Matrix follows from conservation of the Dirac inner product (an ease which applies also to gauge invariance and discrete symmetries, as shown in the next Chapter) and the insights that it affords regarding quantum anomalies. Also, by working directly in terms of the Schrödinger picture states of the system, defined on the observer's hypersurface of simultaneity, my formalism is significantly more useful for the discussion of non-local issues such as state-vector reduction, which have previously been ignored by quantum field theory.

There is also one potential advantage of the formalism of Chapters 2 and 3 which is certainly not just a question of simplicity or conceptual clarity. The Heisenberg picture, and with it the entire formalism of this Chapter, is completely dependent on unitarity of the S-Matrix. The ability to define an inner product between Heisenberg picture states relies on the fact that the inner product between the Schrödinger picture states does not depend on time, and it is not at all clear how this problem could be fixed to allow a Heisenberg picture to work in a non-unitary system. Although non-unitarity has not been encountered outside of the gravitational setting, there has been some work in black hole physics[47], and in quantum gravity generally[82], that seems to suggest that an ability to handle non-unitary systems may be crucial to the advancement of quantum gravity. If this proves true, then the formalism of Chapters 2 and 3 may prove to be applicable in situations where this Chapter's formalism would become redundant.

Chapter 5

Continuous and Discrete Symmetries

In the next three Chapters our attention will be restricted to inertial observers in electromagnetic backgrounds. In this Chapter I will demonstrate that all the physical predictions of this theory are both gauge invariant and Lorentz covariant, and I will examine the consequences of parity reversal, charge conjugation, and time reversal in an electromagnetic background. We will see how the symmetries of the full multi-particle theory follow straightforwardly from the symmetries of the ordinary 'first quantized' Dirac equation. My treatment of discrete symmetries will be different to that used conventionally, for a number of reasons. Firstly, conventional approaches to CPT symmetries in Q.F.T are heavily dependent on momentum eigenstates, and reference to the free theory, and I will remove this dependence here. Secondly, conventional treatments are not suited to working in the Schrödinger picture. This is most evident in the treatment of time-reversal, where we often see the time-reversal 'operator' $\hat{T}$ defined on Dirac states by[83] $\hat{T}\psi(\mathbf{x},t) = T\psi^*(\mathbf{x},-t)$, where in the Dirac representation the matrix T is $T = i\gamma_1\gamma_3$. But in the Schrödinger picture a 'state' is a spinor valued function of space, $\psi(\mathbf{x})$, while time is an external variable, used to describe the evolution of the state. An operator on state space must map a given $\psi(\mathbf{x})$ to another spinor-valued function of space, $\hat{T}\psi(\mathbf{x})$; it cannot depend upon the time-dependence of the state! The fact that time-reversed states evolve backwards in time is a property of the evolution of these states, and should be *deduced* as a *consequence* of the time-reversal operation, rather than as part of the definition of this operation.

5.1 Gauge Invariance and Lorentz Covariance

I now demonstrate that all the physical predictions of this theory are both gauge invariant and Lorentz covariant. First notice that physical predictions come entirely from inner products on state space, and expectation values of operators on state space, and that each of these can be expressed in terms of inner products and expectation values on $\mathcal{H}$. This fact makes the study of continuous symmetries much more straightforward than in conventional treatments, since we need only demonstrate invariance at the level of the ordinary Dirac equation.

For gauge invariance, consider equation (2.4), but with an electromagnetic potential

$A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu\Omega(x)$ for some scalar field Ω.

$$i\gamma^\mu(\partial_\mu + ieA_\mu + \partial_\mu\Omega)\psi = m\psi \tag{5.1}$$

We know already that we can generate a complete set of solutions $\{\psi'_\alpha(x) : \alpha \in I\}$ of equation (5.1) from a complete set of solutions $\{\psi_\alpha(x) : \alpha \in I\}$ of equation (2.4) simply by putting $\psi'_\alpha(x) = \psi_\alpha(x)e^{i\Omega(x)}$. It is clear that the Dirac inner product (2.5) is invariant under this transformation. The other important property to demonstrate is that energy, (and hence the choice of $\mathcal{H}^\pm(t_0)$) is invariant under changes of gauge. I.e. that if $\hat{H}_1(t) : \psi \to -i\gamma^0\gamma^k(\partial_k + ieA_k)\psi + m\gamma^0\psi$ and $\hat{H}'(t) : \psi \to -i\gamma^0\gamma^k(\partial_k + ieA'_k)\psi + m\gamma^0\psi$ then $\hat{H}'_1(t)(\psi') = \hat{H}_1(t)(\psi)$ for all t, ψ. This is easily demonstrated by direct calculation. Similarly, we can show that the covariant 3-momenta $\hat{P}_k : \psi \to i\frac{\partial}{\partial x^k}\psi - eA_k\psi$ are also gauge invariant. (Operators such as $\frac{\partial}{\partial x^k}$, and $\hat{H}_{ev}(t)$ are however not gauge invariant.)

It is worth noting now that although all physical predictions are independent of the choice of gauge, the states themselves do of course change under a change in gauge, and in this respect the vacuum state is no exception. The fact that the vacuum state in this construction is not gauge invariant is a fact that differs from conventional treatments, and is at first sight quite alarming. However, there are two reasons why this does not matter. The first is of course that since physical predictions are all independent of gauge, then the dependence, or otherwise, of particular states is of no significance. The second comforting point is that if we so desired, we could still enforce gauge invariance on the vacuum state. To see this, note that the generator of changes of phase in $\mathcal{F}_\wedge(\mathcal{H})$ is just the Hermitian extension of the generator in $\mathcal{H}$, so that by replacing this with the vacuum subtracted Hermitian extension, we could define a (physically equivalent) phase transformation under which the vacuum was invariant. In this way we would conclude that the generator of global phase changes was indeed the charge operator.

Lorentz Covariance

To investigate Lorentz covariance, consider the Lorentz transformation[83]:

$$\psi(x) \to \psi'(x') = S(L)\psi(x) \qquad A^\mu(x) \to A'^\mu(x') = L^\mu{}_\nu A^\nu(x) \tag{5.2}$$

where $x'^\mu = L^\mu{}_\nu x^\nu$ Notice that $n'_\nu A'^\nu(x') = n_\nu A^\nu(x)$ for any vector n_ν which transforms appropriately under Lorentz transformations, so that if $n_\nu = \delta^0_\nu$ is the normal to the hypersurfaces $t = t_0$, then $n'_\nu = L_\nu{}^0$ is the normal to hypersurfaces $t' = t_0$, and $n'_\nu A'^\nu(x') = A^0(x)$ as expected. Notice also that knowing the value of $A'^\nu(x')$ on the hypersurface $t' = t_0$ only requires knowing $A^\nu(x)$ on the hypersurface $t = t_0$.

Now, demonstrating the Lorentz covariance of my construction requires verifying three facts:

1. If $\psi(x)$ is a solution of the Dirac equation in background $A^\mu(x)$ then $\psi'(x')$ is a solution of the Dirac equation in background $A'^\mu(x')$. That is, if $\psi(x)$ solves (2.4), then $\psi'(x')$ satisfies

$$i\gamma^\mu(\partial'_\mu + ieA'_\mu(x'))\psi'(x') = m\psi'(x') \tag{5.3}$$

To show this, start with (2.4), and substitute $A_\mu(x) = L^\nu{}_\mu A'_\nu(x')$, $\psi(x) = S^{-1}(L)\psi'(x')$ (from (5.2)) and $\partial_\mu = L^\nu{}_\mu \partial'_\nu$ (from $x'^\mu = L^\mu{}_\nu x^\nu$) to get:

$$iL^\nu{}_\mu\gamma^\mu(\partial'_\nu + ieA'_\nu(x'))S^{-1}(L)\psi'(x') = mS^{-1}(L)\psi'(x') \tag{5.4}$$

(5.3) then follows if we multiply through by $S(L)$ and use the identity:

$$L^\nu{}_\mu\gamma^\mu = S^{-1}(L)\gamma^\nu S(L) \tag{5.5}$$

which defines the matrices $S(L)$.

2. For all $\phi(x), \psi(x)$

$$\langle\psi'(x')|\phi'(x')\rangle_{t'=t_0} = \langle\psi(x)|\phi(x)\rangle_{t=t_0} \tag{5.6}$$

where ψ', ϕ' are related to ψ, ϕ by equation (5.2) and the subscript $t' = t_0$ denotes that the inner product on the left hand side is over the hypersurface $t' = t_0$ while on the right hand side the integral is over the hypersurface $t = t_0$.

To show this, simply substitute (5.2) and use (5.5) to deduce that:

$$\bar{\psi}'(x')\gamma^\mu\phi'(x') = L^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\phi(x) \tag{5.7}$$

so that, with $n_\mu = \delta^0_\mu$ and $n'_\nu = L_\mu{}^0$ we have $n'_\mu\bar{\psi}'(x')\gamma^\mu\phi'(x') = n_\mu\bar{\psi}(x)\gamma^\mu\phi(x)$ so that:

$$\int d^3\mathbf{x}' n'_\mu\bar{\psi}'(x')\gamma^\mu\phi'(x') = \int d^3\mathbf{x} n_\mu\bar{\psi}(x)\gamma^\mu\phi(x) \tag{5.8}$$

as required. It is important to note here that this result does not depend on whether or not ψ or ϕ actually solve the Dirac equation. Of course, if they solve the Dirac equation then the inner product becomes independent of hypersurface, and we can drop the subscripts in (5.6) - this is important for unitarity, but is not directly relevant to Lorentz Covariance. However, (5.6) will hold (with the subscripts) regardless of the dynamics of ψ and ϕ. The presence of the subscripts in (5.6) is just a reminder that when we transform the system we must also transform any initial conditions that were applied to the system. This can be made clear by considering a simple analogy - say the temperature distribution across a metal plate, when one edge of the plate is kept at a certain temperature. We would certainly not expect this distribution to change upon rotating the plate a fixed amount, although it is clear that the boundary condition is still to be applied to the edge of the plate (in its new rotated position) not to the line in space where the edge used to be!

3. $\mathcal{H}^{\pm'}(t' = t_0) = \mathcal{H}^{\pm}(t = t_0)$. That is, the particle interpretation of the transformed system on the transformed hypersurface is equivalent to the original particle interpretation on the original hypersurface. To show this, consider first the energy-momentum tensor $T^{\mu}{}_{\nu,\psi}(x)$ formed from the state $\psi(x)$. A straightforward calculation, similar to the derivation of (5.7) gives $T'^{\mu}{}_{\nu,\psi'}(x') = L^{\mu}{}_{\sigma} L_{\nu}{}^{\rho} T^{\sigma}{}_{\rho,\psi}(x)$ from which we obtain:

$$\int d^3\mathbf{x}' T'^{\mu}{}_{\nu,\psi'}(x') n'_{\mu} n'^{\nu} = \int d^3\mathbf{x} T^{\mu}{}_{\nu,\psi}(x) n_{\mu} n^{\nu} \tag{5.9}$$

That is:

$$\langle \psi'(x')|\hat{H'}_1(t')|\psi'(x')\rangle = \langle \psi(x)|\hat{H}_1(t)|\psi(x)\rangle \tag{5.10}$$

That $\mathcal{H}^{\pm'}(t' = t_0) = \mathcal{H}^{\pm}(t = t_0)$ now follows directly from the alternative definition of $\mathcal{H}^{\pm}(t = t_0)$ given on page 13.

So, I have shown that the Lorentz transformed system, as observed by the Lorentz transformed observer (i.e. in the Lorentz transformed reference frame), is identical to the original system as observed by the original observer. This is precisely what is meant by Lorentz covariance, as discussed in Bjorken and Drell[83] for instance. Notice that it contains no statements about the relation between results of local measurements (at a point) made by observers in relative motion (at that point). Indeed, in Bjorken and Drell's discussion[83] of Lorentz covariance, they require only "that there must be an explicit prescription which allows observer O', given the $\psi(x)$ of observer O, to compute the $\psi'(x')$ which describes to O' the same physical state", and that "$\psi'(x')$ will be a solution of an equation which takes the form of [(2.4)] in the primed system". Questions such as how two observers in relative motion would relate measurements of the Energy momentum tensor ($T_{\mu\nu}(x)$ and $T'_{\mu\nu}(x')$) at their point of crossing, which are commonly asked of classical systems, can not be asked of quantum mechanical systems. This is because the 4-momentum operators $\hat{p}_{\mu}$ (the expectation value of which is the spatial integral of the energy momentum tensor) do not commute with the position operators $\hat{x}^{\mu}$, so we cannot simply 'knock off the spatial integral' in order to get a measure of $T_{\mu\nu}(x)$. This is also demonstrated at the 'second quantized' level, by the fact that, even when we can define a finite quantity $\langle \hat{T}_{\mu\nu}(x)\rangle$ to be interpreted as the 'expected energy momentum tensor', we find that the fluctuations of this quantity at a point are infinite, and can only be made finite by averaging the quantity $\langle \hat{T}_{\mu\nu}(x)\hat{T}_{\mu\nu}(y)\rangle$ over a finite region about the point x.

5.2 Parity Reversal

Define the (linear) *parity reversal operator* $\hat{P}_1 : \mathcal{H} \to \mathcal{H}$ by:

$$\hat{P}_1 : \psi(\mathbf{x}) \to \gamma_0 \psi(-\mathbf{x}) \tag{5.11}$$

By substituting this into the inner product, (2.5) it is easy to see that

$$\hat{P}_1^{\dagger} = \hat{P}_1 = \hat{P}_1^{-1}$$

so that $\hat{P}_1$ is unitary. We can also substitute $\hat{P}_1\psi(\mathbf{x})$ into equation (2.8) to see that:

$$\hat{P}_1 \hat{H}_1^{A^\mu(\mathbf{x})} = \hat{H}_1^{A_P^\mu(\mathbf{x})} \hat{P}_1 \tag{5.12}$$

$$\text{where } A_P^0(\mathbf{x}) \equiv A^0(-\mathbf{x}) \quad \text{and} \quad A_P^k(\mathbf{x}) \equiv -A^k(-\mathbf{x}, t_0) \tag{5.13}$$

That is: $A_P^\mu(\mathbf{x}) = A_\mu(-\mathbf{x})$. $\hat{H}_1^{A^\mu(\mathbf{x})}$ is the Dirac Hamiltonian in the presence of the background $A^\mu(\mathbf{x})$, and the same result holds for $\hat{H}_{ev}$.

This has a number of consequences:

- If $\psi(\mathbf{x})$ is an eigenstate of $\hat{H}_1^{A^\mu(\mathbf{x})}$ with eigenvalue E, then $\hat{P}_1\psi(\mathbf{x})$ is an eigenstate of $\hat{H}_1^{A_P^\mu(\mathbf{x})}$, also with eigenvalue E. In particular $\hat{P}_1 : \mathcal{H}^\pm(A^\mu(\mathbf{x})) \to \mathcal{H}^\pm(A_P^\mu(\mathbf{x}))$, so that parity reversal maps particles to particles, and antiparticles to antiparticles. Note that I have used the notation $\mathcal{H}^\pm(A^\mu(\mathbf{x}))$ here, (rather than $\mathcal{H}^\pm(t)$) to denote the positive/negative spectrum of $\hat{H}_1^{A^\mu(\mathbf{x})}$.

- If $\psi(\mathbf{x}, t)$ is a solution of the Dirac equation in potential $A^\mu(\mathbf{x}, t)$ then $\hat{P}_1\psi(\mathbf{x}, t)$ is a solution of the Dirac equation in potential $A_P^\mu(\mathbf{x}, t)$. To see that this is a consequence of (5.13), operate on (2.7) with $\hat{P}_1$ and use (5.13) to deduce that $i\frac{\partial}{\partial t}\hat{P}_1\psi(\mathbf{x}, t) = \hat{H}_{ev}^{A_P(\mathbf{x},t)}\hat{P}_1\psi(\mathbf{x}, t)$ as required. (We could also deduce the same result by substituting $\hat{P}_1\psi(\mathbf{x}, t)$ directly into the Dirac equation (2.4).) In terms of the evolution operator $\hat{U}_1(t_1, t_0)$ this can be written as:

$$\hat{P}_1\hat{U}_1^A(t_1, t_0) = \hat{U}_1^{A_P}(t_1, t_0)\hat{P}_1 \tag{5.14}$$

  In terms of S-matrix elements (on $\mathcal{H}$) this is

$$\begin{aligned} S^A_{\phi_{t_1},\psi_{t_0}}(t_1, t_0) &\equiv \langle\phi_{t_1}|\hat{U}_1^A(t_1, t_0)|\psi_{t_0}\rangle \\ &= \langle\phi_{t_1}|\hat{P}_1^\dagger\hat{U}_1^{A_P}(t_1, t_0)\hat{P}_1|\psi_{t_0}\rangle \qquad \text{from (5.14)} \\ &= \langle\hat{P}_1\phi_{t_1}|\hat{U}_1^{A_P}(t_1, t_0)|\hat{P}_1\psi_{t_0}\rangle \qquad \equiv S^{A_P}_{\hat{P}_1\phi_{t_1},\hat{P}_1\psi_{t_0}}(t_1, t_0) \end{aligned} \tag{5.15}$$

  Combined with the fact that $\hat{P}_1 : \mathcal{H}^\pm(A(\mathbf{x}, t_0)) \to \mathcal{H}^\pm(A_P(\mathbf{x}, t_0))$ this gives:

$$\begin{aligned} \alpha^A_{\phi_{t_1},\psi_{t_0}}(t_1, t_0) &= \alpha^{A_P}_{\hat{P}_1\phi_{t_1},\hat{P}_1\psi_{t_0}}(t_1, t_0) \qquad & \beta^A_{\phi_{t_1},\psi_{t_0}}(t_1, t_0) &= \beta^{A_P}_{\hat{P}_1\phi_{t_1},\hat{P}_1\psi_{t_0}}(t_1, t_0) \\ \gamma^A_{\phi_{t_1},\psi_{t_0}}(t_1, t_0) &= \gamma^{A_P}_{\hat{P}_1\phi_{t_1},\hat{P}_1\psi_{t_0}}(t_1, t_0) \qquad & \epsilon^A_{\phi_{t_1},\psi_{t_0}}(t_1, t_0) &= \epsilon^{A_P}_{\hat{P}_1\phi_{t_1},\hat{P}_1\psi_{t_0}}(t_1, t_0) \end{aligned}$$

To extend the action of $\hat{P}_1$ to $\mathcal{F}_\wedge(\mathcal{H})$ simply define $\hat{P} : \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$ by $\hat{P} \equiv \hat{P}_{1,U}$ This is also unitary, and satisfies

$$\hat{P}\hat{U}^A(t_1, t_0) = \hat{U}^{A_P}(t_1, t_0)\hat{P} \tag{5.16}$$

where $\hat{U}^A(t_1, t_0)$ is now the full evolution operator on $\mathcal{F}_\wedge(\mathcal{H})$. Again we see here how straightforward it is to move from the first quantized to the second quantized treatment of symmetries, something which is quite non-trivial in conventional formulations of quantum field theory. A consequence of (5.16) is of course that (5.15) holds on all of state space. Also, since $\hat{P}_1 : \mathcal{H}^+(A(\mathbf{x}, t_0)) \to \mathcal{H}^+(A_P(\mathbf{x}, t_0))$ we get:

$$\hat{P}|vac_{A,t_0}\rangle = \eta_{vac}|vac_{A_P,t_0}\rangle$$

where η is some unknown, and unimportant phase. I will assume $\eta_{vac} = 1$ from now on (we can easily redefine $\hat{P}_{phys} = \eta^*_{vac}\hat{P}$ to achieve this anyway).

Now, since parity reversing a solution of the Dirac equation in a background $A^\mu(\mathbf{x}, t)$ gives a solution of the Dirac equation in a *different* background $A^\mu_P(\mathbf{x}, t)$, I refer to the Dirac equation in an arbitrary electromagnetic background as *Parity-reversal Covariant.* This is again slightly unconventional, since the term 'Parity-reversal Invariant' is often used for this. I prefer to define the term 'Parity-reversal Invariant' more restrictively. Namely " The Dirac equation in an electromagnetic background $A^\mu(\mathbf{x}, t)$ is *Parity-reversal Invariant* IFF $A^\mu_P(\mathbf{x}, t) = A^\mu(\mathbf{x}, t)$." Thus, in a parity-reversal invariant theory we can use parity-reversal invariance to deduce relationships between the S-Matrix elements of the system. If the theory is just parity-reversal covariant, then we can use this to deduce the S-Matrix elements of the parity-reversed system from those of the original system. For instance, in section 7.4 I consider a constant electric field in the negative z-direction, the parity reverse of which is a constant electric field in the positive z-direction. Hence, although this theory is not parity-reversal invariant, we can use parity-reversal covariance to allow us to deduce everything we need to know about the 'positive z-direction' case, from the corresponding results in the 'negative z-direction' case.

It is also worth noting:

1. I could easily have included an arbitrary phase in (5.11), by writing $\hat{P}_1\psi(\mathbf{x}) = \gamma_0\psi(-\mathbf{x})\eta^*_P$. Notice that then $\hat{P}_1^\dagger\psi(\mathbf{x}) = \hat{P}_1^{-1}\psi(\mathbf{x}) = \gamma_0\psi(-\mathbf{x})\eta_P$ so that $\hat{P}_1$ would still be unitary, but would not be Hermitian (this of course has no effect on any of its physical properties, except that its eigenvalues would be $\pm\eta_P$ rather than ± 1). Also, if we expanded state space at some time t_0 in terms of a set of normal modes $\psi_{\alpha,t_0}(\mathbf{x})$, then we could even allow η_P to be different on each normal mode. This would be slightly cumbersome however, since then to find the action of $\hat{P}$ on some state $\psi_t(\mathbf{x})$ defined at some later time t we would have to expand it in terms of the evolved normal mode solutions $\psi_{\alpha,t_0}(\mathbf{x}, t)$ and use $\eta_P(\psi_{\alpha,t_0}(\mathbf{x}, t)) = \eta_P(\psi_{\alpha,t_0}(\mathbf{x}))$.

2. In the case of free Dirac theory (which is of course parity reversal invariant) , we can define the plane wave states (in the Dirac representation):

$$u_{\mathbf{p},\pm 1/2;t_0}(\mathbf{x}) = \sqrt{\frac{E_\mathbf{p} + m}{2E_\mathbf{p}}} \begin{bmatrix} \phi_{\pm 1/2} \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_\mathbf{p}+m}\phi_{\pm 1/2} \end{bmatrix} e^{-i(E_\mathbf{p}t_0 - \mathbf{p}\cdot\mathbf{x})} \tag{5.17}$$

and

$$v_{\mathbf{p},\pm 1/2;t_0}(\mathbf{x}) = \mp\sqrt{\frac{E_\mathbf{p} + m}{2E_\mathbf{p}}} \begin{bmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_\mathbf{p}+m}\phi_{\mp 1/2} \\ \phi_{\mp 1/2} \end{bmatrix} e^{i(E_\mathbf{p}t_0 - \mathbf{p}\cdot\mathbf{x})} \tag{5.18}$$

where $\boldsymbol{\sigma}\cdot\mathbf{p} \equiv p^k\sigma^k$, σ^k are the 2×2 Pauli matrices, $\phi_{+1/2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\phi_{-1/2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. From these we can define the 1-particle states:

$$|(^{\mathbf{p},\lambda})\, in_{t_0}\rangle \equiv \overrightarrow{u}_{\mathbf{p},\lambda;t_0} \wedge |vac_{t_0}\rangle \qquad |(_{\mathbf{p},\lambda})\, in_{t_0}\rangle \equiv i_{\overrightarrow{v}_{\mathbf{p},\lambda;t_0}} |vac_{t_0}\rangle \tag{5.19}$$

At the first quantized level, $u_{\mathbf{p},\pm 1/2;t_0}(\mathbf{x})$ represents states of momentum $\mathbf{p}$ and spin 'up' or 'down' respectively, while $v_{\mathbf{p},\pm 1/2;t_0}(\mathbf{x})$ represents negative-energy states of momentum $-\mathbf{p}$ and spin 'down', or 'up' respectively. Then, at the second quantized level, $|(^{\mathbf{p},\pm 1/2})\, in_{t_0}\rangle$ represents particles of momentum $\mathbf{p}$, and spin up/down respectively, while since $\overrightarrow{v}_{\mathbf{p},\pm 1/2;t_0}(\mathbf{x})$ is removed from the vacuum in order to get the 1-antiparticle state, $|(_{\mathbf{p},\pm 1/2})\, in_{t_0}\rangle$, then this state represents antiparticles of momentum $\mathbf{p}$, and spin up/down respectively, as required. It is straightforward to verify that $u_{\mathbf{p},\pm 1/2;t_0}(\mathbf{x}), v_{\mathbf{p},\pm 1/2;t_0}(\mathbf{x})$ satisfy:

$$\hat{P}_1 u_{\mathbf{p},\lambda;t_0}(\mathbf{x}) = u_{-\mathbf{p},\lambda;t_0}(\mathbf{x})\eta_P^* \qquad \hat{P}_1 v_{\mathbf{p},\lambda;t_0}(\mathbf{x}) = -v_{-\mathbf{p},\lambda;t_0}(\mathbf{x})\eta_P^*$$

where I have included an overall η_P^* as in the previous note. In terms of creation and annihilation operators this gives:

$$\hat{P} a_\lambda(\mathbf{p}) = \eta_P a_\lambda(-\mathbf{p})\hat{P} \qquad \hat{P} b_\lambda(\mathbf{p}) = -\eta_P^* b_\lambda(-\mathbf{p})\hat{P}$$

in agreement with conventional treatments.

5.3 Time Reversal

Define the *time reversal operator* $\hat{T}_1 : \mathcal{H} \to \mathcal{H}$ by:

$$\hat{T}_1 : \psi(\mathbf{x}) \to T\psi^*(\mathbf{x}) \tag{5.20}$$

where in the Dirac representation the matrix T is given by $T = i\gamma_1\gamma_3$. Since this operation is *antilinear*, then its adjoint is defined not by $\langle \hat{A}^\dagger\phi|\psi\rangle = \langle\phi|\hat{A}\psi\rangle$, but rather, by:

$$\langle\phi|\hat{T}_1\psi\rangle = \overline{\langle\hat{T}_1^\dagger\phi|\psi\rangle} = \langle\psi|\hat{T}_1^\dagger\phi\rangle$$

That is, although we still have $\langle\phi|\hat{T}_1|\psi\rangle \equiv \langle\phi|\hat{T}_1\psi\rangle$ we now have $\langle\phi|\hat{T}_1|\psi\rangle = \overline{\langle\hat{T}_1^\dagger\phi|\psi\rangle}$. It is now straightforward to verify that

$$\hat{T}_1^\dagger = -\hat{T}_1 = \hat{T}_1^{-1}$$

so that $\hat{T}_1$ is antiunitary. We can substitute $\hat{T}_1\psi(\mathbf{x})$ into equation (2.8) as before, to see that:

$$\hat{T}_1 \hat{H}_1^{A^\mu(\mathbf{x})} = \hat{H}_1^{A_\mu(\mathbf{x})}\hat{T}_1 \tag{5.21}$$

and the same result holds for $\hat{H}_{ev}$. This means that $\hat{T}_1$ maps particles to particles, and antiparticles to antiparticles, just as with $\hat{P}_1$. However, because of the antilinearity of $\hat{T}_1$ we now have:

$$\hat{T}_1(-i\hat{H}_1^{A^\mu(\mathbf{x})}) = -(-i\hat{H}_1^{A_\mu(\mathbf{x})})\hat{T}_1 \tag{5.22}$$

This is why time-reversed solutions of the Dirac equation evolve backwards in time! To examine this in more detail, it is convenient to treat the cases of time-reversal invariance, and time-reversal covariance separately.

Time Reversal Invariance: We say that the Dirac equation in a background $A^\mu(\mathbf{x}, t)$ possesses *a time-reversal invariance* if for some t_0 and t_1 we have:

$$A^\mu(\mathbf{x}, t_0 + \tau) = A_\mu(\mathbf{x}, t_1 - \tau) \text{ for } \tau \in [0, T] \text{ for some } T > 0$$

Notice that a particular background can possess many time-reversal invariances, and that a time-reversal invariance is useful even when it exists only for a finite period of time.

Now (5.21) gives

$$\hat{T}_1\hat{H}_1^{A^\mu(\mathbf{x},t_0+\tau)} = \hat{H}_1^{A^\mu(\mathbf{x},t_1-\tau)}\hat{T}_1$$

so that $\hat{T}_1 : \mathcal{H}^\pm(t_0 + \tau) \to \mathcal{H}^\pm(t_1 - \tau)$. That is, $\hat{T}_1$ maps particle/antiparticle 'in' states to particle/antiparticle 'out' states, and vice-versa. Now (5.22) gives:

$$\hat{U}_1(t_1 - \tau, t_1)\hat{T}_1 = \hat{T}_1\hat{U}_1(t_0 + \tau, t_0) \text{ for } \tau \in [0, T] \tag{5.23}$$

In particular, if $t_1 > t_0$ and $T \geq t_1 - t_0$ then $\hat{U}_1(t_0, t_1)\hat{T}_1 = \hat{T}_1\hat{U}_1(t_1, t_0)$ or in a more familiar form $\hat{T}_1\hat{U}_1(t_1, t_0)\hat{T}_1^\dagger = \hat{U}_1^\dagger(t_1, t_0)$. In terms of S-Matrix elements, we can now write:

$$\begin{aligned}
S_{\phi_{t_0+\tau},\psi_{t_0}}(t_0 + \tau, t_0) &= \langle\phi_{t_0+\tau}|\hat{U}_1(t_0 + \tau, t_0)|\psi_{t_0}\rangle \\
&= \langle\phi_{t_0+\tau}|\hat{T}_1^\dagger\hat{U}_1(t_1 - \tau, t_1)\hat{T}_1|\psi_{t_0}\rangle \\
&= \overline{\langle\hat{T}_1\phi_{t_0+\tau}|\hat{U}_1(t_1 - \tau, t_1)|\hat{T}_1\psi_{t_0}\rangle} \\
&= \langle\hat{T}_1\psi_{t_0}|\hat{U}_1(t_1, t_1 - \tau)|\hat{T}_1\phi_{t_0+\tau}\rangle \\
&= S_{\hat{T}_1\psi_{t_0},\hat{T}_1\phi_{t_0+\tau}}(t_1, t_1 - \tau)
\end{aligned} \tag{5.24}$$

From which we get

$$\begin{aligned}
\alpha_{\phi_{t_0+\tau},\psi_{t_0}}(t_0 + \tau, t_0) &= \alpha_{\hat{T}_1\psi_{t_0},\hat{T}_1\phi_{t_0+\tau}}(t_1, t_1 - \tau) \\
\beta_{\phi_{t_0+\tau},\psi_{t_0}}(t_0 + \tau, t_0) &= \gamma_{\hat{T}_1\psi_{t_0},\hat{T}_1\phi_{t_0+\tau}}(t_1, t_1 - \tau) \\
\gamma_{\phi_{t_0+\tau},\psi_{t_0}}(t_0 + \tau, t_0) &= \beta_{\hat{T}_1\psi_{t_0},\hat{T}_1\phi_{t_0+\tau}}(t_1, t_1 - \tau) \\
\epsilon_{\phi_{t_0+\tau},\psi_{t_0}}(t_0 + \tau, t_0) &= \epsilon_{\hat{T}_1\psi_{t_0},\hat{T}_1\phi_{t_0+\tau}}(t_1, t_1 - \tau)
\end{aligned}$$

We can extend the action of $\hat{T}_1$ to $\mathcal{F}_\wedge(\mathcal{H})$ by defining $\hat{T} \equiv \hat{T}_{1,U}$ just as with parity reversal. $\hat{T}$ is still (anti)unitary, and (5.23) and (5.24) both hold on all of state space. We then have

$$\hat{T}|vac_{t_0+\tau}\rangle = \eta_{vac}|vac_{t_1-\tau}\rangle \text{ and } \hat{T}|vac_{t_0}(t_0 + \tau)\rangle = \eta_{vac}|vac_{t_1}(t_1 - \tau)\rangle$$

where η_{vac} is some unknown, and unimportant phase, which I put equal to 1 for convenience.

Time Reversal Covariance: Since the time-reversed system will be different to the original system now, there is little use in keeping arbitrary t_0, t_1, etc, so I will just consider time reversal about $t = 0$ now, as in conventional treatments. For this purpose, define the *time-reversed background* $A^\mu_T(\mathbf{x}, t) \equiv A_\mu(\mathbf{x}, -t)$ so that (5.21) becomes:

$$\hat{T}_1 \hat{H}_1^{A^\mu(\mathbf{x},t)} = \hat{H}_1^{A^\mu_T(\mathbf{x},-t)} \hat{T}_1$$

Then $\hat{T}_1 : \mathcal{H}^\pm(A^\mu(\mathbf{x}, t)) \to \mathcal{H}^\pm(A^\mu_T(\mathbf{x}, -t))$ and $\hat{T}_1 \hat{U}_1^A(t_1, t_0) = \hat{U}_1^{A_T}(-t_1, -t_0)\hat{T}_1$. That is, if $\psi(\mathbf{x}, t)$ is a solution of the Dirac equation in potential $A^\mu(\mathbf{x}, t)$, then $\hat{T}_1\psi(\mathbf{x}, -t)$ is a solution of the Dirac equation in potential $A^\mu_T(\mathbf{x}, t)$ In terms of S-Matrix elements we now have:

$$S^A_{\phi_{t_1},\psi_{t_0}}(t_1, t_0) = S^{A_T}_{\hat{T}_1\psi_{t_0},\hat{T}_1\phi_{t_1}}(-t_0, -t_1) \tag{5.25}$$

so that

$$\begin{aligned}
\alpha^A_{\phi_{t_1},\psi_{t_0}}(t_1, t_0) &= \alpha^{A_T}_{\hat{T}_1\psi_{t_0},\hat{T}_1\phi_{t_1}}(-t_0, -t_1) \\
\beta^A_{\phi_{t_1},\psi_{t_0}}(t_1, t_0) &= \gamma^{A_T}_{\hat{T}_1\psi_{t_0},\hat{T}_1\phi_{t_1}}(-t_0, -t_1) \\
\gamma^A_{\phi_{t_1},\psi_{t_0}}(t_1, t_0) &= \beta^{A_T}_{\hat{T}_1\psi_{t_0},\hat{T}_1\phi_{t_1}}(-t_0, -t_1) \\
\epsilon^A_{\phi_{t_1},\psi_{t_0}}(t_1, t_0) &= \epsilon^{A_T}_{\hat{T}_1\psi_{t_0},\hat{T}_1\phi_{t_1}}(-t_0, -t_1)
\end{aligned}$$

Defining $\hat{T} = \hat{T}_{1,U}$ as before, (5.25) follows over to all of state space, and we have

$$\hat{T}|vac_{A,t_0}\rangle = \eta_{vac}|vac_{A_T,-t_0}\rangle \text{ and } \hat{T}|vac_{A,t_0}(t)\rangle = \eta_{vac}|vac_{A_T,-t_0}(-t)\rangle$$

as expected.

Now, just as in the previous Section, it is worth noting:

1. I could have included an arbitrary phase in (5.20), just as with $\hat{P}_1$ earlier, by writing $\hat{T}_1 : \psi(\mathbf{x}) \to T\psi^*(\mathbf{x})\eta^*_T$ as before. Due to the antilinearity of $\hat{T}_1$ we would still have $\hat{T}_1^\dagger = -\hat{T}_1 = \hat{T}_1^{-1}$ so we would still have $\hat{T}_1^2 = -1$. On $\mathcal{F}_\wedge(\mathcal{H})$ this gives $\hat{T}^2 = (-1)^F$ (where F counts the number of fermions in a given state, and I assume that the number of degrees of freedom in the vacuum is even[1], so that these can be ignored) as expected.

2. Considering the action of $\hat{T}_1$ on the plane wave states of free Dirac theory, as defined in (5.17) and (5.19), we find:

$$\hat{T}_1 u_{\mathbf{p},\lambda;t_0}(\mathbf{x}) = (-1)^{\frac{1}{2}+\lambda} i u_{-\mathbf{p},-\lambda;t_0}(\mathbf{x})\eta^*_T \qquad \hat{T}_1 v_{\mathbf{p},\lambda;t_0}(\mathbf{x}) = (-1)^{\frac{1}{2}+\lambda} i v_{-\mathbf{p},-\lambda;t_0}(\mathbf{x})\eta^*_T$$

[1]Since the grade of the vacuum is generally infinite, it may seem unreasonable to refer to it as even. However, this is justified by the fact that we have two (negative energy) spin degrees of freedom for each momentum, so that the grade of the vacuum is an infinite multiple of two. I will use this assumption again in the charge conjugation Section.

where $\lambda = \pm\frac{1}{2}$ for spin up/down respectively. In terms of creation and annihilation operators this gives:

$$\hat{T}a_\lambda(\mathbf{p}) = (i\eta_T)(-1)^{\frac{1}{2}+\lambda}a_{-\lambda}(-\mathbf{p})\hat{T} \qquad \hat{T}b_\lambda(\mathbf{p}) = (i\eta_T)^*(-1)^{\frac{1}{2}+\lambda}b_{-\lambda}(-\mathbf{p})\hat{T}$$

which agrees with conventional treatments, except for the overall factor of i, which can easily be absorbed into η_T, so has no physical significance.

5.4 Charge Conjugation

Define the *charge conjugation operator* $\hat{C}_1 : \mathcal{H} \to \mathcal{H}$ by:

$$\hat{C}_1 : \psi(\mathbf{x}) \to C\bar{\psi}(\mathbf{x})^T \tag{5.26}$$

where in the Dirac representation the matrix C is given by $C = i\gamma_0\gamma_2$. (So $\hat{C}_1\psi(\mathbf{x}) = -i\gamma_2\psi^*(\mathbf{x})$ in the Dirac representation.) We see here that this operation is also *antilinear* on solutions of the Dirac equation. However, the action of $\hat{C}$ on multiparticle states is generally linear. We will see soon how this difference will be resolved, leading to an operator $\hat{C} : \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$ which is indeed linear.

It is easily checked that:

$$\hat{C}_1^\dagger = \hat{C}_1 = \hat{C}_1^{-1} \tag{5.27}$$

so that $\hat{C}_1$ is antiunitary. Substituting $\hat{C}_1\psi(\mathbf{x})$ into equation (2.8) as before, we see that:

$$\hat{C}_1\hat{H}_1^{A^\mu(\mathbf{x})} = -\hat{H}_1^{-A^\mu(\mathbf{x})}\hat{C}_1 \tag{5.28}$$

That is, $A_C^\mu(\mathbf{x}) \equiv -A^\mu(\mathbf{x})$, so although the Dirac equation is charge-conjugation covariant, it is only charge-conjugation invariant if $A = 0$! However, charge conjugation covariance is very useful, since $A^\mu \to -A^\mu$ is equivalent to $e \to -e$, so we can relate the charge-conjugated S-matrix elements to the originals simply by substituting $e \to -e$. We can also see from (5.28) that $\hat{C}_1 : \mathcal{H}^\pm(A^\mu) \to \mathcal{H}^\mp(-A^\mu)$ so that $\hat{C}_1$ maps particles to antiparticles, and vice versa. Also, due to the antilinearity, (5.28) leads to

$$\hat{C}_1(-i\hat{H}_1^{A^\mu}) = (-i\hat{H}_1^{-A^\mu})\hat{C}_1 \tag{5.29}$$

so that

$$\hat{C}_1\hat{U}_1^A(t_1,t_0) = \hat{U}_1^{-A}(t_1,t_0)\hat{C}_1 \tag{5.30}$$

That is; if $\psi(\mathbf{x},t)$ is a solution of the Dirac equation in background $A^\mu(\mathbf{x},t)$, then $\hat{C}_1\psi(\mathbf{x},t)$ is a solution of the Dirac equation in background $-A^\mu(\mathbf{x},t)$. In terms of S-matrix elements on $\mathcal{H}$ this is:

$$S^A_{\phi_{t_1},\psi_{t_0}}(t_1,t_0) = \overline{S^{-A}_{\hat{C}_1\phi_{t_1},\hat{C}_1\psi_{t_0}}(t_1,t_0)} \tag{5.31}$$

which gives:

$$
\begin{aligned}
\alpha^A_{\phi_{t_1},\psi_{t_0}}(t_1,t_0) &= \overline{\epsilon^{-A}_{\hat{C}_1\phi_{t_1},\hat{C}_1\psi_{t_0}}(t_1,t_0)} \\
\beta^A_{\phi_{t_1},\psi_{t_0}}(t_1,t_0) &= \overline{\gamma^{-A}_{\hat{C}_1\phi_{t_1},\hat{C}_1\psi_{t_0}}(t_1,t_0)} \\
\gamma^A_{\phi_{t_1},\psi_{t_0}}(t_1,t_0) &= \overline{\beta^{-A}_{\hat{C}_1\phi_{t_1},\hat{C}_1\psi_{t_0}}(t_1,t_0)} \\
\epsilon^A_{\phi_{t_1},\psi_{t_0}}(t_1,t_0) &= \overline{\alpha^{-A}_{\hat{C}_1\phi_{t_1},\hat{C}_1\psi_{t_0}}(t_1,t_0)}
\end{aligned}
\tag{5.32}
$$

The extension of $\hat{C}_1$ to $\mathcal{F}_\wedge(\mathcal{H})$ is slightly more subtle than for $\hat{P}_1$ or $\hat{T}_1$, since notice that the state $\hat{C}_{1,U}|vac_{t_0}\rangle_A$ is the Slater determinant of a basis of $\mathcal{H}^+(-A^\mu)$, which is certainly not a vacuum state!

Before describing how I fix this problem, it is convenient for the moment to return to the case where $\mathcal{H}$ has finitely many degrees of freedom. Then we can introduce a basis $\{u^A_{i,t_0}, v^A_{i,t_0}; i=1\ldots N\}$ as in Section 2.3, and let $\{u^{-A}_{i,t_0}, v^{-A}_{i,t_0}; i=1\ldots N\}$ be a basis for the charge conjugated system, defined (for convenience) such that $\hat{C}_1 u^A_{i,t_0} = v^{-A}_{i,t_0}$ and $\hat{C}_1 v^A_{i,t_0} = u^{-A}_{i,t_0}$. (It is easy to check that in the case of free Dirac field theory this requirement is satisfied by the choice of $u_{\mathbf{p},\lambda;t_0}(\mathbf{x}), v_{\mathbf{p},\lambda;t_0}(\mathbf{x})$ given earlier.)

Define:

$$
I^+_A \equiv u^A_{1,t_0} \wedge \cdots \wedge u^A_{N,t_0} = \hat{C}_{1,U}|vac_{-A,t_0}\rangle \tag{5.33}
$$

$$
\text{and } I \equiv I^+_A \wedge |vac_{A,t_0}\rangle \tag{5.34}
$$

So I is the outer product of a basis for $\mathcal{H}$. Hence it only differs from $I^+_{-A} \wedge |vac_{t_0}\rangle_{-A}$ by an unimportant phase.

Now, define $\hat{C}: \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$ by:

$$
\hat{C}F \equiv i_{\hat{C}_{1,U}F} I
$$

Notice that the antilinearity of the map $F \to i_F$ cancels the antilinearity of $\hat{C}_1$, so that $\hat{C}$ is now linear on state space. To consider the action of $\hat{C}$ in more detail, consider a state:

$$
|\tbinom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} in_{A,t_0}\rangle = (-)^J u^A_{i_1,t_0} \wedge \ldots u^A_{i_m,t_0} \wedge v^A_{1,t_0} \wedge \ldots \check{v}^A_{j_1,t_0} \cdots \wedge \check{v}^A_{j_n,t_0} \cdots \wedge v^A_{N,t_0}
$$

as in (2.68). Then

$$
\hat{C}_{1,U}|\tbinom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} in_{A,t_0}\rangle = (-)^J v^{-A}_{i_1,t_0} \wedge \ldots v^{-A}_{i_m,t_0} \wedge u^{-A}_{1,t_0} \wedge \ldots \check{u}^{-A}_{j_1,t_0} \cdots \wedge \check{u}^{-A}_{j_n,t_0} \cdots \wedge u^{-A}_{N,t_0}
$$

So that:

$$
\begin{aligned}
\hat{C}|\tbinom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} in_{A,t_0}\rangle = (-)^J i_{u^{-A}_{N,t_0}} \cdots \tbinom{j_n \ldots j_1}{\text{missing}} \cdots i_{u^{-A}_{1,t_0}} i_{v^{-A}_{i_m,t_0}} \cdots \\
\wedge\, i_{v^{-A}_{i_1,t_0}} (u^{-A}_{1,t_0} \wedge \cdots \wedge u^{-A}_{N,t_0} \wedge v^{-A}_{1,t_0} \wedge \cdots \wedge v^{-A}_{N,t_0}) \\
= (-)^{J+\sum_1^m (i_k - k)} i_{u^{-A}_{N,t_0}} \cdots \tbinom{j_n \ldots j_1}{\text{missing}} i_{u^{-A}_{1,t_0}} (u^{-A}_{1,t_0} \wedge \cdots \wedge u^{-A}_{N,t_0} \wedge v^{-A}_{1,t_0} \wedge \ldots \tbinom{i_1 \ldots i_m}{\text{missing}} \wedge v^{-A}_{N,t_0}) \\
= (-)^{\sum_1^n (j_k + k) + \sum_1^m (i_k - k) + \sum_1^n (j_k - k) + n} u^{-A}_{j_1,t_0} \wedge \ldots u^{-A}_{j_n,t_0} \wedge v^{-A}_{1,t_0} \wedge \ldots \check{v}^{-A}_{i_1,t_0} \cdots \wedge \check{v}^{-A}_{i_m,t_0} \cdots \wedge v^{-A}_{N,t_0} \\
= (-)^n |\tbinom{j_1 j_2 \ldots j_n}{i_1 i_2 \ldots i_m} in_{-A,t_0}\rangle
\end{aligned}
\tag{5.35}
$$

where I have again assumed that $grade(|vac\rangle) = even$. A consequence of (5.35) is that $\hat{C}^2 = (-)^F$ so that $\hat{C}^{-1} = (-)^F\hat{C} = \hat{C}(-)^F$. In terms of creation and annihilation operators (5.35) is:

$$\begin{aligned} \hat{C}a^\dagger(\psi)\hat{C} &= b^\dagger(\hat{C}_1\psi) \qquad & \hat{C}b^\dagger(\psi)\hat{C} &= -a^\dagger(\hat{C}_1\psi) \\ \hat{C}a(\psi)\hat{C} &= -b(\hat{C}_1\psi) \qquad & \hat{C}b(\psi)\hat{C} &= a(\hat{C}_1\psi) \end{aligned} \tag{5.36}$$

The consistency of these is easily checked by using $\hat{C}^2 = (-)^F$, which anticommutes with all creation and annihilation operators, (and squares to 1). To demonstrate the agreement between (5.36) and (5.35) insert $1 = \hat{C}^2(-)^F$ between each creation and annihilation operator, and use (5.36) to write:

$$\begin{aligned} \hat{C}|\left({}^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n}\right) in_A\rangle &= (-)^n \hat{C}a^\dagger_{i_1,A}\hat{C}(-)^F \ldots \hat{C}a^\dagger_{i_m,A}\hat{C}(-)^F\hat{C}b^\dagger_{j_n,A}\hat{C}(-)^F \ldots \hat{C}b^\dagger_{j_1,A}\hat{C}(-)^F\hat{C}|vac_A\rangle \\ &= (-)^{\sum_{k=1}^{n+m-1} k}(-)^n b^\dagger_{i_1,-A} \ldots b^\dagger_{i_m,-A} a^\dagger_{j_n,-A} \ldots a^\dagger_{j_1,-A}|vac_{-A}\rangle \\ &= (-)^n a^\dagger_{j_1,-A} \ldots a^\dagger_{j_n,-A} b^\dagger_{i_m,-A} \ldots b^\dagger_{i_1,-A}|vac_{-A}\rangle \\ &= (-)^n|\left({}^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n}\right) in_{-A}\rangle \end{aligned}$$

as required.

To show that $\hat{C}$ is unitary, we must show that $\langle F_{1,-A}|\hat{C}F_{2,A}\rangle = \langle \hat{C}^{-1}F_{1,-A}|F_{2,A}\rangle$. Using the notation of (2.67) and (2.38), we can write $\hat{C}F_{2,A} = (-)^{n_2}G^+_{-A,n_2} \wedge (i_{F^-_{-A,m_2}}|vac_{-A}\rangle)$, so that :

$$\langle F_{1,-A}|\hat{C}F_{2,A}\rangle = (-)^{n_2}(i_{F^-_{-A,m_2}} G^-_{-A,n_1})_0(i_{F^+_{-A,m_1}} G^+_{-A,n_2})_0$$

while

$$\langle \hat{C}^{-1}F_{1,-A}|F_{2,A}\rangle = (-)^{m_1+n_1}(-)^{n_1}(i_{G^-_{A,n_2}} F^-_{A,m_1})_0(i_{G^+_{A,n_1}} F^+_{A,m_2})_0$$

Now, since $\hat{C}_{1,U}$ is antiunitary, we can deduce that:

$$\begin{aligned} (i_{F^+_{-A,m_1}} G^+_{-A,n_2})_0 &= \overline{(i_{F^-_{A,m_1}} G^-_{A,n_2})_0} = (i_{G^-_{A,n_2}} F^-_{A,m_1})_0 \\ \text{and } (i_{G^+_{A,n_1}} F^+_{A,m_2})_0 &= \overline{(i_{G^-_{-A,n_1}} F^-_{-A,m_2})_0} = (i_{F^-_{-A,m_2}} G^-_{-A,n_1})_0 \end{aligned}$$

So that $\langle F_{1,-A}|\hat{C}F_{2,A}\rangle = \langle \hat{C}^{-1}F_{1,-A}|F_{2,A}\rangle$, and $\hat{C}$ is unitary, as required.

Now, from (5.30) we have $\hat{C}_1 u^A_{t_0}(t) = v^{-A}_{t_0}(t)$, so that using similar methods to (5.35) we get:

$$\hat{C}|\left({}^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n}\right) in_{A,t_0}(t)\rangle = (-)^n \eta_{vac,t_0}(t)|\left({}^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_m}\right) in_{-A,t_0}(t)\rangle$$

where I have used conservation of the inner product to write $I^+_{-A,t_0} \wedge |vac_{-A,t_0}\rangle = \eta_{vac,t_0}(t) I^+_{-A,t_0}(t) \wedge |vac_{-A,t_0}(t)\rangle$ where $\eta_{vac,t_0}(t)$ is an unimportant phase factor. Thus, up to this time-dependent phase factor we have $\hat{C}\hat{U}^A(t,t_0) = \hat{U}^{-A}(t,t_0)\hat{C}$ on all of state space, as required. This of course means that:

$$S^A_{F,G}(t_1,t_0) = S^{-A}_{\hat{C}F,\hat{C}G}(t_1,t_0)$$

on all of state space.

Some notes are again in order:

1. To demonstrate that the action of $\hat{C}$ on $\mathcal{F}_{\wedge}(\mathcal{H})$ is consistent with equations (5.32) (defined by the action of $\hat{C}_1$ on $\mathcal{H}$) it is instructive to expand $|vac_{A,t_0}(t)\rangle$ in terms of Bogoliubov coefficients, and to demonstrate explicitly using (5.32) that $\hat{C}|vac_{A,t_0}(t)\rangle = |vac_{-A,t_0}(t)\rangle$ as expected. From equation (2.106) we have:

$$|vac_{A,t_0}(t)\rangle = \det(\boldsymbol{\epsilon}^A(t,t_0))\{|vac_{A,t}\rangle + \sum_{i,j} V^A_{i,j}(t,t_0)|\binom{i}{j}out_{A,t}\rangle - \sum_{i_1<i_2,j_1<j_2} \det\begin{bmatrix} V_{i_1j_1} & V_{i_1j_2} \\ V_{i_2j_1} & V_{i_2j_2}\end{bmatrix} |\binom{i_1,i_2}{j_1,j_2}out_{A,t}\rangle + \ldots\} \tag{5.37}$$

so

$$\hat{C}|vac_{A,t_0}(t)\rangle = \det(\boldsymbol{\epsilon}^A(t,t_0))\{|vac_{-A,t}\rangle - \sum_{i,j} V^A_{i,j}(t,t_0)|\binom{j}{i}out_{-A,t}\rangle - \sum_{i_1<i_2,j_1<j_2} \det\begin{bmatrix} V_{i_1j_1} & V_{i_1j_2} \\ V_{i_2j_1} & V_{i_2j_2}\end{bmatrix} |\binom{j_1,j_2}{i_1,i_2}out_{-A,t}\rangle + \ldots\}$$

Substituting (5.32) and using (2.111) to write $\boldsymbol{V}_A(t,t_0) = \boldsymbol{V}_{-A}(t,t_0)^T$ we get:

$$\begin{aligned}\hat{C}|vac_{A,t_0}(t)\rangle &= \overline{\det(\boldsymbol{\alpha}^{-A}(t,t_0))}\{|vac_{-A,t}\rangle + \sum_{i,j} V^{-A}_{j,i}(t,t_0)|\binom{j}{i}out_{-A,t}\rangle \\ &\quad - \sum_{i_1<i_2,j_1<j_2} \det\begin{bmatrix} V_{j_1i_1} & V_{j_1i_2} \\ V_{j_2i_1} & V_{j_2i_2}\end{bmatrix} |\binom{j_1,j_2}{i_1,i_2}out_{-A,t}\rangle + \ldots\} \\ &= \eta_{vac,t_0}(t)\det(\boldsymbol{\epsilon}^{-A}(t,t_0))\{|vac_{-A,t}\rangle + \sum_{i,j} V^{-A}_{i,j}(t,t_0)|\binom{i}{j}out_{-A,t}\rangle \\ &\quad - \sum_{i_1<i_2,j_1<j_2} \det\begin{bmatrix} V^{-A}_{i_1j_1} & V^{-A}_{i_1j_2} \\ V^{-A}_{i_2j_1} & V^{-A}_{i_2j_2}\end{bmatrix} |\binom{i_1,i_2}{j_1,j_2}out_{-A,t}\rangle + \ldots\} \\ &= \eta_{vac,t_0}(t)|vac_{-A,t_0}(t)\rangle\end{aligned}$$

where $\eta_{vac,t_0}(t) = \frac{\overline{\det(\boldsymbol{\alpha}^{-A}(t,t_0))}}{\det(\boldsymbol{\epsilon}^{-A}(t,t_0))}$ which is easily shown to be a phase by using the Bogoliubov conditions. A similar calculation to this can also be used to verify that the action of $\hat{C}$ on any chosen state in $\mathcal{F}_{\wedge}(\mathcal{H})$ is consistent with (5.30). We could also carry out a similar verification for parity-reversal and time-reversal, but these cases are more trivial, and are omitted.

2. As with $\hat{P}_1$ and $\hat{T}_1$, we can include an arbitrary phase η_C into the definition of $\hat{C}_1$, by defining $\hat{C}_1 : \psi(\mathbf{x}) \to C\bar{\psi}(\mathbf{x})^T\eta_C^* = C\overline{\eta_C\psi}(\mathbf{x})^T$. We can then either absorb this into our definition of particle/antiparticle (so that we still put $\hat{C}_1u^A_{i,t_0} = v^{-A}_{i,t_0}$ and $\hat{C}_1v^A_{i,t_0} = u^{-A}_{i,t_0}$) in which case it has no further effect, or we can keep the original

definition for the u_{i,t_0}'s and v_{i,t_0}'s, so that we get $\hat{C}_1 u^A_{i,t_0} = v^{-A}_{i,t_0}\eta^*_C$ and $\hat{C}_1 v^A_{i,t_0} = u^{-A}_{i,t_0}\eta^*_C$. Then (5.35) would become:

$$\hat{C}|\binom{i_1 i_2 \ldots i_m}{j_1 j_2 \ldots j_n} in_{A,t_0}\rangle = (-)^n \eta_C^{n-m} |\binom{j_1 j_2 \ldots j_n}{i_1 i_2 \ldots i_m} in_{-A,t_0}\rangle$$

while (5.36) would become:

$$\begin{array}{lll} \hat{C}a^\dagger(\psi)\hat{C} = \eta_C b^\dagger(\psi\sigma_1) & \text{and} & \hat{C}b^\dagger(\psi)\hat{C} = -\eta^*_C a^\dagger(\psi\sigma_1) \\ \hat{C}a(\psi)\hat{C} = -\eta^*_C b(\psi\sigma_1) & \text{and} & \hat{C}b(\psi)\hat{C} = \eta_C a(\psi\sigma_1) \end{array}$$

and we would still have $\hat{C}^2 = (-)^F$.

3. In the Majorana representation, we have $C = \begin{bmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{bmatrix}$ and $\gamma^0 = \begin{bmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{bmatrix}$ so that $\hat{C}\psi = i\psi^*$ in this representation. Hence, by a suitable choice of η_C, we can have $\hat{C}\psi = \psi^*$ in the Majorana representation. This is of course what allows us to define Majorana spinors to be 'spinors invariant under charge conjugation' or to be 'spinors which are real in the Majorana representation'.

4. It is now a straightforward exercise to combine $\hat{C}$, $\hat{P}$ and $\hat{T}$ to form $\hat{\Theta} \equiv \hat{C}\hat{P}\hat{T}$, and to investigate the effect of this transformation on S-matrix elements, Bogoliubov matrices etc.

Chapter 6

Some Perturbative Calculations

Given the non-perturbative treatment of S-Matrix elements and expectation values presented in Section 2.3, and the strong connection it makes with the 'first quantized' treatment, we see that performing perturbative calculations simply requires expanding the 'first quantized' solutions $\psi_{t_0}(\mathbf{x}, t)$ in powers of the coupling constant e. In this Chapter I will describe briefly how this is done, and will perform some perturbative calculations, to verify that the results of this agree with conventional[41] methods. Again I will consider only the case of an electromagnetic background (as viewed by an inertial observer), and will not consider back-reaction. In order to make connection with conventional techniques, I start by assuming that the background electromagnetic field tends to zero sufficiently quickly as $t \to \pm\infty$ that the 'in' and 'out' states $u_{\mathbf{p},\lambda;\pm\infty}(\mathbf{x}), v_{\mathbf{p},\lambda;\pm\infty}(\mathbf{x})$ can be approximated by the plane wave states of free Dirac theory, as given in (5.17) and (5.18) (for asymptotically early/late t_0 respectively).

Now, start by writing equation (2.4) in the form:

$$(i\gamma^\mu(x)\partial_\mu - m)\psi(x) = e\gamma^\mu A_\mu(x)\psi(x)$$

A straightforward iterative procedure[81, 41, 84] allows us to write a solution $\psi(\mathbf{x}, t)$ to the full Dirac equation in terms of a solution $\psi^f(\mathbf{x}, t)$ of the free ($A(x) = 0$) Dirac equation, as:

$$\psi(x) = \psi^f(x) + e\int d^4x' G(x, x')\gamma^\mu A_\mu(x')\psi^f(x') + \tag{6.1}$$

$$e^2 \int d^4x' \int d^4x'' G(x, x')\gamma^\mu A_\mu(x')G(x', x'')\gamma^\mu A_\mu(x'')\psi^f(x'') + \dots \tag{6.2}$$

where the 'Greens function' $G(x, x')$ must satisfy:

$$(i\gamma^\mu\partial_\mu - m)G(x, x') = \delta(x - x') \tag{6.3}$$

There are an infinite number of possible choices for $G(x, x')$, some of which were described in Section 4.5[1]. As also mentioned in Section 4.5, the conventional choice is $G(x, x') = S_F(x, x')$, since this avoids the possibility of positive energy states evolving to include a

[1]Notice however, that in Section 4.5 I was describing Greens functions for the full Dirac equation, whereas $G(x, x')$ is a Greens function for the free Dirac equation.

negative energy component, although it has problems with causality, which are reflected in the fact that $\psi^f(x)$ cannot be identified with the 'in' state or with the 'out' state. I will instead choose:

$$G(x,x') = -iS_R(x,x') \tag{6.4}$$

$$= \theta(t-t') \int \frac{d^3\mathbf{q}}{(2\pi)^3 2E_{\mathbf{q}}} \sum_{\sigma} \{u^f_{\mathbf{q},\sigma}(\mathbf{x},t)\bar{u}^f_{\mathbf{q},\sigma}(\mathbf{x}',t') + v_{\mathbf{q},\sigma}(\mathbf{x},t)\bar{v}^f_{\mathbf{q},\sigma}(\mathbf{x}',t')\} \tag{6.5}$$

With this choice we can obtain $u_{\mathbf{p},\lambda;-\infty}(\mathbf{x},t)$ simply by substituting $u^f_{\mathbf{p},\lambda}(\mathbf{x},t)$ for $\psi^f(x)$ (which would not have been the case with any of the other Greens functions) Substituting this into (6.2) allows us to calculate, for instance:

$$\alpha_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty) = \langle u_{\mathbf{p},\lambda;+\infty}(\mathbf{x}) | u_{\mathbf{q},\sigma;-\infty}(\mathbf{x},\infty)\rangle \tag{6.6}$$

$$= (2\pi)^3 2E_{\mathbf{q}}\delta_{\lambda\sigma}\delta(\mathbf{p}-\mathbf{q}) - i\int_{-\infty}^{+\infty} dt \boldsymbol{H}^{++}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) \tag{6.7}$$

$$-\int_{-\infty}^{+\infty} dt \int_{-\infty}^{t} dt' \int \frac{d^3\mathbf{p}'}{(2\pi)^3 2E_{\mathbf{p}'}} \sum_{\lambda'} \{\boldsymbol{H}^{++}_{I;\mathbf{p},\lambda;\mathbf{p}',\lambda'}(t)\boldsymbol{H}^{++}_{I;\mathbf{p}',\lambda';\mathbf{q},\sigma}(t') + \boldsymbol{H}^{+-}_{I;\mathbf{p},\lambda;\mathbf{p}',\lambda'}(t)\boldsymbol{H}^{-+}_{I;\mathbf{p}',\lambda';\mathbf{q},\sigma}(t')\} \tag{6.8}$$

$$+ \text{ higher order terms} \tag{6.9}$$

where I have defined:

$$\boldsymbol{H}^{++}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) = e\int d^3\mathbf{x}\bar{u}^f_{\mathbf{p},\lambda}(\mathbf{x},t)\gamma^\mu A_\mu(\mathbf{x},t)u^f_{\mathbf{q},\sigma}(\mathbf{x},t) \tag{6.10}$$

$$\boldsymbol{H}^{+-}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) = e\int d^3\mathbf{x}\bar{u}^f_{\mathbf{p},\lambda}(\mathbf{x},t)\gamma^\mu A_\mu(\mathbf{x},t)v^f_{\mathbf{q},\sigma}(\mathbf{x},t) \tag{6.11}$$

$$\boldsymbol{H}^{-+}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) = e\int d^3\mathbf{x}\bar{v}^f_{\mathbf{p},\lambda}(\mathbf{x},t)\gamma^\mu A_\mu(\mathbf{x},t)u^f_{\mathbf{q},\sigma}(\mathbf{x},t) \tag{6.12}$$

$$\boldsymbol{H}^{--}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) = e\int d^3\mathbf{x}\bar{v}^f_{\mathbf{p},\lambda}(\mathbf{x},t)\gamma^\mu A_\mu(\mathbf{x},t)v^f_{\mathbf{q},\sigma}(\mathbf{x},t) \tag{6.13}$$

$\beta_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty)$, $\gamma_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty)$ and $\epsilon_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty)$ can be calculated using the same method, and we get, in matrix form:

$$\begin{bmatrix} \alpha_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty) & \beta_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty) \\ \gamma_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty) & \epsilon_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty) \end{bmatrix} = \begin{bmatrix} (2\pi)^3 2E_{\mathbf{p}}\delta_{\lambda\sigma}\delta(\mathbf{p}-\mathbf{q}) & 0 \\ 0 & (2\pi)^3 2E_{\mathbf{p}}\delta_{\lambda\sigma}\delta(\mathbf{p}-\mathbf{q}) \end{bmatrix}$$
$$- i\int_{-\infty}^{\infty} dt \begin{bmatrix} \boldsymbol{H}^{++}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) & \boldsymbol{H}^{+-}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) \\ \boldsymbol{H}^{-+}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) & \boldsymbol{H}^{--}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) \end{bmatrix} \tag{6.14}$$
$$- \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' \begin{bmatrix} \boldsymbol{H}^{++}_{I;\mathbf{p},\lambda;\mathbf{p}',\lambda'}(t) & \boldsymbol{H}^{+-}_{I;\mathbf{p},\lambda;\mathbf{p}',\lambda'}(t) \\ \boldsymbol{H}^{-+}_{I;\mathbf{p},\lambda;\mathbf{p}',\lambda'}(t) & \boldsymbol{H}^{--}_{I;\mathbf{p},\lambda;\mathbf{p}',\lambda'}(t) \end{bmatrix} \begin{bmatrix} \boldsymbol{H}^{++}_{I;\mathbf{p}',\lambda';\mathbf{q},\sigma}(t) & \boldsymbol{H}^{+-}_{I;\mathbf{p}',\lambda';\mathbf{q},\sigma}(t) \\ \boldsymbol{H}^{-+}_{I;\mathbf{p}',\lambda';\mathbf{q},\sigma}(t) & \boldsymbol{H}^{--}_{I;\mathbf{p}',\lambda';\mathbf{q},\sigma}(t) \end{bmatrix}$$
$$+ \text{ higher order terms}$$

where the $\int \frac{d^3\mathbf{p}'}{(2\pi)^3 2E_{\mathbf{p}'}} \sum_{\lambda'}$ of (6.7) is now implicit in the definition of matrix multiplication.

By substituting these expressions for $\alpha_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty)$, $\beta_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty)$, $\gamma_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty)$ and $\epsilon_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty)$ into the various formulae of Section 2.3, we can now calculate any S-matrix elements or expectation values, to any order of e, and in any background $A(\mathbf{x},t)$ which satisfies the assumption made at the start of this Section.

As an example of this, I now turn my attention to the calculation of the total number of pairs created by an arbitrary electromagnetic background $A_\mu(\mathbf{x},t)$, to lowest order in the coupling constant e.

After this, I will demonstrate the general equivalence of this procedure with conventional methods.

6.1 Pair Creation To Lowest Order

Now, from (2.115) we have:

$$N_{pair} = \int \frac{d^3\mathbf{q}}{(2\pi)^3 2E_\mathbf{q}} \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \sum_{\sigma,\lambda} |\gamma_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty)|^2 \tag{6.15}$$

where

$$\gamma_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty,-\infty) = -ie\int d^4x \bar{v}^f_{\mathbf{p},\lambda}(x)\gamma^\mu A_\mu(x) u^f_{\mathbf{q},\sigma}(x) + O(e^2) \tag{6.16}$$

$$= -ie\int d^4x e^{-i(p+q)\cdot x}\bar{v}_\lambda(\mathbf{p}) \not{A}(x) u_\sigma(\mathbf{q}) + O(e^2) \tag{6.17}$$

where I have written $u^f_{\mathbf{q},\sigma}(x) = u_\sigma(\mathbf{q})e^{-iq\cdot x}$, and have introduced the 'Feynman slash' $\not{A} \equiv \gamma^\mu A_\mu$. Substituting (6.17) into (6.15) and using $\sum_\sigma u_\sigma(\mathbf{q})\bar{u}_\sigma(\mathbf{q}) = \not{q} + m$ and $\sum_\lambda v_\lambda(\mathbf{p})\bar{v}_\lambda(\mathbf{p}) = \not{p} - m$ we get:

$$N_{pair} = e^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3 2E_\mathbf{q}} \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \sum_{\sigma,\lambda} \int d^4x d^4y \tag{6.18}$$

$$e^{i(p+q)\cdot(y-x)} Trace((\not{q}+m)\not{A}(y)(\not{p}-m)\not{A}(x)) \tag{6.19}$$

Now define $a_\mu(k)$ by:

$$A_\mu(x) = \int d^4k \frac{a_\mu(k)}{(2\pi)^2} e^{-ik\cdot x} \tag{6.20}$$

Substituting this into (6.19) gives:

$$N_{pair} = \frac{\alpha}{\pi}\int d^4k \delta(p+q-k) \int \frac{d^3\mathbf{q}}{2E_\mathbf{q}}\frac{d^3\mathbf{p}}{2E_\mathbf{p}} Trace((\not{q}+m)\not{A}(y)(\not{p}-m)\not{A}(x)) \tag{6.21}$$

where $\alpha \equiv \frac{e^2}{4\pi}$ is the fine structure constant. This equation can be identified with equation (4.102) of Itzykson and Zuber[41]. The rest of the derivation is exactly as in Itzykson and Zuber[41], and leads to:

$$N_{pair} = \frac{\alpha}{3} \int d^4q \;\; \theta(q^2 - 4m^2)(|\boldsymbol{E}(q)|^2 - |\boldsymbol{B}(q)|^2)(1 - \frac{4m^2}{q^2})^{\frac{1}{2}}(1 + \frac{2m^2}{q^2}) \qquad (6.22)$$

where:

$$|\boldsymbol{B}(q)|^2 - |\boldsymbol{E}(q)|^2 = q^2 a(q) \cdot a(-q) - q \cdot a(q) q \cdot a(-q) \qquad (6.23)$$

$$= \tfrac{1}{2} F_{\mu\nu}(q) F^{\mu\nu}(-q) \qquad (6.24)$$

$$\text{and } F_{\mu\nu} = -i(q_\mu a_\nu(q) - q_\nu a_\mu(q)) \qquad (6.25)$$

is the Fourier transform of the electromagnetic field strength.

6.2 Equivalence To Conventional Methods

Start by writing the Dirac equation in the form:

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}_0|\psi(t)\rangle + \hat{H}_I(t)|\psi(t)\rangle \qquad (6.26)$$

$$\text{where } \hat{H}_0(\mathbf{x})\psi = (-i\gamma^0\gamma^k\partial_k + m\gamma^0)\psi \qquad (6.27)$$

$$\text{and } \hat{H}_I(\mathbf{x},t)\psi = e\gamma_0 \not{A}(\mathbf{x},t)\psi \qquad (6.28)$$

Write

$$u^f_{\mathbf{p},\lambda}(\mathbf{x},t) = e^{-i\hat{H}_0\,(t-t_0)} u_{\mathbf{p},\lambda;t_0}(\mathbf{x})$$

where the 'in time' t_0 is in the asymptotically distant past, the 'in state' $u_{\mathbf{p},\lambda;t_0}(\mathbf{x})$ is given by (5.17), and $e^{-i\hat{H}_0\,(t-t_0)}$ is the free propagator, from t_0 to t. Write $v^f_{\mathbf{p},\lambda}(\mathbf{x},t)$ similarly. Let $|\psi(t)\rangle$ be an arbitrary solution to the Dirac equation, and expand $|\psi(t)\rangle$ as:

$$|\psi(t)\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \sum_\lambda \{|u^f_{\mathbf{p},\lambda}(t)\rangle c_{\mathbf{p},\lambda}(t) + |v^f_{\mathbf{p},\lambda}(t)\rangle d_{\mathbf{p},\lambda}(t)\} \qquad (6.29)$$

where $c_{\mathbf{p},\lambda}(t) = \langle u^f_{\mathbf{p},\lambda}(t)|\psi(t)\rangle$ and $d_{\mathbf{p},\lambda}(t) = \langle v^f_{\mathbf{p},\lambda}(t)|\psi(t)\rangle$. Differentiating this gives:

$$i\frac{d}{dt}|\psi(t)\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \sum_\lambda \{(\frac{d}{dt}|u^f_{\mathbf{p},\lambda}(t)\rangle) c_{\mathbf{p},\lambda}(t) + |u^f_{\mathbf{p},\lambda}(t)\rangle \frac{dc_{\mathbf{p},\lambda}(t)}{dt}$$

$$+ (\frac{d}{dt}|v^f_{\mathbf{p},\lambda}(t)\rangle) d_{\mathbf{p},\lambda}(t) + |v^f_{\mathbf{p},\lambda}(t)\rangle \frac{dd_{\mathbf{p},\lambda}(t)}{dt}\} \qquad (6.30)$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \sum_\lambda \{|u^f_{\mathbf{p},\lambda}(t)\rangle \frac{dc_{\mathbf{p},\lambda}(t)}{dt} + |v^f_{\mathbf{p},\lambda}(t)\rangle \frac{dd_{\mathbf{p},\lambda}(t)}{dt}\} + \hat{H}_0|\psi(t)\rangle \qquad (6.31)$$

so that (6.26) gives

$$\hat{H}_I(t)|\psi(t)\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \sum_\lambda \{|u^f_{\mathbf{p},\lambda}(t)\rangle \frac{dc_{\mathbf{p},\lambda}(t)}{dt} + |v^f_{\mathbf{p},\lambda}(t)\rangle \frac{dd_{\mathbf{p},\lambda}(t)}{dt}\} \tag{6.32}$$

This motivates defining the *Interaction Picture* by

$$|\psi(t)\rangle_i \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \sum_\lambda \{|u_{\mathbf{p},\lambda;t_0}\rangle c_{\mathbf{p},\lambda}(t) + |v_{\mathbf{p},\lambda;t_0}\rangle d_{\mathbf{p},\lambda}(t)\} = e^{i\hat{H}_0\,(t-t_0)}|\psi(t)\rangle \tag{6.33}$$

$$\hat{H}_{Ii}(t) = e^{-i\hat{H}_0\,(t-t_0)}\hat{H}_I(t)e^{i\hat{H}_0\,(t-t_0)} \tag{6.34}$$

Notice that the matrix entries of the operator $\hat{H}_{Ii}(t)$ are:

$$\begin{aligned}
\langle u_{\mathbf{p},\lambda;t_0}|\hat{H}_{Ii}(t)|u_{\mathbf{q},\sigma;t_0}\rangle = \langle u^f_{\mathbf{p},\lambda}(t)|\hat{H}_I(t)|u^f_{\mathbf{q},\sigma}(t)\rangle &= e\int d^3\mathbf{x}\bar{u}^f_{\mathbf{p},\lambda}(\mathbf{x},t)\gamma^\mu A_\mu(\mathbf{x},t)u^f_{\mathbf{q},\sigma}(\mathbf{x},t) \\
&= \boldsymbol{H}^{++}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) \qquad (6.35)\\
\text{and similarly} \qquad \langle u_{\mathbf{p},\lambda;t_0}|\hat{H}_{Ii}(t)|v_{\mathbf{q},\sigma;t_0}\rangle &= \boldsymbol{H}^{+-}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) \\
\langle v_{\mathbf{p},\lambda;t_0}|\hat{H}_{Ii}(t)|u_{\mathbf{q},\sigma;t_0}\rangle &= \boldsymbol{H}^{-+}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t) \\
\langle v_{\mathbf{p},\lambda;t_0}|\hat{H}_{Ii}(t)|v_{\mathbf{q},\sigma;t_0}\rangle &= \boldsymbol{H}^{--}_{I;\mathbf{p},\lambda;\mathbf{q},\sigma}(t)
\end{aligned}$$

That is, they are simply the matrix entries of $\hat{H}_I(t)$ with respect to the *freely comoving basis* $\{|u^f_{\mathbf{p},\lambda}(t)\rangle, |v^f_{\mathbf{p},\lambda}(t)\rangle\}$. The equivalent fact in conventional treatments is to notice that $\hat{H}_{Ii}(t)$ can be obtained from $\hat{H}_I(t)$ by replacing the field operator $\hat{\psi}(\mathbf{x},t)$ with the interaction picture field operator $\hat{\psi}_i(\mathbf{x},t)$. Now we can write equation (6.32) as

$$i\frac{d}{dt}|\psi(t)\rangle_i = \hat{H}_{Ii}(t)|\psi(t)\rangle_i \tag{6.36}$$

So far we have only considered the evolution of 'first quantized' states. We can now extend equation (6.36) to all of state space, where it will clearly be:

$$i\frac{d}{dt}|F(t)\rangle_i = \hat{H}_{Ii,H}(t)|F(t)\rangle_i \tag{6.37}$$

where $\hat{H}_{Ii,H}(t)$ is the Hermitian extension of $\hat{H}_{Ii}(t)$, and the interaction picture states $|F(t)\rangle_i = e^{i\hat{H}_{0,H}(t-t_0)}|F_{t_0}\rangle_i$ are just Slater determinants of 'first quantized interaction picture states' This can be solved in the conventional way, giving $|F(t)\rangle_i = \hat{U}_i(t,t_0)|F_{t_0}\rangle$ where:

$$\begin{aligned}
\hat{U}_i(t,t_0) &= \hat{1} + \sum_{n=1}^\infty (-i)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \hat{H}_{Ii,H}(t_1)\ldots\hat{H}_{Ii,H}(t_n) \qquad (6.38)\\
&= \hat{1} + \sum_{n=1}^\infty \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n T(\hat{H}_{Ii,H}(t_1)\ldots\hat{H}_{Ii,H}(t_n)) \qquad (6.39)\\
&= Te^{-i\int_{t_0}^t \hat{H}_{Ii,H}(\tau)d\tau} \qquad (6.40)
\end{aligned}$$

in agreement with conventional methods. However, it remains to show that the operator $\hat{H}_{Ii,H}(\tau)$ above is the same as the operator of conventional approaches. To see this, first use (6.35) to write:

$$\hat{H}_{Ii}(t) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\lambda} \int \frac{d^3\mathbf{q}}{(2\pi)^3 2E_{\mathbf{q}}} \sum_{\sigma} \{|u_{\mathbf{p},\lambda;t_0}\rangle\langle u^f_{\mathbf{p},\lambda}(t)|\hat{H}_I(t)|u^f_{\mathbf{q},\sigma}(t)\rangle\langle u_{\mathbf{q},\sigma;t_0}|$$
$$+|u_{\mathbf{p},\lambda;t_0}\rangle\langle u^f_{\mathbf{p},\lambda}(t)|\hat{H}_I(t)|v^f_{\mathbf{q},\sigma}(t)\rangle\langle v_{\mathbf{q},\sigma;t_0}| + |v_{\mathbf{p},\lambda;t_0}\rangle\langle v^f_{\mathbf{p},\lambda}(t)|\hat{H}_I(t)|u^f_{\mathbf{q},\sigma}(t)\rangle\langle u_{\mathbf{q},\sigma;t_0}|$$
$$+|v_{\mathbf{p},\lambda;t_0}\rangle\langle v^f_{\mathbf{p},\lambda}(t)|\hat{H}_I(t)|v^f_{\mathbf{q},\sigma}(t)\rangle\langle v_{\mathbf{q},\sigma;t_0}|$$

Now use the fact that the Hermitian extension of $|\phi\rangle\langle\psi|$ is $\overrightarrow{\phi} \wedge (i_{\overrightarrow{\psi}}$ to convert $|u_{\mathbf{p},\lambda;t_0}\rangle\langle u_{\mathbf{q},\sigma;t_0}| \to a^{\dagger}_{\lambda,in}(\mathbf{p})a_{\sigma,in}(\mathbf{q})$, and similarly for the others. This allows us to write the Hermitian extension $\hat{H}_{Ii,H}(t)$ in terms of the standard expression for the interaction picture field operator $\hat{\psi}_i(\mathbf{x},t)$ as:

$$\hat{H}_{Ii,H}(t) = e\int d^3\mathbf{x}\hat{\bar{\psi}}_i(\mathbf{x},t)\gamma^{\mu}A_{\mu}(\mathbf{x},t)\hat{\psi}_i(\mathbf{x},t) \tag{6.41}$$

as required.

Finally, it is worth noting one minor difference between (6.40) and the equivalent conventional equation. Namely, that I have $|F(t)\rangle_i = e^{i\hat{H}_{0,H}(t-t_0)}|F_{t_0}\rangle_i$ (where $\hat{H}_{0,H}$ is not normal ordered) whereas conventional methods use the normal ordered free Hamiltonian here. This difference is only an overall phase, so has no effect on S-Matrix elements or expectation values. There is also of course a slight difference, as mentioned in Section 2.2.3, between the vacuum subtraction scheme employed in Section 2.2.3 for defining physical operators at finite times, and the normal ordering prescription used in conventional methods[41]. This will have no effect on expectation values calculated at asymptotically early or late times, (under the assumption made at the start of this Section) but would affect the calculation of expectation values at finite times.

Chapter 7

Application to Uniform Electric Fields

This Chapter is devoted to a simple application of the formalism introduced in Chapter 2. I consider particle creation in time varying, spatially uniform electric fields, with specific results presented for the case of a constant (in time and space) electric field.

Particle creation in a constant electric field was originally studied by Schwinger[28] in 1951. It has since been rederived a number of ways[16, 14, 11, 66, 65, 85], and has proved to be a useful testing ground for ideas in semiclassical physics.

The 'Schwinger mechanism' for pair creation, which has found many applications, such as in heavy ion collisions[21, 19] and in astrophysical plasmas[86], is based on a pair creation rate which has been averaged over an infinite period of time. The problem of pair creation after a finite evolution time has only been considered quite recently[16, 14]. In these cases, an electric field was considered which was constant over a certain time range (say for $0 < t < T$), but was zero outside of this time range. The 'in' and 'out' states were defined in the 'free regions' before the field had been turned on ($t < 0$) , and after it had been turned off ($t > T$), where the solutions were simple plane wave states. They then investigated the dependence of their results on T

In this Chapter I show that, by using the finite time particle definition introduced in Chapter 2, we can derive the amount of pair creation in a state which has evolved from vacuum for a finite time, and we can do so without having to 'switch off the field to measure the particles'. That is, I consider the background $\boldsymbol{E} = (0, 0, E)$ for all time, and I ask the question. "Suppose the system is in the vacuum state at some time t_0. What will be the properties of the system at time $t_0 + T$." Obviously, if my particle definition is correct, then the answer to this question will be the same as the answers found by previous authors[16, 14] to the question above. I show in this Chapter that this is indeed the case. Although the difference between these two questions is quite academic here, it is vitally important if we wish to consider pair creation in strong gravitational fields, which cannot simply be 'switched off whilst making measurements'.

I also consider in this Chapter the pair creation rate, as a function of time, in a time varying electric field, a result which is much more difficult to achieve by other methods. Also, unlike all previous 'Bogoliubov coefficient' methods[34] the methods employed here are (as demonstrated in Chapter 5) completely independent of gauge.

Sections 7.1 - 7.3 are devoted to calculating the finite time Bogoliubov coefficients in an arbitrarily time varying electric field, in terms of the solution of a simple linear, second order equation. In Section 7.4 I solve this equation in the case of a constant electric field, and use this to explicitly calculate the number density of created pairs, after a finite evolution time $\triangle t$. This is the first time that this result has been calculated without having to 'turn off the electric field when making measurements'. My results agree with the results obtained for an electric field that was only on for the time $\triangle t$, thus demonstrating the consistency of my particle definition. Also calculated in this Section is the current density due to these created pairs, as a function of the evolution time. I demonstrate that using the vacuum subtraction procedure introduced in Section 2.2.3 gives a finite result for this current density, which is in agreement with that calculated in Mottola et. al.[87], by other methods.

7.1 A Complete Set of Solutions

Consider a time varying, but spatially uniform, electric field in the z-direction, $\boldsymbol{E}(t) = (0,0,E(t))$ (and $\boldsymbol{B}(t) = \mathbf{0}$). This can be described by a vector potential of the form $A_\mu(t) = (0,0,0,A(t))$ where $\dot{A}(t) = E(t)$. Thus, in order to use the procedure described in Chapter 2, we first need a complete set of solutions to the Dirac equation in this background. That is, we need a complete set of solutions to:

$$i\gamma^\mu(\partial_\mu + ieA_\mu)\psi(\mathbf{x},t) = m\psi(\mathbf{x},t) \tag{7.1}$$

with

$$A_\mu = (0,0,0,A(t)) \tag{7.2}$$

I have chosen a specific gauge in 7.2, but have demonstrated thoroughly in Section 5.1 that all physical predictions are indeed independent of gauge. To solve (7.1), consider solutions $\psi(x)$ of the form:

$$\psi_{\mathbf{p}}(\mathbf{x},t) = \psi_{\mathbf{p}}(t)e^{i\mathbf{p}\cdot\mathbf{x}} \tag{7.3}$$

where $\mathbf{p}\cdot\mathbf{x} = \sum_{k=1}^{3} p^k x^k$. Substituting this into (7.1) gives:

$$i\frac{d\psi_{\mathbf{p}}(t)}{dt} = m\gamma_0\psi_{\mathbf{p}}(t) + \bar{\mathbf{p}}_\perp\psi_{\mathbf{p}}(t) + (p^3 + eA(t))\bar{\sigma}_3\psi_{\mathbf{p}}(t) \tag{7.4}$$

where $\bar{\sigma}_k \equiv \gamma^0\gamma^k = \gamma_k\gamma_0$ and $\bar{\mathbf{p}}_\perp \equiv p^1\bar{\sigma}_1 + p^2\bar{\sigma}_2$. Using the standard 'Dirac representation' of the Dirac matrices we have:

$$\bar{\sigma}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad \bar{\mathbf{p}}_\perp = \begin{bmatrix} 0 & 0 & 0 & \bar{z} \\ 0 & 0 & z & 0 \\ 0 & \bar{z} & 0 & 0 \\ z & 0 & 0 & 0 \end{bmatrix} \tag{7.5}$$

where $z \equiv p^1 + ip^2$ and $\bar{z}$ is the complex conjugate of z. A convenient choice of basis spinors is:

$$\phi_{++}(\mathbf{p}) = \frac{1}{2}\begin{bmatrix} 1 \\ \frac{z}{|z|} \\ 1 \\ \frac{-z}{|z|} \end{bmatrix} \quad \phi_{+-}(\mathbf{p}) = \frac{1}{2}\begin{bmatrix} -1 \\ \frac{z}{|z|} \\ -1 \\ \frac{-z}{|z|} \end{bmatrix} \quad \phi_{-+}(\mathbf{p}) = \frac{1}{2}\begin{bmatrix} 1 \\ \frac{z}{|z|} \\ -1 \\ \frac{z}{|z|} \end{bmatrix} \quad \phi_{--}(\mathbf{p}) = \frac{1}{2}\begin{bmatrix} -1 \\ \frac{z}{|z|} \\ 1 \\ \frac{z}{|z|} \end{bmatrix} \tag{7.6}$$

These are orthonormal, and satisfy

$$\begin{aligned} &\bar{\sigma}_3\phi_{\pm\lambda}(\mathbf{p}) &&= \pm\phi_{\pm\lambda}(\mathbf{p}) && \text{for } \lambda = \pm \text{ and all } \mathbf{p} \\ &\gamma_0\phi_{\pm\lambda}(\mathbf{p}) &&= \phi_{\mp\lambda}(\mathbf{p}) && \text{for } \lambda = \pm \text{ and all } \mathbf{p} \\ &\bar{\mathbf{p}}_\perp\phi_{s\lambda}(\mathbf{p}) &&= s\lambda|\mathbf{p}_\perp|\phi_{-s-\lambda}(\mathbf{p}) && \text{for } s = \pm, \lambda = \pm, \text{ and all } \mathbf{p} \end{aligned} \tag{7.7}$$

where $|\mathbf{p}_\perp|^2 \equiv (p^1)^2 + (p^2)^3$. We can now expand $\psi_{\mathbf{p}}(t)$ as:

$$\psi_{\mathbf{p}}(t) = \sum_\lambda \{\phi_{+\lambda}(\mathbf{p})f_\lambda(t) + \phi_{-\lambda}(\mathbf{p})g_\lambda(t)\} \tag{7.8}$$

The inner product of two solutions $\psi^1_{\mathbf{p}}(\mathbf{x},t), \psi^2_{\mathbf{q}}(\mathbf{x},t)$ expanded as in (7.3) and (7.8) is given by:

$$\langle\psi^1_{\mathbf{p}}(\mathbf{x},t)|\psi^2_{\mathbf{q}}(\mathbf{x},t)\rangle = (2\pi)^3\delta(\mathbf{p}-\mathbf{q})\sum_\lambda(\bar{f}^1_\lambda(t)f^2_\lambda(t) + \bar{g}^1_\lambda(t)g^2_\lambda(t)) \tag{7.9}$$

Substituting (7.8) into (7.4) gives:

$$\begin{aligned} i\frac{df_+}{dt} - (p^3 + eA(t))f_+ &= mg_+ + |\mathbf{p}_\perp|g_- \\ i\frac{df_-}{dt} - (p^3 + eA(t))f_- &= mg_- - |\mathbf{p}_\perp|g_+ \\ i\frac{dg_+}{dt} + (p^3 + eA(t))g_+ &= mf_+ - |\mathbf{p}_\perp|f_- \\ i\frac{dg_-}{dt} + (p^3 + eA(t))g_- &= mf_- + |\mathbf{p}_\perp|f_+ \end{aligned} \tag{7.10}$$

In anticipation of the constant electric field case, I now write $A(t) \equiv Et + \frac{E}{\sqrt{eE}}a(t)$ where $a(t)$ is some dimensionless function of time, which represents the deviation of $E(t) = \dot{A}(t)$ from the constant E. Since e in (7.1) is the actual charge of the electron, $e = -|e|$, then it is convenient to consider $E < 0$ (an electric field in the negative z-direction) so that $eE > 0$. We can now define the dimensionless quantities:

$$\xi \equiv \frac{eEt + p^3}{\sqrt{eE}} \quad M \equiv \frac{m}{\sqrt{eE}} \quad p \equiv \frac{|\mathbf{p}_\perp|}{\sqrt{eE}} \text{ and } \lambda_p \equiv \frac{m^2 + |\mathbf{p}_\perp|^2}{eE} = M^2 + p^2 \tag{7.11}$$

so that (7.10) becomes:

$$
\begin{aligned}
(i\frac{d}{d\xi} - (\xi + a(\xi)))f_+(\xi) &= Mg_+(\xi) + pg_-(\xi) \\
(i\frac{d}{d\xi} - (\xi + a(\xi)))f_-(\xi) &= Mg_-(\xi) - pg_+(\xi) \\
(i\frac{d}{d\xi} + (\xi + a(\xi)))g_+(\xi) &= Mf_+(\xi) - pf_-(\xi) \qquad (7.12) \\
(i\frac{d}{d\xi} + (\xi + a(\xi)))g_-(\xi) &= Mf_-(\xi) + pf_+(\xi)
\end{aligned}
$$

where $a(\xi) = a(t(\xi))$. From this we get:

$$\frac{d^2 f_\lambda}{d\xi^2} + (\lambda_p + (\xi + a(\xi))^2 + i(1 + \frac{da}{d\xi}))f_\lambda = 0 \qquad (7.13)$$

$$\frac{d^2 g_\lambda}{d\xi^2} + (\lambda_p + (\xi + a(\xi))^2 - i(1 + \frac{da}{d\xi}))g_\lambda = 0 \qquad (7.14)$$

for all λ. In order to construct a complete set of solutions to (7.12) let $f(\xi)$ be any solution of (7.13). The we can find a solution of (7.12) by putting $f_-(\xi) = 0$, $f_+(\xi) = f(\xi)$ and use (7.12) to get:

$$g_+(\xi) = \frac{M}{\lambda_p} h(\xi) \quad \text{and } g_-(\xi) = \frac{p}{\lambda_p} h(\xi) \qquad (7.15)$$

where $h(\xi) \equiv (i\frac{d}{d\xi} - (\xi + a(\xi)))f(\xi)$. From this we get:

$$\psi'_{\mathbf{p},1}(\mathbf{x}, t) = C[\phi_{++}(\mathbf{p})\sqrt{\lambda_p} f(\xi) + \frac{M\phi_{-+}(\mathbf{p}) + p\phi_{--}(\mathbf{p})}{\sqrt{\lambda_p}} h(\xi)] e^{i\mathbf{p}\cdot\mathbf{x}} \qquad (7.16)$$

where $C = (\lambda_p |f|^2 + |h|^2)^{-\frac{1}{2}}$ is constant by virtue of conservation of the norm. Similarly, $\psi'_{\mathbf{p},2}(\mathbf{x}, t)$ can be found by putting $f_+(\xi) = 0$ and $f_-(\xi) = f(\xi)$, and using (7.12) to get $g_+(\xi)$ and $g_-(\xi)$ (in terms of $h(\xi)$). $\psi_{\mathbf{p},3}(\mathbf{x}, t)$ is found by putting $g_-(\xi) = 0$, $g_+(\xi) = \bar{f}(\xi)$ (which solves equation (7.14)) and using equations (7.12) to specify $f_\pm(\xi)$ in terms of $\bar{h}(\xi)$, while $\psi_{\mathbf{p},4}(\mathbf{x}, t)$ is found similarly, by putting $g_+(\xi) = 0$, $g_-(\xi) = \bar{f}(\xi)$. This allows us to generate the complete set of solutions:

$$
\begin{aligned}
\psi_{\mathbf{p},1}(\mathbf{x}, t) &= \frac{1}{\sqrt{\lambda_p}}(M\psi'_{\mathbf{p},1}(\mathbf{x}, t) - p\psi'_{\mathbf{p},2}(\mathbf{x}, t)) \\
&= C[(M\phi_{++}(\mathbf{p}) - p\phi_{+-}(\mathbf{p}))f(\xi) + \phi_{-+}(\mathbf{p})h(\xi)]e^{i\mathbf{p}\cdot\mathbf{x}} \\
\psi_{\mathbf{p},2}(\mathbf{x}, t) &= \frac{1}{\sqrt{\lambda_p}}(p\psi'_{\mathbf{p},1}(\mathbf{x}, t) + M\psi'_{\mathbf{p},2}(\mathbf{x}, t)) \\
&= C[(p\phi_{++}(\mathbf{p}) + M\phi_{+-}(\mathbf{p}))f(\xi) + \phi_{--}(\mathbf{p})h(\xi)]e^{i\mathbf{p}\cdot\mathbf{x}} \qquad (7.17) \\
\psi_{\mathbf{p},3}(\mathbf{x}, t) &= C[\frac{(-M\phi_{++}(\mathbf{p}) + p\phi_{+-}(\mathbf{p}))}{\sqrt{\lambda_p}}\bar{h}(\xi) + \phi_{-+}(\mathbf{p})\sqrt{\lambda_p}\bar{f}(\xi)]e^{i\mathbf{p}\cdot\mathbf{x}} \qquad (7.18) \\
\psi_{\mathbf{p},4}(\mathbf{x}, t) &= C[\frac{(-p\phi_{++}(\mathbf{p}) - M\phi_{+-}(\mathbf{p}))}{\sqrt{\lambda_p}}\bar{h}(\xi) + \phi_{--}(\mathbf{p})\sqrt{\lambda_p}\bar{f}(\xi)]e^{j\mathbf{p}\cdot\mathbf{x}}
\end{aligned}
$$

It is easy to verify that these solutions are mutually orthogonal, and are each normalised to $(2\pi)^3\delta^{(3)}(0)$. The choice of $\psi_{\mathbf{p},1}$ and $\psi_{\mathbf{p},2}$ here (in terms of $\psi'_{\mathbf{p},1}$ and $\psi'_{\mathbf{p},2}$) is for later convenience.

7.2 Energy Eigenstates

In order to define $\mathcal{H}^{\pm}(t_0)$ at a given time t_0, we need to decompose the space of all spinor-valued functions $\psi(\mathbf{x}, t_0)$ (of space, at time t_0) in terms of the spectrum of the operator $\hat{H}(t_0)$ given by:

$$\hat{H}(t_0) : \psi(\mathbf{x}, t_0) \to -i \sum_k \bar{\sigma}_k \partial_k \psi(\mathbf{x}, t_0) + eA(t_0)\bar{\sigma}_3\psi(\mathbf{x}, t_0) + m\gamma_0\psi(\mathbf{x}, t_0) \tag{7.19}$$

For this, consider the action of $\hat{H}(t_0)$ on plane wave states of the form of (7.3). It is easy to see that:

$$\begin{gathered} \hat{H}(t_0)\psi_{\mathbf{p}}(\mathbf{x}, t_0) = E_{\mathbf{p},t_0}\psi_{\mathbf{p}}(\mathbf{x}, t_0) \text{ if and only if} \\ m\gamma_0\psi_{\mathbf{p}}(t_0) + \bar{\mathbf{p}}_\perp\psi_{\mathbf{p}}(t_0) + (p^3 + eA(t_0))\bar{\sigma}_3\psi_{\mathbf{p}}(t_0) = E_{\mathbf{p},t_0}\psi_{\mathbf{p}}(t_0) \end{gathered} \tag{7.20}$$

Substituting $\psi_{\mathbf{p},t_0} = \sum_\lambda \{\phi_{+\lambda}(\mathbf{p}) f_{\lambda,t_0} + \phi_{-\lambda}(\mathbf{p}) g_{\lambda,t_0}\}$ and using (7.11) allows us to turn this into the eigenvalue problem:

$$\sqrt{eE}\begin{bmatrix} \xi_0 + a(\xi_0) & 0 & M & p \\ 0 & \xi_0 + a(\xi_0) & -p & M \\ M & -p & -\xi_0 - a(\xi_0) & 0 \\ p & M & 0 & -\xi_0 - a(\xi_0) \end{bmatrix} \begin{bmatrix} f_{+,t_0} \\ f_{-,t_0} \\ g_{+,t_0} \\ g_{-,t_0} \end{bmatrix} = E_{\mathbf{p},t_0} \begin{bmatrix} f_{+,t_0} \\ f_{-,t_0} \\ g_{+,t_0} \\ g_{-,t_0} \end{bmatrix} \tag{7.21}$$

This matrix squares to $eE(\lambda + (\xi_0 + a(\xi_0))^2)I_4 = (m^2 + |\mathbf{p}_\perp|^2 + (p^3 + eEA(t_0))^2)I_4$. Hence its eigenvalues are $E_{\mathbf{p},t_0} = \pm\sqrt{m^2 + |\mathbf{p}_\perp|^2 + (p^3 + eEA(t_0))^2} = \pm\sqrt{eE}E_{\xi_0}$ where I have defined the dimensionless quantity $E_\xi \equiv \sqrt{\lambda + (\xi + a(\xi))^2}$. Consider the following cases:

Case 1: $E_{\mathbf{p},t_0} = \sqrt{eE}E_{\xi_0}$

We need two independent solutions of (7.21) with $E_{\mathbf{p},t_0} = \sqrt{eE}E_{\xi_0}$. Desiring a similarity between these and the $\psi_{\mathbf{p},i}(\mathbf{x}, t)$ motivates choosing $u_{\mathbf{p},1,t_0}(\mathbf{x})$ such that $g_{+,t_0} = 1$ and $g_{-,t_0} = 0$. Substituting into (7.21), this gives:

$$u_{\mathbf{p},1,t_0}(\mathbf{x}) = \frac{D}{E_{\xi_0} - \xi_0 - a(\xi_0)}\{M\phi_{++}(\mathbf{p}) - p\phi_{+-}(\mathbf{p}) + (E_{\xi_0} - \xi_0 - a(\xi_0))\phi_{-+}(\mathbf{p})\}e^{i\mathbf{p}\cdot\mathbf{x}} \tag{7.22}$$

where $D =$ constant, and we have used the identity $(E_{\xi_0} - \xi_0 - a(\xi_0))(E_{\xi_0} + \xi_0 + a(\xi_0)) = \lambda$. Requiring that $u_{\mathbf{p},1,t_0}(\mathbf{x})$ have norm $(2\pi)^3\delta^3(0)$ gives $D = \sqrt{\frac{E_{\xi_0} - \xi_0 - a(\xi_0)}{2E_{\xi_0}}}$.

Similarly, $u_{\mathbf{p},2,t_0}(\mathbf{x})$ can be found by putting $g_{+,t_0} = 0$, $g_{-,t_0} = 1$, and then normalising to $(2\pi)^3\delta^3(0)$. This gives:

$$u_{\mathbf{p},1,t_0}(\mathbf{x}) = \frac{M\phi_{++}(\mathbf{p}) - p\phi_{+-}(\mathbf{p}) + (E_{\xi_0} - \xi_0 - a(\xi_0))\phi_{-+}(\mathbf{p})}{\sqrt{2E_{\xi_0}(E_{\xi_0} - \xi_0 - a(\xi_0))}} e^{i\mathbf{p}\cdot\mathbf{x}} \tag{7.23}$$

$$u_{\mathbf{p},2,t_0}(\mathbf{x}) = \frac{p\phi_{++}(\mathbf{p}) + M\phi_{+-}(\mathbf{p}) + (E_{\xi_0} - \xi_0 - a(\xi_0))\phi_{--}(\mathbf{p})}{\sqrt{2E_{\xi_0}(E_{\xi_0} - \xi_0 - a(\xi_0))}} e^{i\mathbf{p}\cdot\mathbf{x}} \tag{7.24}$$

Case 2: $E_{\mathbf{p},t_0} = -\sqrt{eE}E_{\xi_0}$
A similar procedure to above gives:

$$v_{\mathbf{p},1,t_0}(\mathbf{x}) = \frac{-M\phi_{++}(\mathbf{p}) + p\phi_{+-}(\mathbf{p}) + (E_{\xi_0} + \xi_0 + a(\xi_0))\phi_{-+}(\mathbf{p})}{\sqrt{2E_{\xi_0}(E_{\xi_0} + \xi_0 + a(\xi_0))}} e^{i\mathbf{p}\cdot\mathbf{x}} \tag{7.25}$$

$$v_{\mathbf{p},2,t_0}(\mathbf{x}) = \frac{-p\phi_{++}(\mathbf{p}) - M\phi_{+-}(\mathbf{p}) + (E_{\xi_0} + \xi_0 + a(\xi_0))\phi_{--}(\mathbf{p})}{\sqrt{2E_{\xi_0}(E_{\xi_0} + \xi_0 + a(\xi_0))}} e^{i\mathbf{p}\cdot\mathbf{x}} \tag{7.26}$$

Before proceeding, some comments are in order:

1. We could have found $\mathcal{H}^{\pm}(t_0)$ by writing $\psi_{\mathbf{p},t_0} = \begin{bmatrix} \phi_{\mathbf{p},t_0} \\ \kappa_{\mathbf{p},t_0} \end{bmatrix}$ in the similar way to that used to deduce the plane wave solutions to the free Dirac equation. In this way we see that the positive energy eigenstates are of the form:

$$u_{\mathbf{p},\lambda,t_0}(\mathbf{x}) = \sqrt{\frac{E_{\mathbf{p},t_0} + m}{2E_{\mathbf{p},t_0}}} \begin{bmatrix} \phi_{\lambda,\mathbf{p}'} \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{E_{\mathbf{p},t_0}+m}\phi_{\lambda,\mathbf{p}'} \end{bmatrix} e^{i\mathbf{p}\cdot\mathbf{x}} \tag{7.27}$$

and the negative energy eigenstates of the form

$$v_{\mathbf{p},\lambda,t_0}(\mathbf{x}) = \sqrt{\frac{E_{\mathbf{p},t_0} + m}{2E_{\mathbf{p},t_0}}} \begin{bmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{E_{\mathbf{p},t_0}+m}\phi_{\lambda,\mathbf{p}'} \\ \phi_{\lambda,\mathbf{p}'} \end{bmatrix} e^{i\mathbf{p}\cdot\mathbf{x}} \tag{7.28}$$

where[1] $\boldsymbol{\sigma}\cdot\mathbf{p}' \equiv p^1\sigma^1 + p^2\sigma^2 + (p^3 + eA(t_0))\sigma^3$ and the $\phi_{\lambda,\mathbf{p}'}$ are 2-spinors, chosen for convenience. Although these expressions are less convenient than (7.23) - (7.26) for our purposes, they do reveal that these eigenstates are just like the plane wave states of free Dirac field theory, except that the momentum in the spinor does not match that in the exponential. These states are eigenstates of the gauge invariant 3-momentum operator $-i\partial_k + eA_k$ of momentum, $\mathbf{p}' = (p^1, p^2, p^3 + eA(t_0))$. Hence the 1-electron state $\overrightarrow{u}_{\mathbf{p},\lambda,t_0} \wedge |vac_{t_0}\rangle$ has physical (gauge invariant) momentum $\mathbf{p}'$, while the 1-positron state $i_{\overrightarrow{v}_{\mathbf{p},\lambda,t_0}}|vac_{t_0}\rangle$ has physical momentum $-\mathbf{p}'$. With this choice of gauge, $\mathbf{p}$ represents the physical momentum only at $t = 0$. Also, by acting on these states with the current operator $\hat{J}^k = e\bar{\sigma}_k$ we find that, although none of these states are eigenstates of $\hat{J}^k$ (this is only possible if $m = 0$, or $E_{\mathbf{p}} = \infty$) their expected

[1] The σ^k are the 2×2 Pauli matrices. They are related to the $\bar{\sigma}_k$ by $\bar{\sigma}_k = \begin{bmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{bmatrix}$.

current is $\hat{J}^k(u_{\mathbf{p},\lambda,t_0}) = \frac{e}{E_{\mathbf{p}'}}(p^1, p^2, p^3 + eA(t_0))$, $\hat{J}^k(v_{\mathbf{p},\lambda,t_0}) = -\frac{e}{E_{\mathbf{p}'}}(p^1, p^2, p^3 + eA(t_0))$, so that:

$$\hat{J}^k(\overrightarrow{u}_{\mathbf{p},\lambda,t_0} \wedge |vac_{t_0}\rangle) = \frac{e}{E_{\mathbf{p}'}}(p^1, p^2, p^3 + eA(t_0)) = \hat{J}^k(i_{\overrightarrow{v}^\dagger_{\mathbf{p},\lambda,t_0}} |vac_{t_0}\rangle) \tag{7.29}$$

just as one would expect classically for an electron of momentum $\mathbf{p}'$ (and charge e) or a positron of momentum $-\mathbf{p}'$ (and charge $-e$).

2. Suppose that $A(t_0) \to \mp\infty$ as $t_0 \to \pm\infty$. This is certainly true for the constant electric field $A(t) = Et$ $(E < 0)$ and is also true of many other electric fields. In this case we have $\hat{J}^k(u_{\mathbf{p},\lambda,-\infty}) = (0, 0, e) = -\hat{J}^k(v_{\mathbf{p},\lambda,-\infty})$, while the reverse is true at $t \to +\infty$. That is:

$$\bar{\sigma}_3 u_{\mathbf{p},\lambda,-\infty}(\mathbf{x}) = u_{\mathbf{p},\lambda,-\infty}(\mathbf{x}) \qquad \bar{\sigma}_3 v_{\mathbf{p},\lambda,-\infty}(\mathbf{x}) = -v_{\mathbf{p},\lambda,-\infty}(\mathbf{x}) \tag{7.30}$$

Similarly, as $t_0 \to +\infty$ u and v become such that:

$$\bar{\sigma}_3 u_{\mathbf{p},\lambda,\infty}(\mathbf{x}) = -u_{\mathbf{p},\lambda,\infty}(\mathbf{x}) \qquad \bar{\sigma}_3 v_{\mathbf{p},\lambda,\infty}(\mathbf{x}) = v_{\mathbf{p},\lambda,\infty}(\mathbf{x}) \tag{7.31}$$

This signifies the fact that any electron (of charge e) in this electric field, that has a finite velocity at finite time, will at late times be moving in the positive x^3 direction with infinite momentum (i.e. at speeds approaching that of light), while at very early times it will have been moving in the negative x^3 direction at speeds approaching that of light, whereas for a positron, with charge $-e$ this situation is reversed. We see here how important it is to keep track of what constitutes particles/antiparticles at different times, since not only does this change with time in an electric field, but we see here that between early and late times the spaces $\mathcal{H}^{\pm}$ have been completely reversed; $\mathcal{H}^{\pm}(-\infty) = \mathcal{H}^{\mp}(\infty)$! The fact that $\mathcal{H}^{\pm}(-\infty) = \mathcal{H}^{\mp}(\infty)$ is also mentioned in [11].

7.3 Finite Time Transition Probabilities

In order to use the methods described in Chapter 2 we wish to calculate the matrix elements of the evolution operator on $\mathcal{H}$:

$$\begin{aligned} S_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t_1,t_0) &= \begin{bmatrix} \alpha_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t_1,t_0) & \beta_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t_1,t_0) \\ \gamma_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t_1,t_0) & \epsilon_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t_1,t_0) \end{bmatrix} \\ &= \begin{bmatrix} \langle u_{\mathbf{p},\lambda;t_1}(\mathbf{x})|\hat{U}_1(t_1,t_0)|u_{\mathbf{q},\sigma;t_0}(\mathbf{x})\rangle & \langle u_{\mathbf{p},\lambda;t_1}(\mathbf{x})|\hat{U}_1(t_1,t_0)|v_{\mathbf{q},\sigma;t_0}(\mathbf{x})\rangle \\ \langle v_{\mathbf{p},\lambda;t_1}(\mathbf{x})|\hat{U}_1(t_1,t_0)|u_{\mathbf{q},\sigma;t_0}(\mathbf{x})\rangle & \langle v_{\mathbf{p},\lambda;t_1}(\mathbf{x})|\hat{U}_1(t_1,t_0)|v_{\mathbf{q},\sigma;t_0}(\mathbf{x})\rangle \end{bmatrix} \end{aligned}$$

where $\hat{U}_1(t_1,t_0) = \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \sum_{i=1}^4 |\psi_{\mathbf{p}',i}(t_1)\rangle\langle\psi_{\mathbf{p}',i}(t_0)|$ is the evolution operator on $\mathcal{H}$, written in terms of the complete set of solutions derived earlier. That is:

$$S_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t_1,t_0) = \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \sum_{i=1}^4 M_{\mathbf{p},\lambda;\mathbf{p}',i}(t_1)\bar{M}_{\mathbf{q},\sigma;\mathbf{p}',i}(t_0) \tag{7.32}$$

where:

$$M_{\mathbf{p},\lambda;\mathbf{p}',i}(t) = \begin{bmatrix} \langle u_{\mathbf{p},\lambda;t}(\mathbf{x})|\psi_{\mathbf{p}',i}(\mathbf{x},t)\rangle \\ \langle v_{\mathbf{p},\lambda;t}(\mathbf{x})|\psi_{\mathbf{p}',i}(\mathbf{x},t)\rangle \end{bmatrix} \tag{7.33}$$

which is unitary for all time, since it describes the overlap between two orthonormal sets of states.

We can now use (7.23) - (7.26), (7.17) and (7.18) to calculate $M(t)$, and we find:

$$M_{\mathbf{p},\lambda;\mathbf{p}',i}(t) = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}') \begin{bmatrix} A(\xi(t))I_2 & -\bar{B}(\xi(t))I_2 \\ B(\xi(t))I_2 & \bar{A}(\xi(t))I_2 \end{bmatrix} \tag{7.34}$$

where

$$A(\xi) = \frac{C}{\sqrt{2E_\xi(E_\xi - \xi - a(\xi))}}((E_\xi - \xi - a(\xi))h(\xi) + \lambda_p f(\xi)) \tag{7.35}$$

$$= C\sqrt{\frac{E_\xi - \xi - a(\xi)}{2E_\xi}}(i\frac{df}{d\xi} + E_\xi f) \tag{7.36}$$

$$B(\xi) = \frac{C}{\sqrt{2E_\xi(E_\xi + \xi + a(\xi))}}((E_\xi + \xi + a(\xi))h(\xi) - \lambda f(\xi)) \tag{7.37}$$

$$= C\sqrt{\frac{E_\xi + \xi + a(\xi)}{2E_\xi}}(i\frac{df}{d\xi} - E_\xi f) \tag{7.38}$$

and I_2 is the 2×2 identity matrix. The unitarity of $M(t)$ can be verified by using $C = (\lambda|f|^2 + |h|^2)^{-\frac{1}{2}}$. From (7.34) we get:

$$S_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t_1,t_0) = (2\pi)^3\delta(\mathbf{p}-\mathbf{q}) \begin{bmatrix} \alpha(\xi(t_1),\xi(t_0))I_2 & \beta(\xi(t_1),\xi(t_0))I_2 \\ -\bar{\beta}(\xi(t_1),\xi(t_0))I_2 & \bar{\alpha}(\xi(t_1),\xi(t_0))I_2 \end{bmatrix} \tag{7.39}$$

where:

$$\alpha(\xi_1,\xi_0) \equiv A(\xi_1)\bar{A}(\xi_0) + \bar{B}(\xi_1)B(\xi_0) \tag{7.40}$$

$$\beta(\xi_1,\xi_0) \equiv A(\xi_1)\bar{B}(\xi_0) - \bar{B}(\xi_1)A(\xi_0) \tag{7.41}$$

We can now use the formulae presented in Section 2.3 to extract from (7.39) any information that we desire about this system of electrons evolving in a background $\boldsymbol{E} = (0,0,\dot{A}(t))$, expressed in terms of an arbitrary solution to equation (7.13). It is worth noting at this point that although $\boldsymbol{M}(t)$ will in general depend on the initial conditions chosen for $f(\xi)$ in (7.13) the elements of $\boldsymbol{S}(t)$ calculated in this Chapter are completely independent of this.

In the following Section I will consider the time evolution of an initial vacuum state, and will consider specifically the example of a constant electric field.

7.4 Constant Electric Fields

In this case we $a(\xi) = 0$, so (7.13) becomes:

$$\frac{d^2 f}{d\xi^2} + (\lambda_p + \xi^2 + i)f = 0 \tag{7.42}$$

Comparison with 8.2 (1) of [88] reveals that solutions of this are linear combinations of $D_{\nu_p - 1}(\pm(1-i)\xi)$ and $D_{-\nu_p}(\pm(1+i)\xi)$, where $\nu_p \equiv \frac{i\lambda_p}{2}$, and $D_\nu(z)$ are Parabolic Cylinder Functions. For convenience we choose $f(\xi) \equiv D_{\nu_p - 1}(-(1-i)\xi)$. This gives (using properties of the parabolic cylinder functions) $h(\xi) = (1+i)D_{\nu_p}(-(1-i)\xi)$ and $C = \frac{e^{\frac{-\pi\lambda_p}{8}}}{\sqrt{2}}$.

The convenience of this particular choice of $f(\xi)$ is that $|f(\xi)| \to 0$ as $\xi \to -\infty$, while $|h(\xi)|$ remains finite in this limit. This results in:

$$\lim_{\xi \to -\infty} \boldsymbol{M}(\xi) = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) \begin{bmatrix} e^{iC} I_2 & 0 \\ 0 & e^{-iC} I_2 \end{bmatrix} \tag{7.43}$$

where C is a physically unimportant phase factor. This allows us to identify $M(\xi(t))$ with $\boldsymbol{S}(t, -\infty)$ (up to this phase factor).

From (7.34) we have

$$S_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty, -\infty) = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) \begin{bmatrix} A I_2 & -\bar{B} I_2 \\ B I_2 & \bar{A} I_2 \end{bmatrix} \tag{7.44}$$

$$\text{where } A = \lim_{t \to \infty} A(\xi(t)) = \frac{\sqrt{\lambda_p}}{\sqrt{2}} e^{\frac{-\pi\lambda_p}{8}} \lim_{t \to \infty} f(\xi(t)) \tag{7.45}$$

$$\text{and } B = \lim_{t \to \infty} B(\xi(t)) = \frac{e^{\frac{-\pi\lambda_p}{8}}}{\sqrt{2}} \lim_{t \to \infty} h(\xi(t)) \tag{7.46}$$

(since $|f|$ and $|h|$ are both bounded above for all time). This gives $\boldsymbol{\beta}_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty, -\infty) = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) \delta_{\lambda\sigma} B$ so that

$$(\boldsymbol{\beta}^\dagger \boldsymbol{\beta})_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty, -\infty) = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) \frac{e^{\frac{-\pi\lambda}{4}}}{2} \lim_{t \to \infty} |h(\xi(t))|^2 \delta_{\lambda\sigma} \tag{7.47}$$

An asymptotic expansion of the parabolic cylinder functions gives $\lim_{t\to\infty} |h(\xi(t))|^2 = 2e^{\frac{-3\pi\lambda}{4}}$ so that

$$(\boldsymbol{\beta}^\dagger \boldsymbol{\beta})_{\mathbf{p},\lambda;\mathbf{q},\sigma}(\infty, -\infty) = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) e^{-\pi\lambda_p} \delta_{\lambda\sigma} \tag{7.48}$$

which would correspond to a number density of created particles (not including anti-particles), per unit volume, of:

$$N^{\infty}_{\mathbf{p},\lambda} = e^{-\pi\lambda_p} \tag{7.49}$$

This agrees with the result given in [16]. However, a closer analysis of $S(t_1,t_0)$ reveals that it is wrong to interpret $S(\infty,-\infty)$ as representing the infinite time transition amplitude. I explain this here.

For the physical significance of $S(t_1,t_0)$, and its dependence on p^3 we must take into account the fact that p^3 appears in S only in the form of $\xi=\frac{p^{3'}}{\sqrt{eE}}$ where $p^{3'}=eEt+p^3$ is the physical 'z-momentum' at time t of the state denoted by the subscript $\mathbf{p}$. Hence, the most physically appropriate representation of $S(t_1,t_0)$ is in terms of the 'elapsed time' $\triangle t\equiv t_1-t_0=\frac{\xi_1-\xi_0}{\sqrt{eE}}(\equiv\frac{\tau}{\sqrt{eE}}$ say) and the gauge covariant 'in' and 'out' momenta $\mathbf{p}_{in}=(p^1,p^2,p^3+eEt_0)$ and $\mathbf{p}_{out}=(p^1,p^2,p^3+eEt_1)=\mathbf{p}_{in}+(0,0,eE\triangle t)$. In this form we have:

$$S_{\mathbf{p}_{out},\lambda;\mathbf{p}_{in},\sigma}(\tau)=(2\pi)^3\delta^{(2)}(\mathbf{p}_{out\perp}-\mathbf{p}_{in\perp})\delta(p^3_{out}-p^3_{in}-eE\triangle t)\begin{bmatrix}\alpha(p_z,p_z-\tau)I_2 & -\bar{\beta}(p_z,p_z-\tau)I_2\\ \beta(p_z,p_z-\tau)I_2 & \bar{\alpha}(p_z,p_z-\tau)I_2\end{bmatrix}\tag{7.50}$$

where I have defined the dimensionless quantity $p_z\equiv\frac{p^3_{out}}{\sqrt{eE}}$ for convenience. Understanding these subtleties allows us to recognise that the physically relevant infinite-time scattering matrix is *not* $S(\infty,-\infty)$, but rather:

$$\lim_{\tau\to\infty}S_{\mathbf{p}_{out},\lambda;\mathbf{p}_{in},\sigma}(\tau)=S(p_z,-\infty)\tag{7.51}$$

The result of this is that the probability of creation per unit volume $N^{\infty}_{\mathbf{p},\lambda}$ is not $e^{-\pi\lambda_p}$ as often quoted, but rather:

$$N^{\infty}_{\mathbf{p},\lambda}=|B(p_z)|^2\tag{7.52}$$

$$=\frac{e^{\frac{-\pi\lambda_p}{4}}}{4E_{p_z}(E_{p_z}+p_z)}|(E_{p_z}+p_z)(1+i)D_{\frac{i\lambda_p}{2}}(-(1-i)p_z)-\lambda_pD_{\frac{i\lambda_p}{2}-1}(-(1-i)p_z)|^2\tag{7.53}$$

Graphs of this, for various values of $\frac{eE}{m^2}$ are shown in Figures 7.1 and 7.2. We see that $N^{\infty}_{\mathbf{p},\lambda}$ is effectively zero for negative p_z while tending towards $e^{-\pi\lambda_p}$ for large p_z. This result agrees with that found in 1998 by Kluger, Mottola et. al.[87], and is of course much more reasonable than (7.49) when we consider that any particle, once created, will be accelerated to ever increasing p_z by the electric field. The number density of created antiparticles is identical, but with the sign of p_z reversed. A more relevant result than (7.49), which allows Schwinger's formula to be derived in the large time limit, will be described soon.

7.4.1 Number Density In The Evolved Vacuum

If we start the system at some initial time in the vacuum state, and let this state evolve for a time $\triangle t=\frac{\tau}{\sqrt{eE}}$, then this evolved state will in general contain created particle/antiparticle pairs, and we can now investigate the distribution of these pairs. Figure 7.3 is a graph of the expected number density in the evolved vacuum, for $eE=m^2$

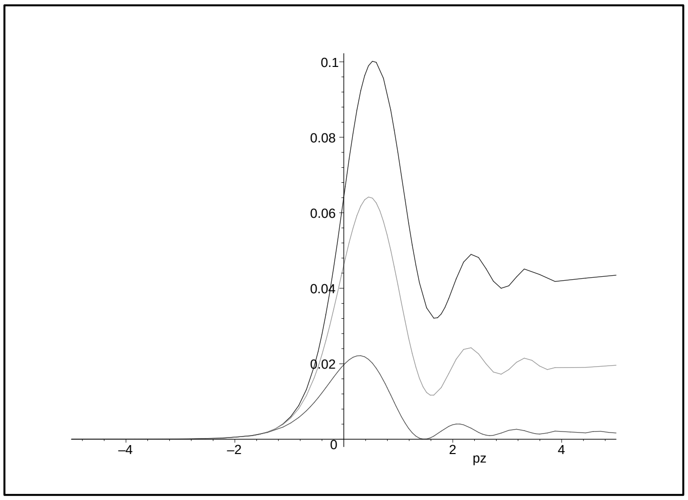

Figure 7.1: Number density in an infinitely evolved vacuum, as a function of $p_z = \frac{p_{out}^3}{\sqrt{eE}}$ for $M = 1$ and $p = 0$ (top curve) , 0.5 (middle curve), 1 (bottom curve).

and $|\mathbf{p}_\perp| = m$, as a function of p_z, for $\triangle t = \frac{7}{\sqrt{eE}}$ (right), $\frac{4}{\sqrt{eE}}$, $\frac{2}{\sqrt{eE}}$ and $\frac{0.5}{\sqrt{eE}}$ (single peak). A graph for $\triangle t = \infty$ is also included, which is seen to fit the left peak of the $\triangle t = 7\sqrt{eE}$ line, while becoming constant at large p_z. The finite time behaviour demonstrated here agrees with that presented in [14]. The physical interpretation of Figure 7.3 is quite interesting. We see that at early times, particles are created mainly with p_z close to zero (Figure 7.2 shows that p close to zero is also favoured) as would be expected, since these pairs require the least creation energy. At later times, the right peak represents the same early time particles (which have of course been accelerated by the electric field) while the left peak represents recently created pairs, again at low momenta. The low oscillations in between (about $e^{-\pi\lambda_p}$) are due to quantum interference effects.

Summing over all spin and momentum channels gives the total number of pairs created:

$$N_{TOT}(\tau) \equiv V \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_\lambda N_{\mathbf{p},\lambda}(\tau) = \frac{V(eE)^{\frac{3}{2}}}{4\pi^2} \int_{-\infty}^{\infty} \int_{M^2}^{\infty} d\lambda_p dp_z |\beta(p_z, p_z - \tau)|^2 \quad (7.54)$$

In Figure 7.4 we have graphed $\frac{N_{TOT}(\tau)}{V(eE)^{\frac{3}{2}}}$ for $M^2 = \frac{1}{2}$, 1 and 2.

To help put this in perspective, it is worth considering the units involved here. The value of E for which $M = 1$ is $-E = E_c = \frac{m^2c^3}{|e|\hbar} \approx 1.3 \times 10^{16} V/cm$, where we have substituted the mass and charge of the electron. This is too large to be produced on a macroscopic scale in the laboratory (it corresponds to an electromagnetic energy density of approximately 137 electron rest mass energies per Compton volume), although astrophysical examples[89, 86] have been considered. Also, on a microscopic scale fields of this strength are present in superheavy nuclei, with charges greater than $137|e|$. To see the time scales involved, note that $\tau = 1$ corresponds to $\triangle t = \frac{\lambda_c}{2\pi c}\sqrt{\frac{E_c}{-E}} \approx 1.288 \times 10^{-21}\sqrt{\frac{E_c}{-E}}$ seconds. The length scale corresponding to $\frac{1}{\sqrt{eE}}$ is $\triangle l = \sqrt{\frac{\hbar c}{eE}} = \frac{\lambda_c}{2\pi}\sqrt{\frac{E_c}{-E}} \approx 3.862 \times 10^{-11}\sqrt{\frac{E_c}{-E}}$ cm.

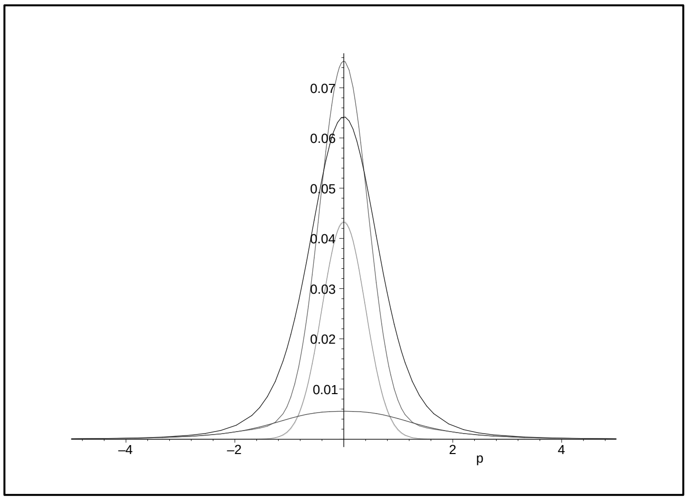

Figure 7.2: Number density in an infinitely evolved vacuum, as a function of p for $M = 1$ and $\frac{p_{out}^3}{\sqrt{eE}} = -1$ (bottom curve) , 0 (outermost curve), 1 (top curve) and 5 (innermost curve). A plot of $e^{-\pi\lambda_p}$ is also included, which coincides almost exactly with the curve for $\frac{p_{out}^3}{\sqrt{eE}} = 5$.

Now, from Figure 7.3 we see that for large τ, $N_{\mathbf{p},\lambda}(\tau) = |\beta_{\mathbf{p},\lambda}(p_z, p_z - \tau)|^2$ can be well approximated by

$$N_{\mathbf{p},\lambda}(\tau) \approx e^{-\pi\lambda_p}(\theta(p_z) - \theta(\tau - p_z)) \tag{7.55}$$

where $\theta(p_z)$ is the Heavyside step function. This allows us to approximate:

$$\int N_{\mathbf{p},\lambda}(\tau) dp^3 \approx \sqrt{eE}\tau e^{-\pi\lambda_p} \tag{7.56}$$

from which we get the total number of pairs created:

$$N_{TOT}(\tau) \approx \frac{(eE)^{\frac{3}{2}} V\tau}{4\pi^3} e^{\frac{-\pi m^2}{eE}} \text{ for } \tau \text{ large} \tag{7.57}$$

This correctly predicts the final gradient of the graphs in Figure 7.4 and is the result most commonly referred to in the literature[14, 16, 21]. However, we see in Figure 7.4 that for $|E| << E_c$ the dominant contribution to the total number of produced pairs comes from the initial burst of pair creation (the late time rate of creation is exponentially small in this case). Even for $|E| = \frac{1}{2}E_c$, we have ~ 0.44 particles created in a volume λ_c^3 in the first 10^{-20} seconds (i.e. up until $\tau = 4$), with only ~ 0.005 particles every 10^{-20} seconds after that. For $|E| = \frac{E_c}{10}$ we have ~ 0.014 particles created in a volume λ_c^3 in the first 2.5×10^{-20} seconds (up until $\tau = 6$) with only 8.6×10^{-15} particles created every 2.5×10^{-20} seconds after that. This difference becomes exponentially more pronounced as the ratio $|\frac{E_c}{E}|$ increases.

This fact could possibly help in the experimental detection of this effect, although it should be pointed out that this initial burst of particles will accelerate out of the region of interest soon after the electric field is 'switched on' (in any finite laboratory experiment) so it may be difficult to detect.

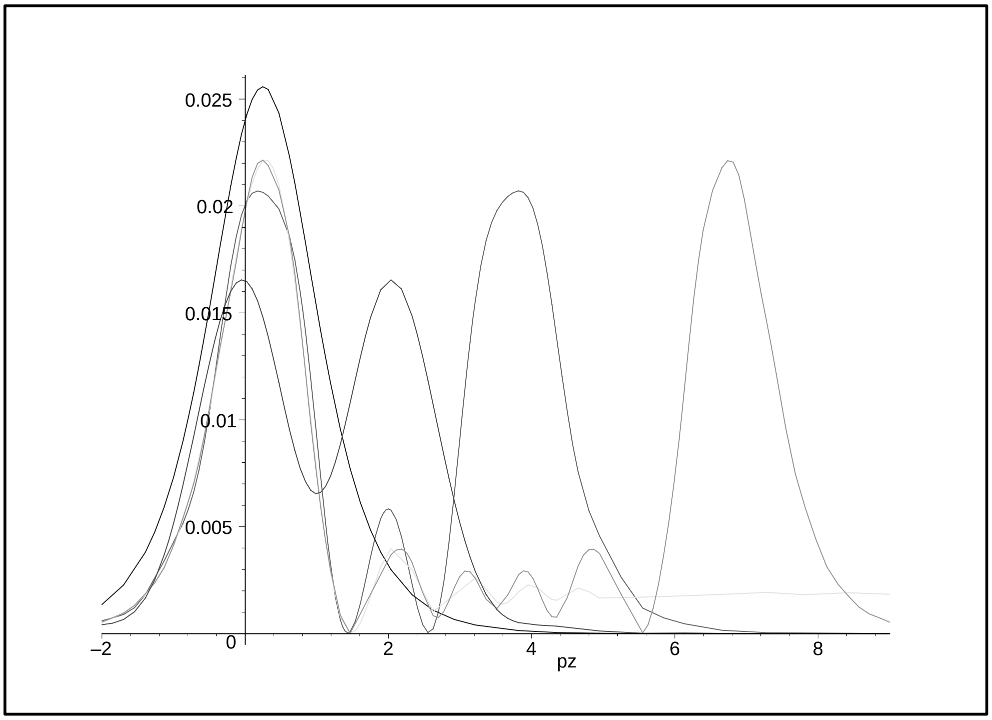

Figure 7.3: Number density in the evolved vacuum, as a function of $\frac{p^3_{out}}{\sqrt{eE}}$, for $M = 1 = p$, and $\tau = 7$ (right), 4, 2 and .5 (single peak). A graph for $\tau = \infty$ is included, which is seen to fit the left peak of the $\tau = 7$ line, while becoming constant at large $\frac{p^3_{out}}{\sqrt{eE}}$.

Similarly, we can use equation (2.101) to calculate the vacuum - vacuum transition probability:

$$\mathcal{P}_{vac\to vac}(\tau) = \exp(V \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{\lambda} \log(1 - N_{\mathbf{p},\lambda}(\tau))) \tag{7.58}$$

$$= \exp(\frac{(eE)^{\frac{3}{2}} V}{4\pi^2} \int_{-\infty}^{\infty} \int_{M^2}^{\infty} d\lambda_p dp_z \log(1 - N_{\mathbf{p},\lambda}(\tau)) \tag{7.59}$$

Substituting (7.55) and writing $\log(1 - e^{-\pi\lambda}) = -\sum_{n=1}^{\infty} \frac{e^{-\pi n\lambda}}{n}$ gives:

$$\mathcal{P}_{vac\to vac}(\tau) \approx \exp(\frac{-(eE)^{\frac{3}{2}} V\tau}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{\frac{-\pi n m^2}{eE}}) \text{ for } \tau \text{ large.} \tag{7.60}$$

in agreement with Schwinger's original calculation[28] (also in [41], [16], [14] and others). In Figure 7.5 we show the exact vacuum - vacuum transition probability, and the large time approximation of (7.60) for $|E| = E_c$ and $V = 10\lambda_c^3$.

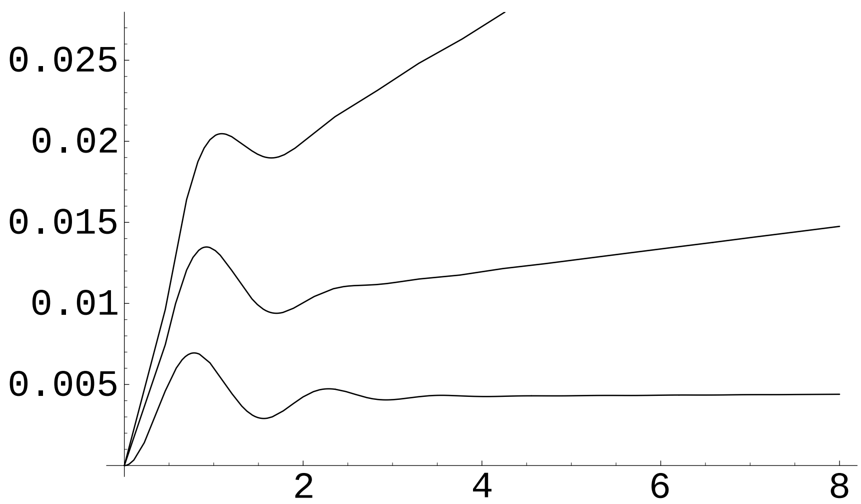

Figure 7.4: Total particle number (in a volume $(\triangle l)^3$) in the evolved vacuum, as a function of τ, for $\frac{eE}{m^2} = \frac{1}{2}$ (bottom curve), 1 and 2 (top curve).

7.4.2 Vacuum Current, and Induced Electric Field

From Section 2.3.2 we have:

$$\langle vac_{t_0}(t)|\hat{J}_{phys}(t)|vac_{t_0}(t)\rangle = Trace(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger \boldsymbol{J}^{++} - \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger \boldsymbol{J}^{--} + \boldsymbol{\epsilon}\boldsymbol{\beta}^\dagger \boldsymbol{J}^{+-} + \boldsymbol{\beta}\boldsymbol{\epsilon}^\dagger \boldsymbol{J}^{-+}) \quad (7.61)$$

for any operator $\hat{J}_{phys}$, where we have defined:

$$\boldsymbol{J}^{++}_{kj} \equiv \langle u_{k,t}|\hat{J}_1(t)|u_{j,t}\rangle \quad (7.62)$$

$$\boldsymbol{J}^{--}_{kj} \equiv \langle v_{k,t}|\hat{J}_1(t)|v_{j,t}\rangle \quad (7.63)$$

$$\boldsymbol{J}^{+-}_{kj} \equiv \langle u_{k,t}|\hat{J}_1(t)|v_{j,t}\rangle \quad (7.64)$$

$$\text{and } \boldsymbol{J}^{-+}_{kj} \equiv \langle v_{k,t}|\hat{J}_1(t)|u_{j,t}\rangle \quad (7.65)$$

$$= \overline{\boldsymbol{J}^{+-}_{jk}} \text{ if } \hat{J}_1 \text{ is Hermitian}$$

and $\hat{J}_1$ is the operator on $\mathcal{H}$ from which $\hat{J}_{phys}$ is defined. Taking $\hat{J}_1$ to be the z-component of the current density (so that $\hat{J}_1 = e\bar{\sigma}_3\delta(\mathbf{x}-\mathbf{y})$ as in Section[2] 2.4) we can evaluate these

[2] I acknowledge here that strictly speaking this is not well-defined (for the reasons mentioned at the end of section 5.1 for instance), although since the problem is spatially uniform, then we can interpret the results as being total current divided by total volume (rather than the current density at a point).

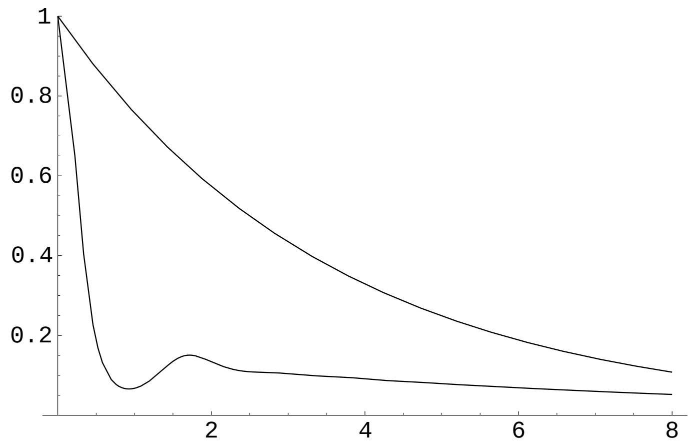

Figure 7.5: Vacuum - vacuum transition probability for $|E| = E_c$ and $V = 10\lambda_c^3$. The initial burst of pair creation cause the exact transition probability (bottom curve) to be significantly lower than Schwinger's prediction (top curve) at early times.

matrix elements easily using (7.9) and (7.7). Then, using

$$(\beta\beta^\dagger)_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t,t_0) = (2\pi)^3\delta(\mathbf{p}-\mathbf{q})\delta_{\lambda\sigma}|\beta(t,t_0)|^2 = (\gamma\gamma^\dagger)_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t,t_0) \tag{7.66}$$

$$(\beta\epsilon^\dagger)_{\mathbf{p},\lambda;\mathbf{q},\sigma}(t,t_0) = (2\pi)^3\delta(\mathbf{p}-\mathbf{q})\delta_{\lambda\sigma}\beta(t,t_0)\alpha(t,t_0) \tag{7.67}$$

we find the current density in a vacuum that has evolved for a time $\triangle t = \frac{\tau}{\sqrt{eE}}$ to be:

$$J_{3,vac}(\tau) \equiv \frac{\langle vac_{t_0}(t_0+\triangle t)|\hat{J}_{phys}|vac_{t_0}(t_0+\triangle t)\rangle}{\langle vac_{t_0}(t_0+\triangle t)|vac_{t_0}(t_0+\triangle t)\rangle} \tag{7.68}$$

$$= 4e\int \frac{d^3\mathbf{p}_{out}}{(2\pi)^3 E_{p_z}}(p_z|\beta(p_z,p_z-\tau)|^2 - \sqrt{\lambda_p}\Re(\beta(p_z,p_z-\tau)\alpha(p_z,p_z-\tau))) \tag{7.69}$$

$$= \frac{e(eE)^{\frac{3}{2}}}{2\pi^2}\int_{-\infty}^{\infty}\int_{M^2}^{\infty}\frac{d\lambda_p dp_z}{E_{p_z}}(p_z|\beta(p_z,p_z-\tau)|^2 - \sqrt{\lambda_p}\Re(\beta(p_z,p_z-\tau)\alpha(p_z,p_z-\tau))) \tag{7.70}$$

where $\Re$ denotes taking the real component.

To compare this with the current before vacuum subtraction $J_{3,naive}(\tau) = \sum_i J(v_{i,t_0}(\mathbf{x},t_0+\triangle t))$ and with the alternative definition $J'_{3,vac}(\tau) = \frac{1}{2}\sum_i\{J(v_{i,t_0}(\mathbf{x},t_0+\triangle t)) - J(u_{i,t_0}(\mathbf{x},t_0+\triangle t))\}$, as mentioned in Section 2.4, notice that:

$$J(v_{i,t_0}(\mathbf{x},t_0+\triangle t)) = \frac{2e}{E_{p_z}}[p_z(|\beta(p_z,p_z-\tau)|^2 - \tfrac{1}{2}) - \sqrt{\lambda_p}\Re(\beta(p_z,p_z-\tau)\alpha(p_z,p_z-\tau))] \tag{7.71}$$

$$= -J(u_{i,t_0}(\mathbf{x},t_0+\triangle t)) \tag{7.72}$$

Hence:

$$J'_{3,vac}(\tau) = J_{3,naive}(\tau) \tag{7.73}$$

$$= \frac{e(eE)^{\frac{3}{2}}}{2\pi^2} \int_{-\infty}^{\infty} \int_{M^2}^{\infty} \frac{d\lambda_p dp_z}{E_{p_z}} [p_z(|\beta(p_z, p_z - \tau)|^2 - \tfrac{1}{2}) - \sqrt{\lambda_p} \Re(\beta(p_z, p_z - \tau)\alpha(p_z, p_z - \tau))] \tag{7.74}$$

$$= J_{3,vac}(\tau) - \frac{e(eE)^{\frac{3}{2}}}{4\pi^2} \int_{-\infty}^{\infty} \int_{M^2}^{\infty} \frac{p_z d\lambda_p dp_z}{E_{p_z}} \tag{7.75}$$

The final term in (7.75) corresponds to the vacuum subtraction. It is easily seen to be independent of time, as claimed in Section 2.3.2. Also, as mentioned in Section 2.4, although it is clearly divergent, it is often discarded due to being odd in p_z.

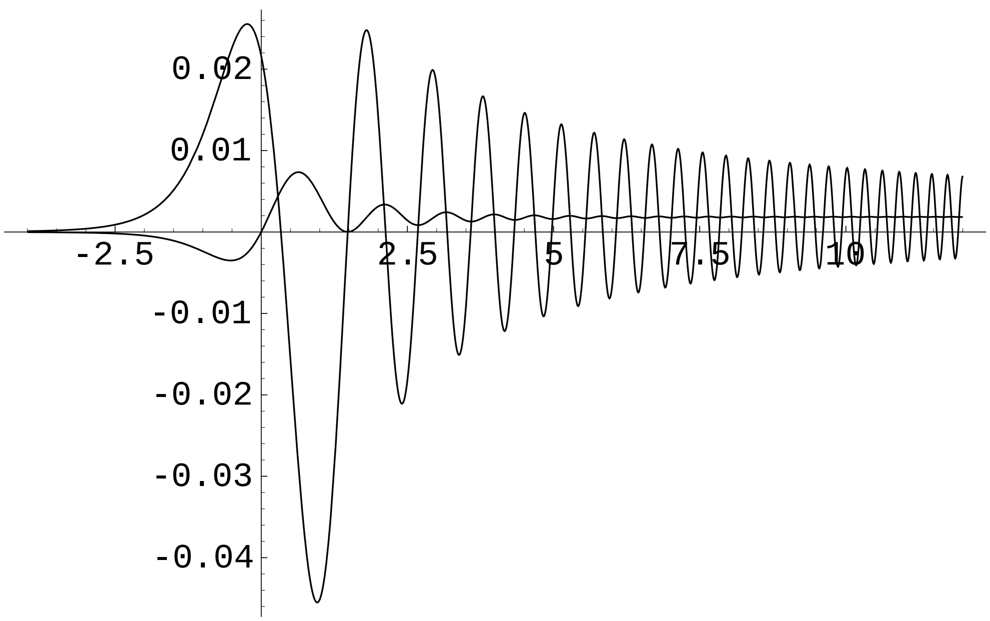

Figure 7.6: Current density in an infinitely evolved vacuum, as a function of p_z, for $\lambda_p = 2$. The $|\beta|^2$ contribution is also shown (the smoother of the two curves). The $\Re(\beta\alpha)$ term produces rapid oscillations about the mean contribution.

In Figures 7.6, 7.7 and 7.8 we have graphed the integrand of (7.70), as a function of p_z, for various evolution times, and have included on these a graph of the $|\beta|^2$ contribution. We can see from these that at late times the dominant contribution to the integral comes from the $|\beta|^2$ term, while the other term simply generates rapid oscillations about the $|\beta|^2$ value. The $|\beta|^2$ contribution is the obvious, essentially classical contribution, of the form 'sum over created particles of the current of each particle', while the $\Re(\beta\alpha)$ term represents quantum interference effects. These oscillations are on a time scale of $t_{qu} \sim \frac{\lambda_c}{2\pi c} = \frac{\hbar}{m}$, and a length scale of $\triangle z \sim \lambda_c$, as are the smaller, more strongly damped oscillations already present in the $|\beta|^2$ term. These can also be seen in Figure 7.9, which plots the integrand as a function of τ, for $p_z = 5$ and $\lambda_p = 2$. Although the total integrand here deviates noticeably from the $|\beta|^2$ contribution at late times, the amount (and sign)

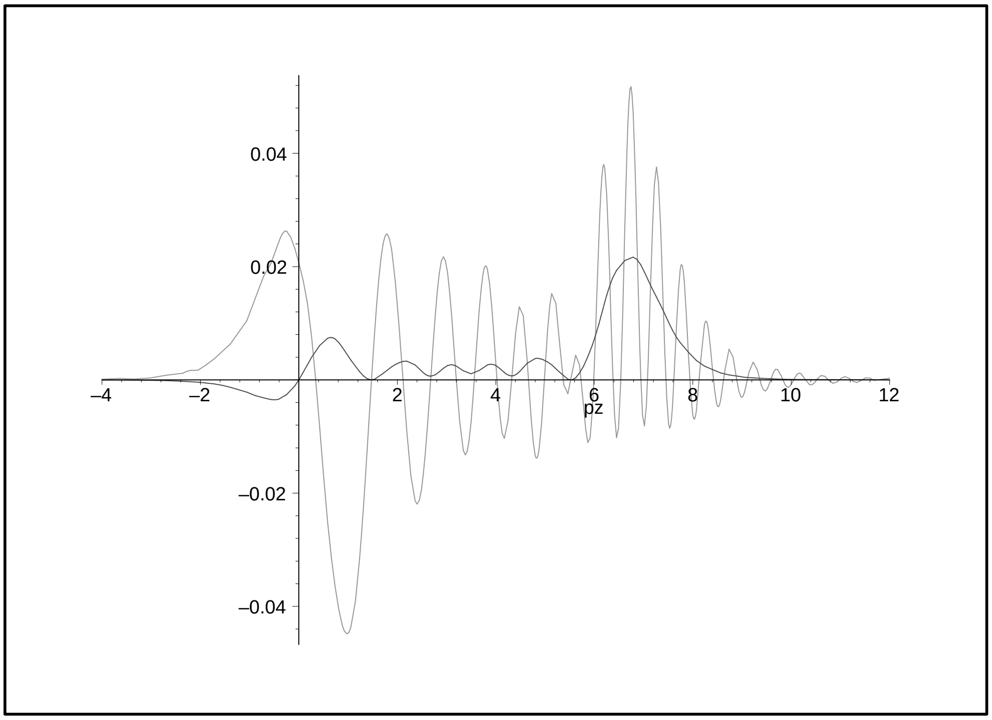

Figure 7.7: Current density in an evolved vacuum, after a time $\triangle t = 7\sqrt{eE}$, as a function of p_z, for $\lambda_p = 2$. The $|\beta|^2$ contribution in also shown (the darker curve). Again, the $\Re(\beta\alpha)$ term produces rapid oscillations about the mean contribution.

of this deviation is sensitively dependent on the chosen value of p_z. These oscillations on the time scale of t_{qu} can be identified with those mentioned in [87] and [90] which play an important role in the quantum decoherence[91] and effective dissipation processes discussed in these papers. Mottola et. al.[87] comment that when studying the back-reaction problem (where the effect of the created particles on the electric field is taken into account) for scalar electrodynamics in 1+1 dimensions, neglecting these rapid oscillations results in the reversible Hamiltonian evolution being replaced by an irreversible kinetic evolution. Figure 7.10 shows the total vacuum current density as a function of τ, along with the $|\beta|^2$ contribution, and the approximation that results from not only ignoring the $\Re(\beta\alpha)$ terms, but also using (7.55).

The $|\beta|^2$ contribution shows strong agreement with Figure 5 of Kluger, Mottola et. al.[87] (although a different scale is used in [87]). We also see that, although the 'semiclassical' $|\beta|^2$ term dominates in all but the first 10^{-20} seconds, the 'quantum interference' term is significantly more important in this first instant. Finally, it is worth noting that due to symmetry, we have $J_{1,vac}(\tau) = 0 = J_{2,vac}(\tau)$, although the integrand of $J_{1,vac}(\tau)$ and $J_{2,vac}(\tau)$ will still display rapid oscillatory dependence on p_z.

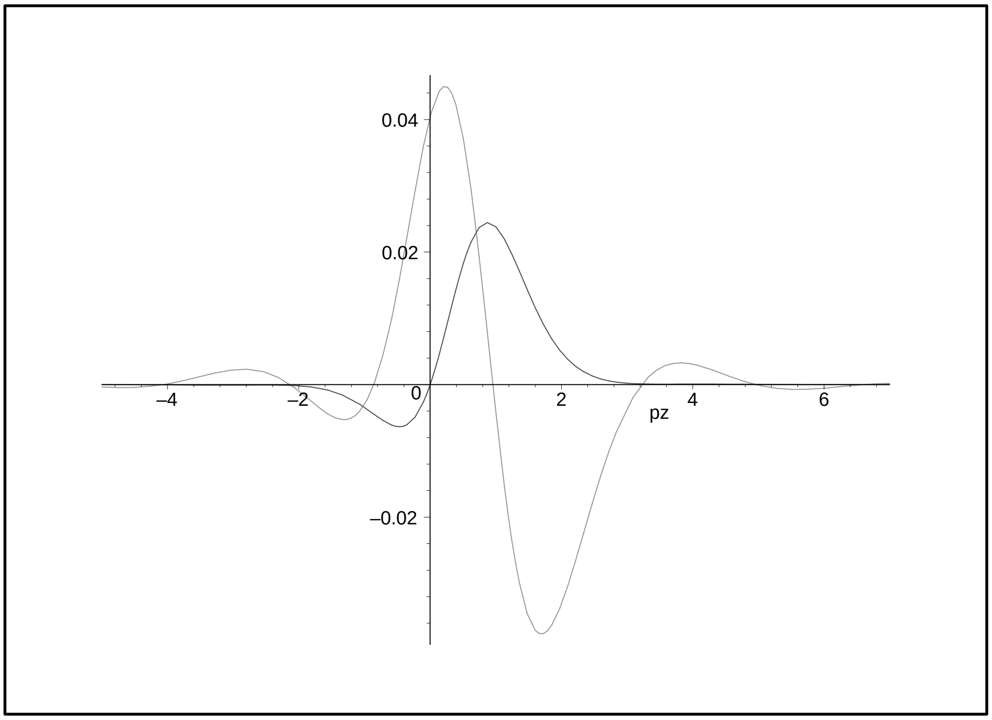

Figure 7.8: Current density in an evolved vacuum, after a time $\triangle t = \sqrt{eE}$, as a function of p_z, for $\lambda_p = 2$. The $|\beta|^2$ contribution in also shown (the darker curve). We can no longer say that the $\Re(\beta\alpha)$ term produces rapid oscillations about the mean contribution.

The large time approximation is:

$$J_{3,vac}(\tau) \approx \frac{e(eE)^{\frac{3}{2}}\tau}{2\pi^3} e^{\frac{-\pi m^2}{eE}} \tag{7.76}$$

This would generate an induced electric field in the positive z-direction:

$$E_{ind}(\tau) \approx \frac{e^2|E|\tau^2}{4\pi^3} e^{\frac{-\pi m^2}{eE}} \tag{7.77}$$

Since we have not taken back-reaction into account, then our results can only be considered realistic during times when $E_{ind} << |E|$ That is, when $\tau << \sqrt{\frac{\pi}{\alpha}} e^{\frac{\pi}{2}|\frac{E_c}{E}|}$. Thus, although for small electric fields (where $eE << m^2$) the above approximation is valid for an exponentially long time, we see that it is never valid for an infinite time! To calculate infinite time pair creation effects without including back-reaction is meaningless! Also, for $|E| \sim E_c$ this approximation is only valid for $\triangle t \sim 10^{-19}$ seconds!

7.4.3 Non-Vacuum Initial Conditions

So far we have only considered the properties of an evolved vacuum state. We could easily have considered an initial state which is not vacuum. For instance, consider an initial state,

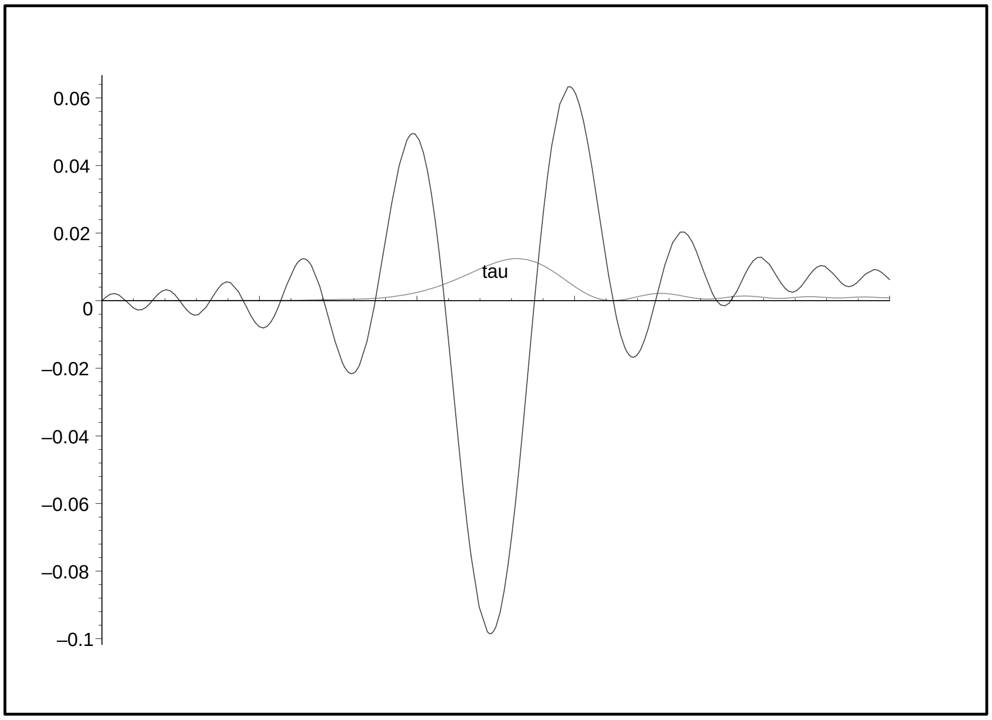

Figure 7.9: Current density in an evolved vacuum, as a function of τ, for $p_z = 5$, and $\lambda_p = 2$. The $|\beta|^2$ contribution in also shown.

at time t_0, consisting of N particles, of momenta $\mathbf{p}_1, \ldots \mathbf{p}_N$ and spins $\lambda_1, \ldots \lambda_N$. Then from equations (2.86) and (7.39), the expected number of out particles (after evolving for a time $\frac{\tau}{\sqrt{eE}}$) is:

$$N_{out}(\tau) = \sum_{i=1}^{N} (|\alpha_{\mathbf{p}_i,\lambda_i}(\tau)|^2 - |\beta_{\mathbf{p}_i,\lambda_i}(\tau)|^2) + N_{vac}(\tau) \tag{7.78}$$

which is less than $N_{vac}(\tau) + N$, due to Pauli exclusion. The probability that the out state would contain exactly N particles is:

$$\mathcal{P}_{N\to N}(\tau) = \frac{\mathcal{P}_{vac\to vac}(\tau)}{\prod_{i=1}^{N} |\alpha_{\mathbf{p}_i,\lambda_i}(\tau)|^2} \tag{7.79}$$

which again shows the well-known fact that particle creation is less likely in a state that already contains particles (recall that $|\alpha_{\mathbf{p}_i,\lambda_i}(\tau)|^2 + |\beta_{\mathbf{p}_i,\lambda_i}(\tau)|^2 = 1$). The current density of this evolved state is:

$$J_{out}(\tau) = \sum_{i=1}^{N} J(u_{\mathbf{p}_i,\lambda_i,t_0}(\mathbf{x}, t)) + J_{vac}(\tau) \tag{7.80}$$

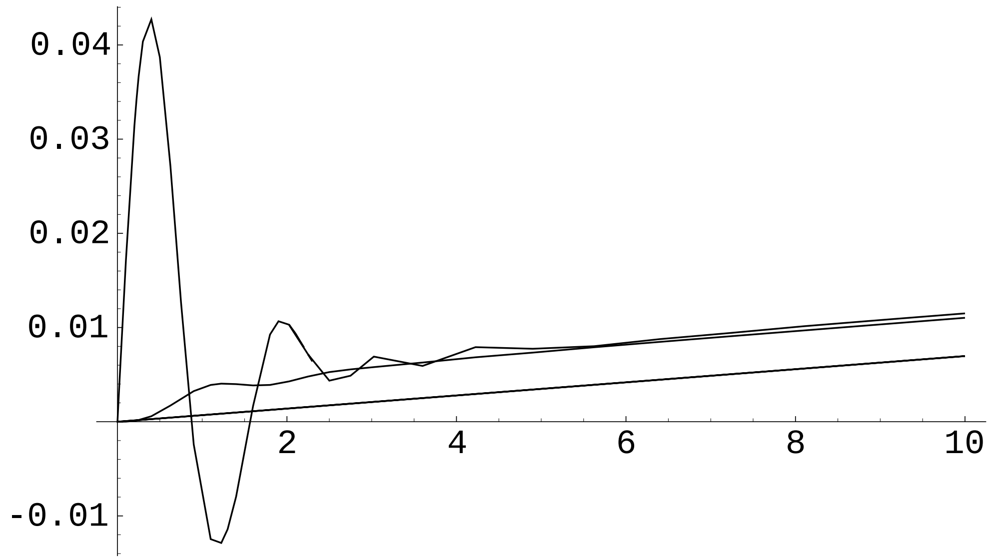

Figure 7.10: Total current (in a volume $(\triangle l)^3$) in an evolved vacuum, as a function of τ. The $|\beta|^2$ contribution in also shown, which provides an accurate approximation except at very early times. The late time approximation of (7.76) is also included, which gets the final gradient correct, but misses the initial 'burst' of creation, just as was the case with the total number of created particles.

where

$$|u_{\mathbf{p}_i,\lambda_i,t_0}(\mathbf{x},t)\rangle = \hat{U}_1(t,t_0)|u_{\mathbf{p}_i,\lambda_i,t_0}(\mathbf{x})\rangle = \alpha_{\mathbf{p}_i,\lambda_i}(t,t_0)|u_{\mathbf{p}_i,\lambda_i,t}(\mathbf{x})\rangle - \bar{\beta}_{\mathbf{p}_i,\lambda_i,t}(\mathbf{x})|v_{\mathbf{p}_i,\lambda_i,t}(\mathbf{x})\rangle \quad (7.81)$$

which gives (for the constant electric field)

$$J(u_{\mathbf{p}_i,\lambda_i,t_0}(\mathbf{x},t)) = \frac{e}{E_{p_{z,i}}}\{(1-2|\beta(p_{z,i}^{in}+\tau,p_{z,i}^{in})|^2)(p_{z,i}^{in}+\tau)+ \quad (7.82)$$

$$2\sqrt{\lambda_p}\Re(\alpha(p_{z,i}^{in}+\tau,p_{z,i}^{in})\beta(p_{z,i}^{in}+\tau,p_{z,i}^{in}))\} \quad (7.83)$$

Note that $J(u_{\mathbf{p}_i,\lambda_i,t_0}(\mathbf{x},t))$ does not represent the current of 1 particle. Rather, $J(u_{\mathbf{p}_i,\lambda_i,t_0}(\mathbf{x},t)) + J_{vac}(\mathbf{x},t)$ represents the expected current at time t in a state that at time t_0 contained one particle. It will contain contributions not just from the 1-particle component of the evolved state, but from the entire evolved state. If, before measuring the current of the evolved state, we measured the particle number, and found exactly one particle present, then the current we would measure would be that of the 'collapsed state' $u_{\mathbf{p}_i,\lambda_i,t}(\mathbf{x}) \wedge |vac_t\rangle$ (which is the one-particle component of the evolved state, appropriately normalised). This current is easily deduced to be $J = e\frac{p_{z,i}^{out}}{E_{p_{z,i}^{out}}}$, which is simply the classical result for a single particle, accelerating in this electric field. This is, to the best of my knowledge, the first time that 'collapse of the wave-function' has been applied to a relativistic system, and it provides a simple example of the insights that can be gained from working directly with the Schrödinger picture states of the system, rather than via a field operator.

$J(u_{\mathbf{p}_i,\lambda_i,t_0}(\mathbf{x},t))$ is graphed in Figures 7.11 and 7.12, as a function of $\tau = \sqrt{eE}(t-t_0)$, for $N = 1$, $\mathbf{p}_{in} = 0$, and $|E| = E_c$ (Figure 7.11), and $\frac{E_c}{10}$ (Figure 7.12). Also included in these graphs is the classical current for a single particle, accelerating from rest in the relevant electric field. We see from these that for $|E| = E_c$ the classical result differs significantly from the quantum result, even when the vacuum contribution, $J_{vac}(\mathbf{x}, t)$, from Figure 7.10, is included. However, for $|E| = \frac{E_c}{10}$ the quantum result agrees well with the classical prediction. The result for $|E| = \frac{E_c}{100}$ was also calculated, but the graph is not included here, since in this case the quantum result was indistinguishable from the classical result.

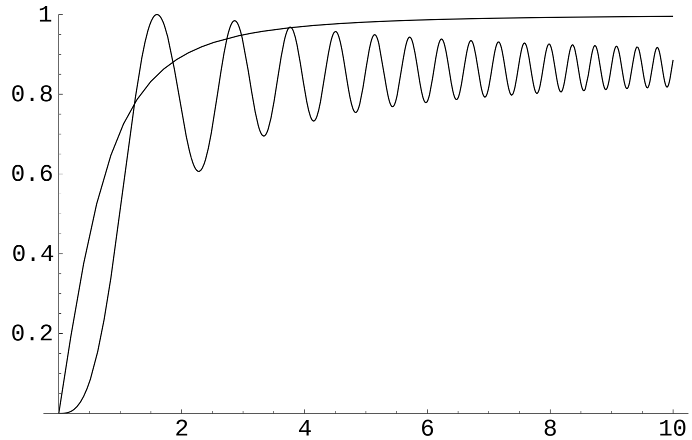

Figure 7.11: $J(u_{\mathbf{p},\lambda;t_0}(\mathbf{x}, t_0 + \frac{\tau}{\sqrt{eE}}))$ as a function of τ, for $\mathbf{p}_{in} = 0$ and $|E| = E_c$. Also included is the classically expected current for a particle accelerated from rest for a time $\frac{\tau}{\sqrt{eE}}$)).

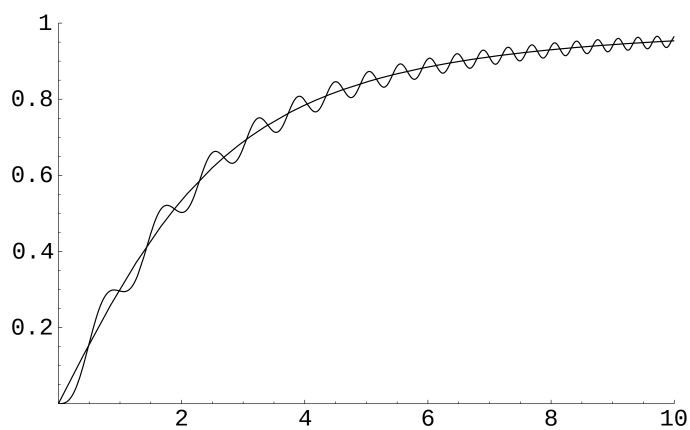

Figure 7.12: $J(u_{\mathbf{p},\lambda;t_0}(\mathbf{x}, t_0 + \frac{\tau}{\sqrt{eE}}))$ as a function of τ, for $\mathbf{p}_{in} = 0$ and $|E| = \frac{E_c}{10}$. Also included is the classically expected current for a particle accelerated from rest for a time $\frac{\tau}{\sqrt{eE}}$)). We see, as expected, that the full result more closely resembles the classical result in this case. We also calculated the result for $|E| = \frac{E_c}{100}$, and found no discernible difference between the full result and the classical result in this case.

Chapter 8

Bosonic Systems

In this Chapter I present a description of some of the work that has been done in an attempt to extend the formalism of the previous Chapters to include quantized bosonic systems. Although this extension is far from complete, there are some important results that are worth presenting here.

I restrict my attention to the case of a charged scalar field in the presence of an electromagnetic or gravitational background. The main results of this Chapter are an extension of the particle definition and vacuum subtraction scheme to the bosonic case.

However, since the bosonic equivalent of the formalism of Chapter 2 is not yet complete, then I will use the conventional, field operator based formalism analogous to that used in Chapter 4.

In Section 8.1 I introduce the equations relevant to a charged scalar field in an electromagnetic background, and I show how the particle definition introduced in Chapter 2 can be applied in the bosonic case. In Section 8.2 I define the 'in' and 'out' Fock spaces, Bogoliubov coefficients, etc, and derive the general S-Matrix Element of the theory. Section 8.3 is devoted to the discussion of expectation values, and the introduction of a vacuum subtraction scheme, while gravitational backgrounds and arbitrary observers are considered in Section 8.4.

8.1 The Equations

The Lagrangian for a charged scalar field in an electromagnetic field $A^\mu(\mathbf{x}, t)$ is[92]:

$$\mathcal{L} = D_\mu \phi^* D^\mu \phi - m^2 \phi^* \phi \tag{8.1}$$

$$\text{where } D_\mu \phi \equiv (\partial_\mu + ieA_\mu)\phi \tag{8.2}$$

$$D_\mu \phi^* \equiv (D_\mu \phi)^* = (\partial_\mu - ieA_\mu)\phi^* \tag{8.3}$$

and e is the charge of the bosonic particle. This leads to the governing equation:

$$(D^\mu D_\mu + m^2)\phi = 0 \tag{8.4}$$

and the conserved current density:

$$J^\mu = i(\phi^* D^\mu \phi - (D^\mu \phi^*)\phi) \equiv i\phi^* \overleftrightarrow{D^\mu} \phi \tag{8.5}$$

from which we can deduce that the inner product:

$$\langle\psi(\mathbf{x},t)|\phi(\mathbf{x},t)\rangle = i\int d^3\mathbf{x}\psi^*\overleftrightarrow{D^0}\phi = \int d^3\mathbf{x}(i\psi^*\overleftrightarrow{\partial_t}\phi - 2eA^0(\mathbf{x},t)\psi^*\phi) \tag{8.6}$$

is conserved, but not positive definite. Notice also that now the norm of a state at a given time is dependent on the value of $A^0(\mathbf{x},t)$ at that time.

We now wish to express equation (8.4) in Hamiltonian form, with respect to a linear Hamiltonian operator. Start by defining $\pi \equiv D_0\phi$ (so $\pi^* = D_0\phi^*$). The 'state at time t' is then defined by the doublet $\Psi(\mathbf{x},t) = \begin{bmatrix}\phi(\mathbf{x},t)\\ \pi(\mathbf{x},t)\end{bmatrix}$, and equation (8.4) can be written as:

$$(i\frac{\partial}{\partial t} - eA_0(\mathbf{x},t))\begin{bmatrix}\phi(\mathbf{x},t)\\ \pi(\mathbf{x},t)\end{bmatrix} = \begin{bmatrix}i\pi(\mathbf{x},t)\\ i(\sum_k D_k^2 - m^2)\phi(\mathbf{x},t)\end{bmatrix} = \hat{H}_1(\mathbf{x},t)\begin{bmatrix}\phi(\mathbf{x},t)\\ \pi(\mathbf{x},t)\end{bmatrix} \tag{8.7}$$

where $k = 1,2,3$, and the gauge invariant, Hermitian[1] 'first quantized' Hamiltonian operator $\hat{H}_1(\mathbf{x},t)$ can be written in matrix form as:

$$\hat{H}_1(\mathbf{x},t) = i\begin{bmatrix}0 & 1\\ (\sum_k D_k^2 - m^2) & 0\end{bmatrix} \tag{8.8}$$

Note that $\pi(\mathbf{x},t)$ defined above is not the 'conjugate variable' to $\phi(\mathbf{x},t)$. That is, $\pi(\mathbf{x},t) \neq \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$. It is well-known[45] that defining a Canonical Hamiltonian is unreliable in gravitational backgrounds, since not only is it sensitively dependent upon a choice of time coordinate, but it also has little to do with the energy-momentum tensor.

The inner product can be written in terms of $\begin{bmatrix}\phi(\mathbf{x},t)\\ \pi(\mathbf{x},t)\end{bmatrix}$ as:

$$\langle\Psi_1(\mathbf{x},t)|\Psi_2(\mathbf{x},t)\rangle = i\int d^3\mathbf{x}(\phi_1^*\pi_2 - \pi_1^*\phi_2) \tag{8.9}$$

and the expectation value of the Hamiltonian in the state $\Psi(\mathbf{x},t)$ is:

$$\langle\Psi(\mathbf{x},t)|\hat{H}_1(\mathbf{x},t)|\Psi(\mathbf{x},t)\rangle = \int d^3\mathbf{x}(\pi^*\pi - \phi^*(\sum_k D_k^2 - m^2)\phi) \tag{8.10}$$

$$= \int d^3\mathbf{x}(\pi^*\pi + \sum_k D_k\phi^* D_k\phi + m^2\phi^*\phi) \tag{8.11}$$

which is easily seen to be the Hamiltonian of classical field theory. We can now define $\mathcal{H}^+(t)$ to be the span of the positive spectrum of the operator $\hat{H}_1(\mathbf{x},t)$, and define $\mathcal{H}^-(t)$ to be the span of the negative spectrum of $\hat{H}_1(\mathbf{x},t)$, just as in the fermionic case. To see how this relates to the conventional choice involving spaces of positive definite/negative definite norm, notice that an eigenstate Ψ_i of $\hat{H}_1(\mathbf{x},t)$ of eigenvalue E_i has norm $2E_i\int d^3\mathbf{x}|\phi_i(\mathbf{x},t)|^2$, which is positive for $E_i > 0$ and negative for $E_i < 0$. From this we can deduce that:

$$\langle\hat{P}^+(t_0)\Psi|\hat{P}^+(t_0)\Psi\rangle \geq \langle\Psi|\Psi\rangle \geq \langle\hat{P}^-(t_0)\Psi|\hat{P}^-(t_0)\Psi\rangle \tag{8.12}$$

[1] This is Hermitian despite the definition of inner product being time dependent, since the time dependence of that definition is balanced by the $eA_0(\mathbf{x},t)$ term present in the left hand side of (8.7).

for all $\Psi \in \mathcal{H}$, where $\hat{P}^{\pm}(t_0) : \mathcal{H} \to \mathcal{H}^{\pm}(t_0)$ projects out the positive/negative energy components of the state Ψ. Hence, the definition of $\mathcal{H}^{\pm}(t_0)$ in terms of the spectrum of $\hat{H}_1(t_0)$ is consistent with the conventional requirement[24, 22] (namely, that $\mathcal{H}^{\pm}(t_0)$ be orthogonal spaces, spanning $\mathcal{H}$, with positive definite/negative definite norm respectively) but has the advantage of being unique[2]. To see how this definition of $\mathcal{H}^{\pm}(t_0)$ can be expressed in terms of initial conditions on $\phi(\mathbf{x}, t)$, notice that the equation $\hat{H}_1(t_0)|\Psi_{t_0}\rangle = \pm E_{t_0}|\Psi_{t_0}\rangle$ can be rewritten in terms of ϕ and its time derivative as:

$$-\sum_k D_k^2 \phi(\mathbf{x}, t_0) = (E_{t_0}^2 - m^2)\phi(\mathbf{x}, t_0) \tag{8.13}$$

$$D_0\phi(\mathbf{x}, t)|_{t=t_0} = \begin{cases} -iE_{t_0}\phi(\mathbf{x}, t_0) & \text{eigenstate } E_{t_0} > 0 \\ iE_{t_0}\phi(\mathbf{x}, t_0) & \text{eigenstate } -E_{t_0} < 0 \end{cases} \tag{8.14}$$

This allows us, if we desire, to express our definition of $\mathcal{H}^{\pm}(t_0)$ completely independently of the $\begin{bmatrix} \phi(\mathbf{x}, t) \\ \pi(\mathbf{x}, t) \end{bmatrix}$ system.

8.2 Bogoliubov Coefficients and S-Matrix Elements

As in the fermionic case, let $\{u_{i,t_0}(\mathbf{x}, t), v_{i,t_0}(\mathbf{x}, t); i \in I\}$ be a basis of solutions of the 'first quantized' equation (8.4), where I is some index set, which is assumed for convenience to be countable. Also, let:

$$U_{i,t_0}(\mathbf{x}, t) = \begin{bmatrix} u_{i,t_0}(\mathbf{x}, t) \\ D_0 u_{i,t_0}(\mathbf{x}, t) \end{bmatrix} \qquad V_{i,t_0}(\mathbf{x}, t) = \begin{bmatrix} v_{i,t_0}(\mathbf{x}, t) \\ D_0 v_{i,t_0}(\mathbf{x}, t) \end{bmatrix} \tag{8.15}$$

and let these be such that, at time t_0, the set $\{U_{i,t_0}(\mathbf{x}, t_0), i \in I\}$ provides an orthonormal basis for $\mathcal{H}^{+}(t_0)$, while the set $\{V_{i,t_0}(\mathbf{x}, t_0), i \in I\}$ provides a negative orthonormal basis for $\mathcal{H}^{-}(t_0)$. So[3]:

$$\langle U_{i,t_0}(\mathbf{x}, t)|U_{j,t_0}(\mathbf{x}, t)\rangle = \delta_{ij} = -\langle V_{i,t_0}(\mathbf{x}, t)|V_{j,t_0}(\mathbf{x}, t)\rangle \qquad \langle U_{i,t_0}(\mathbf{x}, t)|V_{j,t_0}(\mathbf{x}, t)\rangle = 0 \tag{8.16}$$

Let $\{u_{i,t_1}(\mathbf{x}, t), v_{i,t_1}(\mathbf{x}, t); i \in I\}$ be similarly defined with respect to some 'out time' $t_1 > t_0$. We can expand the 'in' basis in terms of the 'out' basis as:

$$\begin{aligned} u_{i,t_0}(\mathbf{x}, t) &= \sum_j \{\alpha_{ji}(t_1, t_0) u_{j,t_1}(\mathbf{x}, t) - \gamma_{ji}(t_1, t_0) v_{j,t_1}(\mathbf{x}, t)\} \\ v_{i,t_0}(\mathbf{x}, t) &= \sum_j \{\beta_{ji}(t_1, t_0) u_{j,t_1}(\mathbf{x}, t) - \epsilon_{ji}(t_1, t_0) v_{j,t_1}(\mathbf{x}, t)\} \end{aligned} \tag{8.17}$$

[2] To see that the conventional requirement is not unique, consider the simple case where $\mathcal{H}$ has only one positive norm and one negative norm degree of freedom. This is equivalent to 1+1 dimensional spacetime, which certainly does not possess a unique split into 'space' and 'time'.

[3] Notice that we are switching between two different spaces here, one being the set of all (finite norm) solutions of (8.4), with inner product (8.6), and the other being $L^2(\mathbb{R}^3)^2$, with inner product (8.9). Hopefully it is clear in context which is being referred to.

which defines the *Time-Dependent Bogoliubov Coefficients*:

$$\alpha_{ij}(t_1,t_0) = \langle U_{i,t_1}(t)|U_{j,t_0}(t)\rangle \quad \gamma_{ij}(t_1,t_0) = \langle V_{i,t_1}(t)|U_{j,t_0}(t)\rangle \tag{8.18}$$

$$\beta_{ij}(t_1,t_0) = \langle U_{i,t_1}(t)|V_{j,t_0}(t)\rangle \quad \epsilon_{ij}(t_1,t_0) = \langle V_{i,t_1}(t)|V_{j,t_0}(t)\rangle \tag{8.19}$$

(8.17) can then be inverted to get:

$$\begin{aligned} u_{i,t_1}(\mathbf{x},t) &= \sum_j \{\alpha^*_{ij}(t_1,t_0)u_{j,t_0}(\mathbf{x},t) - \beta^*_{ij}(t_1,t_0)v_{j,t_0}(\mathbf{x},t)\} \\ v_{i,t_1}(\mathbf{x},t) &= \sum_j \{\gamma^*_{ij}(t_1,t_0)u_{j,t_0}(\mathbf{x},t) - \epsilon^*_{ij}(t_1,t_0)v_{j,t_0}(\mathbf{x},t)\} \end{aligned} \tag{8.20}$$

and the Bogoliubov conditions can now be written as:

$$\begin{bmatrix} \boldsymbol{\alpha} & -\boldsymbol{\beta} \\ \boldsymbol{\gamma} & -\boldsymbol{\epsilon} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}^\dagger & -\boldsymbol{\gamma}^\dagger \\ \boldsymbol{\beta}^\dagger & -\boldsymbol{\epsilon}^\dagger \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}^\dagger & -\boldsymbol{\gamma}^\dagger \\ \boldsymbol{\beta}^\dagger & -\boldsymbol{\epsilon}^\dagger \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} & -\boldsymbol{\beta} \\ \boldsymbol{\gamma} & -\boldsymbol{\epsilon} \end{bmatrix} \tag{8.21}$$

for all $t_1 > t_0$. The field operator $\hat{\phi}(\mathbf{x},t)$ can be written in terms of these as:

$$\hat{\phi}(\mathbf{x},t) = \sum_i \{u_{i,t_0}(\mathbf{x},t)a_{i,t_0,h} + v_{i,t_0}(\mathbf{x},t)b^\dagger_{i,t_0,h}\} \tag{8.22}$$

where $a_{i,t_0,h}$ is the annihilation operator of the *Heisenberg picture state* representing the solution $U_{i,t_0}(\mathbf{x},t)$, and similarly for $b_{i,t_0,h}$. By requiring that $\hat{\psi}(\mathbf{x},t)$ satisfy the (equal time) Canonical Commutation Relations, we can deduce that $a_{i,t_0,h}$, $b_{i,t_0,h}$ must satisfy (for all t_0):

$$[a_{i,t_0,h}, a^\dagger_{j,t_0,h}] = \delta_{ij} = [b_{i,t_0,h}, b^\dagger_{j,t_0,h}] \tag{8.23}$$

with all other commutators zero. Requiring that $\hat{\phi}(\mathbf{x},t)$ be independent of t_0, we deduce that

$$\begin{aligned} a_{i,t_1,h} &= \sum_j \{\alpha_{ij}(t_1,t_0)a_{j,t_0,h} + \beta_{ij}(t_1,t_0)b^\dagger_{j,t_0,h}\} \\ b_{i,t_1,h} &= \sum_j \{-\gamma^*_{ij}(t_1,t_0)a^\dagger_{j,t_0,h} - \epsilon^*_{ij}(t_1,t_0)b_{j,t_0,h}\} \end{aligned} \tag{8.24}$$

and that:

$$\begin{aligned} a_{i,t_0,h} &= \sum_j \{\alpha^*_{ji}(t_1,t_0)a_{j,t_1,h} + \gamma^*_{ji}(t_1,t_0)b^\dagger_{j,t_1,h}\} \\ b_{i,t_0,h} &= \sum_j \{-\beta_{ji}(t_1,t_0)a^\dagger_{j,t_1,h} - \epsilon_{ji}(t_1,t_0)b_{j,t_1,h}\} \end{aligned} \tag{8.25}$$

The 't_0 vacuum' $|vac_{t_0},h\rangle$ is defined (for each t_0) by the requirement that

$$a_{i,t_0,h}|vac_{t_0},h\rangle = 0 = b_{i,t_0,h}|vac_{t_0},h\rangle \qquad \langle vac_{t_0},h|vac_{t_0},h\rangle = 1 \tag{8.26}$$

The 't_0 Fock space' is spanned by the set of all states of the form:

$$\frac{(a^\dagger_{i_1,t_0,h})^{r_1}\dots(a^\dagger_{i_m,t_0,h})^{r_m}(b^\dagger_{j_n,t_0,h})^{s_n}\dots(b^\dagger_{j_1,t_0,h})^{s_1}}{\sqrt{r_1!r_2!\dots r_m!s_1!\dots s_n!}}|vac_{t_0},h\rangle \tag{8.27}$$

where the factor of $(r_1!r_2!\dots r_m!s_1!\dots s_n!)^{-\frac{1}{2}}$ is included to ensure that these states are all of unit norm. Notice that although the norm on $\mathcal{H}$ is not positive definite, the norm on the t_0 Fock Space is positive definite. It is useful to denote these states by

$$|\binom{i_1^{r_1}i_2^{r_2}\dots i_m^{r_m}}{j_1^{s_1}j_2^{s_2}\dots j_n^{s_n}}in_{t_0},h\rangle \tag{8.28}$$

which we abbreviate to $|\binom{\boldsymbol{i}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}in_{t_0},h\rangle$. With this notation, define

$$c_v \equiv \langle vac_{t_1},h|vac_{t_0},h\rangle \tag{8.29}$$

$$V\binom{\boldsymbol{i}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}} \equiv \frac{1}{c_v}\langle\binom{\boldsymbol{i}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}out_{t_1},h|vac_{t_0},h\rangle \tag{8.30}$$

$$\Lambda\binom{\boldsymbol{i}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}} \equiv \frac{1}{c_v}\langle vac_{t_1},h|\binom{\boldsymbol{i}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}in_{t_0},h\rangle \tag{8.31}$$

in analogy with the treatment of fermions described in Chapter 4. We can now calculate c_v, Λ, and V using the same basic procedure as in Section 4.2. We find

$$\Lambda\binom{\boldsymbol{i}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}} = \delta_{rs}\frac{\mathrm{per}(\boldsymbol{\Lambda}\binom{\boldsymbol{i}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}(t_1,t_0))}{\sqrt{r_1!r_2!\dots r_m!s_1!\dots s_n!}} \tag{8.32}$$

$$\text{where } r\equiv\sum_k r_k \quad s\equiv\sum_k s_k \qquad \boldsymbol{\Lambda}(t_1,t_0)\equiv-\boldsymbol{\epsilon}^{-1}(t_1,t_0)\boldsymbol{\gamma}(t_1,t_0) \tag{8.33}$$

and $\boldsymbol{\Lambda}\binom{\boldsymbol{i}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}(t_1,t_0)$ is the s by r matrix composed of n by m blocks, where the kl^{th} block is the s_k by r_l matrix all entries of which are $\Lambda_{j_k i_l}$. 'per' denotes the permanent of this matrix[4].

$$V\binom{\boldsymbol{i}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}} = \delta_{rs}\frac{\mathrm{per}(\boldsymbol{V}\binom{\boldsymbol{i}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}(t_1,t_0))}{\sqrt{r_1!r_2!\dots r_m!s_1!\dots s_n!}} \tag{8.34}$$

$$\text{where } \boldsymbol{V}(t_1,t_0) = -\boldsymbol{\beta}(t_1,t_0)\boldsymbol{\epsilon}^{-1}(t_1,t_0) \tag{8.35}$$

and $\boldsymbol{V}\binom{\boldsymbol{i}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}(t_1,t_0)$ is the r by s matrix composed of m by n blocks, where the lk^{th} block is the r_l by s_k matrix all entries of which are $V_{i_l j_k}$. Finally,

$$|c_v| = |\det(\boldsymbol{\epsilon}(t_1,t_0))|^{-1} = |\det(\boldsymbol{\alpha}(t_1,t_0))|^{-1} \tag{8.36}$$

Note also that by using the Bogoliubov conditions we can deduce $\boldsymbol{\Lambda}^\dagger = -\boldsymbol{\alpha}^{-1}\boldsymbol{\beta}$ and $\boldsymbol{V}^\dagger = -\boldsymbol{\gamma}\boldsymbol{\alpha}^{-1}$. We will now prove (8.34) and (8.36). (8.32) can be proved similarly. For

[4]The formula for the permanent of a matrix is just like that for the determinant, but without the factors of the sign of the permutations.

this purpose, it is convenient to slightly modify the notation. Implicit to the notation used in (8.28) was the fact that only those i_k are included for which $r_k \neq 0$, (and that there are m such i_k). However, we will find it convenient for these proofs to include all possible i, (from 1 to N) and allow some of the r_i to be 0 (so $m = N$ now) Similarly, we will include all possible j (from 1 to N) and allow some of the s_j to be 0. For instance, the state $|\binom{3^2 5^1}{}in_{t_0}, h\rangle \equiv \frac{1}{\sqrt{2}} a^\dagger_{3,t_0,h} a^\dagger_{3,t_0,h} a^\dagger_{5,t_0,h} |vac_{t_0}, h\rangle$ would be denoted $|\binom{1^0 2^0 3^2 4^0 5^1 6^0 \ldots N^0}{1^0 2^0 \ldots N^0} in_{t_0}, h\rangle$ here. (This is clearly an inefficient notation in practice, but is convenient for the purpose of these proofs.)

Now, as in Chapter 4, expand $|vac_{t_0}, h\rangle$ in terms of the various $|\binom{\boldsymbol{i}^{\mathbf{r}}}{\boldsymbol{j}^{\mathbf{s}}} out_{t_1}, h\rangle$ as

$$|vac_{t_0}, h\rangle = c_v \sum_{\mathbf{r},\mathbf{s}} V\binom{\boldsymbol{i}^{\mathbf{r}}}{\boldsymbol{j}^{\mathbf{s}}}(t_1, t_0) |\binom{\boldsymbol{i}^{\mathbf{r}}}{\boldsymbol{j}^{\mathbf{s}}} out_{t_1}, h\rangle \tag{8.37}$$

Acting on this with $a_{k,t_0,h}$ and using (8.25) gives:

$$\sum_{k=1}^{n} \sum_{\mathbf{r},\mathbf{s}} V\binom{\boldsymbol{i}^{\mathbf{r}}}{\boldsymbol{j}^{\mathbf{s}}}(t_1, t_0) \{\alpha^*_{ki}\sqrt{r_k} |\binom{1^{r_1} \ldots k^{r_k - 1} \ldots}{\boldsymbol{j}^{\mathbf{s}}} out_{t_1}, h\rangle + \gamma^* ki \sqrt{s_k + 1} |\binom{\boldsymbol{i}^{\mathbf{r}}}{1^{s_1} \ldots k^{s_k+1} \ldots} out_{t_1}, h\rangle\} \tag{8.38}$$

which is true for all i. Consider now the coefficient of $|\binom{1^0 2^0 \ldots N^0}{1^0 2^0 \ldots j^1 \ldots N^0} out_{t_1}, h\rangle$ in (8.38), for some j. This gives

$$0 = \gamma^*_{ji} V\binom{1^0 \ldots N^0}{1^0 \ldots N^0} + \sum_k \alpha^*_{ki} V\binom{1^0 \ldots k^1 \ldots N^0}{1^0 \ldots j^1 \ldots N^0}$$

Using $V\binom{1^0 \ldots N^0}{1^0 \ldots N^0} = 1$ and the definition of $\boldsymbol{V}(t_1, t_0)$ we can hence deduce that $V\binom{1^0 \ldots k^1 \ldots N^0}{1^0 \ldots j^1 \ldots N^0} = V_{kj}$ as expected. Looking at the coefficient of $|\binom{\boldsymbol{i}^{\mathbf{u}}}{1^0 \ldots N^0} out_{t_1}, h\rangle$ in (8.38) (for any $\mathbf{u}$) allows us to deduce that $V\binom{\boldsymbol{i}^{\mathbf{r}}}{1^0 \ldots N^0} = 0$ whenever $r \neq 0$, while similarly $V\binom{1^0 \ldots N^0}{\boldsymbol{j}^{\mathbf{s}}} = 0$ whenever $s \neq 0$.

Looking at the coefficient of $|\binom{\boldsymbol{i}^{\mathbf{u}}}{\boldsymbol{j}^{\mathbf{v}}} out_{t_1}, h\rangle$ in (8.38) gives

$$0 = \sum_k \{V\binom{1^{u_1} \ldots k^{u_k+1} \ldots N^{u_N}}{\boldsymbol{j}^{\mathbf{v}}} \alpha^*_{ki}\sqrt{u_k + 1} + V\binom{\boldsymbol{i}^{\mathbf{u}}}{1^{v_1} \ldots j^{v_j - 1} \ldots N^{v_N}} \gamma^* ki \sqrt{v_k}\}$$

Multiplying through by $(\alpha^\dagger)^{-1}_{ji}$, summing over i, and relabelling, gives:

$$V\binom{\boldsymbol{i}^{\mathbf{r}}}{\boldsymbol{j}^{\mathbf{s}}} = \frac{1}{\sqrt{r_j}} \sum_k \sqrt{s_k} V_{jk} V\binom{1^{r_1} \ldots j^{r_j - 1} \ldots N^{r_N}}{1^{s_1} \ldots k^{s_k - 1} \ldots N^{s_N}} \tag{8.39}$$

Equation (8.34) now follows by induction. That is, if $r \neq s$ then we can use (8.39) repeatedly to write $V\binom{\boldsymbol{i}^{\mathbf{r}}}{\boldsymbol{j}^{\mathbf{s}}}$ as a sum of terms in which the smallest of r or s is zero, and this will give $V\binom{\boldsymbol{i}^{\mathbf{r}}}{\boldsymbol{j}^{\mathbf{s}}} = 0$ since $V\binom{\boldsymbol{i}^{\mathbf{r}}}{1^0 \ldots N^0} = 0$ for $r \neq 0$ and $V\binom{1^0 \ldots N^0}{\boldsymbol{j}^{\mathbf{s}}} = 0$ for $s \neq 0$.

Alternatively, if $r = s$ and we assume that (8.34) holds for all r, s up to some value n then for $r = s = n+1$ we can use (8.39) to write:

$$V\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}} = \frac{1}{\sqrt{r_1!r_2!\dots r_N!s_1!\dots s_N!}} \sum_k s_k V_{jk} \mathrm{per}(\boldsymbol{V}\binom{1^{r_1}\dots j^{r_j-1}\dots N^{r_N}}{1^{s_1}\dots k^{s_k-1}\dots N^{s_N}}) \tag{8.40}$$

$$= \frac{\mathrm{per}(\boldsymbol{V}\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}})(t_1,t_0))}{\sqrt{r_1!r_2!\dots r_N!s_1!\dots s_N!}} \tag{8.41}$$

which can be seen by expanding the evaluation of $\mathrm{per}(\boldsymbol{V}\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}))$ along one of the rows in the j^{th} block (the factor of s_k is because there is s_k copies of the same column in the jk^{th} block).

Now we proceed to the proof of (8.36). For this, simply use (8.37) to write:

$$1 = c_v \sum_{\mathbf{r},\mathbf{s}} V\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}(t_1,t_0)\langle vac_{t_0}h|\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}out_{t_1},h\rangle$$

$$= |c_v|^2 \sum_{\mathbf{r},\mathbf{s}} |V\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}|^2$$

$$= |c_v|^2 \sum_{\mathbf{r},\mathbf{s}} \frac{|\mathrm{per}(\boldsymbol{V}\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}))|^2}{r_1!r_2!\dots r_N!s_1!\dots s_N!} \tag{8.42}$$

$$= |c_v|^2 \det((I - \boldsymbol{V}\boldsymbol{V}^\dagger)^{-1}) \tag{8.43}$$

$$= |c_v|^2 \det(\boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger) = |c_v|^2 |\det(\boldsymbol{\alpha})|^2 = |c_v|^2 |\det(\boldsymbol{\epsilon})|^2 \quad \text{as required.} \tag{8.44}$$

(8.44) follows from (8.43) from using the Bogoliubov conditions. Obtaining (8.43) from (8.42) is rather more difficult[5]. To do this, use MacMahons Master Theorem (see corollary 9.4.2 of [93]) to write $\det((I-A)^{-1}) = \sum_{\mathbf{r}} \frac{\mathrm{per}(\boldsymbol{A}\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{i}^{\mathrm{r}}})}{r_1!r_2!\dots r_N!}$ for any N by N matrix $\boldsymbol{A}$. Then put $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{V}^\dagger$ and notice that $\boldsymbol{A}\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{i}^{\mathrm{r}}} = \boldsymbol{V}\binom{\boldsymbol{i}^{\mathrm{r}}}{1^1\dots N^1}(\boldsymbol{V}\binom{\boldsymbol{i}^{\mathrm{r}}}{1^1\dots N^1})^\dagger$. (8.42) now follows by using the Cauchy Binet Theorem for permanents[94] to write $\mathrm{per}(\boldsymbol{V}\binom{\boldsymbol{i}^{\mathrm{r}}}{1^1\dots N^1}(\boldsymbol{V}\binom{\boldsymbol{i}^{\mathrm{r}}}{1^1\dots N^1})^\dagger) = \sum_{\mathbf{s}} \frac{|\mathrm{per}(\boldsymbol{V}\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}})|^2}{s_1!\dots s_N!}$.

Finally, before proceeding to the next section, it is worth pointing out the curious fact that, although all the 'relative' S-Matrix elements $\Lambda\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}$ and $V\binom{\dot{\boldsymbol{i}}^{\mathrm{r}}}{\boldsymbol{j}^{\mathrm{s}}}$ can be expressed in terms of permanents of first quantized solutions, the vacuum - vacuum S-Matrix element is an inverse determinant! This fact is quite counterintuitive, and it is peculiar that this could follow from a completely symmetric state space. Another intriguing fact is that,

[5]Special thanks are due here to Jim Reeds of AT&T Labs, USA, for his help in proving this result. Curiously, although a similar matrix identity is necessary for DeWitts[24] proof for real scalar fields, he seemed to consider the result to be obvious enough to need no justification, and makes no mention of any proof!

if we relate the bosonic Bogoliubov matrices to their fermionic counterparts by $\boldsymbol{\alpha}_B \leftrightarrow (\boldsymbol{\alpha}_F^\dagger)^{-1} \quad \boldsymbol{\beta}_B \leftrightarrow -\boldsymbol{V}_F \quad \boldsymbol{\gamma}_B \leftrightarrow \boldsymbol{\Lambda}_F \quad \boldsymbol{\epsilon}_B \leftrightarrow \boldsymbol{\epsilon}_F^{-1}$, then the bosonic Bogoliubov conditions are satisfied whenever their fermionic counterparts are satisfied, and the vacuum - vacuum S-matrix element maps over appropriately from the fermionic to the bosonic system. This is an interesting result, which is worth pursuing further.

8.3 Expectation Values and Vacuum Subtraction

We will not present a general treatment of vacuum expectation values, but will simply illustrate the procedure with an example. The example we choose here is the vacuum expectation value of the 'second quantized' Hamiltonian operator. The 'naive' (not normal ordered) Hamiltonian is given by equation (8.11), with the functions $\phi(\mathbf{x},t), \pi(\mathbf{x},t)$ replaced by the field operators:

$$\hat{\phi}(\mathbf{x},t) = \sum_i \{u_{i,t_0}(\mathbf{x},t)a_{i,t_0,h} + v_{i,t_0}(\mathbf{x},t)b^\dagger_{i,t_0,h}\} \tag{8.45}$$

$$\text{and } \hat{\pi}(\mathbf{x},t) = \sum_i \{D_0 u_{i,t_0}(\mathbf{x},t)a_{i,t_0,h} + D_0 v_{i,t_0}(\mathbf{x},t)b^\dagger_{i,t_0,h}\} \tag{8.46}$$

Since $\hat{\phi}(\mathbf{x},t)$, $\hat{\pi}(\mathbf{x},t)$ are independent of the chosen value of t_0, then so is $\hat{H}_{naive}(\mathbf{x},t)$. The vacuum expectation value of $\hat{H}_{naive}(\mathbf{x},t)$ is easily found to be:

$$\langle vac_{t_0},h|\hat{H}_{naive}(\mathbf{x},t)|vac_{t_0},h\rangle = \sum_i \langle V_{i,t_0}(\mathbf{x},t)|\hat{H}_1(\mathbf{x},t)|V_{i,t_0}(\mathbf{x},t)\rangle \tag{8.47}$$

where $\langle V_{i,t_0}(\mathbf{x},t)|\hat{H}_1(\mathbf{x},t)|V_{i,t_0}(\mathbf{x},t)\rangle$ is given by (8.11), with $\Psi = V_{i,t_0}(\mathbf{x},t)$. This represents the total energy (before normal ordering) at time t of the state that was vacuum at time t_0. The insights gained from Chapters 2 and 3 allow us to realise that the appropriate normal ordering scheme is not $:\quad:_{t_{in}}$ or $:\quad:_{t_{out}}$, but rather $:\quad:_t$

With $\hat{H}_{phys}(\mathbf{x},t) \equiv\ :\hat{H}_{naive}(\mathbf{x},t):_t$ we get:

$$\langle vac_{t_0},h|\hat{H}_{phys}(\mathbf{x},t)|vac_{t_0},h\rangle = \sum_i \{\langle V_{i,t_0}(\mathbf{x},t)|\hat{H}_1(\mathbf{x},t)|V_{i,t_0}(\mathbf{x},t)\rangle - \langle V_{i,t}(\mathbf{x},t)|\hat{H}_1(\mathbf{x},t)|V_{i,t}(\mathbf{x},t)\rangle\} \tag{8.48}$$

Substituting (8.17) and expanding gives:

$$\langle vac_{t_0},h|\hat{H}_{phys}(\mathbf{x},t)|vac_{t_0},h\rangle \tag{8.49}$$

$$= Trace(\boldsymbol{\beta}^\dagger \boldsymbol{H}^{++}\boldsymbol{\beta} - \boldsymbol{\beta}^\dagger \boldsymbol{H}^{+-}\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^\dagger \boldsymbol{H}^{-+}\boldsymbol{\beta} + \boldsymbol{\epsilon}^\dagger \boldsymbol{H}^{--}\boldsymbol{\epsilon}) - Trace(\boldsymbol{H}^{--}) \tag{8.50}$$

$$= Trace(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger \boldsymbol{H}^{++} + \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger \boldsymbol{H}^{--} - \boldsymbol{\epsilon}\boldsymbol{\beta}^\dagger \boldsymbol{A}^{+-} - \boldsymbol{\beta}\boldsymbol{\epsilon}^\dagger \boldsymbol{H}^{-+}) \tag{8.51}$$

where we have used $\boldsymbol{\epsilon\epsilon}^\dagger - \boldsymbol{\gamma\gamma}^\dagger = \boldsymbol{I}$, and we have defined, for any operator $\hat{A}_1 : \mathcal{H} \to \mathcal{H}$:

$$\begin{aligned}
\boldsymbol{A}^{++}_{jk} &\equiv \langle U_{j,t}(\mathbf{x},t)|\hat{A}_1(\mathbf{x},t)|U_{k,t}(\mathbf{x},t)\rangle \\
\boldsymbol{A}^{--}_{jk} &\equiv \langle V_{j,t}(\mathbf{x},t)|\hat{A}_1(\mathbf{x},t)|V_{k,t}(\mathbf{x},t)\rangle \\
\boldsymbol{A}^{+-}_{jk} &\equiv \langle U_{j,t}(\mathbf{x},t)|\hat{A}_1(\mathbf{x},t)|V_{k,t}(\mathbf{x},t)\rangle \\
\text{and } \boldsymbol{A}^{-+}_{jk} &\equiv \langle V_{j,t}(\mathbf{x},t)|\hat{A}_1(\mathbf{x},t)|U_{k,t}(\mathbf{x},t)\rangle \\
&= \overline{\boldsymbol{A}^{+-}_{kj}} \text{ if } \hat{A}_1 \text{ is Hermitian}
\end{aligned} \tag{8.52}$$

Now, by the definition of $\mathcal{H}^\pm(t)$ we have $\boldsymbol{H}^{+-} = 0 = \boldsymbol{H}^{-+}$, so that:

$$\langle vac_{t_0}, h|\hat{H}_{phys}(\mathbf{x},t)|vac_{t_0}, h\rangle = Trace(\boldsymbol{\beta\beta}^\dagger \boldsymbol{H}^{++} + \boldsymbol{\gamma\gamma}^\dagger \boldsymbol{H}^{--}) \tag{8.53}$$

and it is worth noting that, as in Section 3.3, it is only by using $:\quad:_t$ that we can use $\boldsymbol{\epsilon\epsilon}^\dagger - \boldsymbol{\gamma\gamma}^\dagger = \boldsymbol{I}$ to cancel the divergent $\boldsymbol{\epsilon\epsilon}^\dagger \boldsymbol{H}^{--}$ term in (8.50) in favour of the $\boldsymbol{\gamma\gamma}^\dagger \boldsymbol{H}^{--}$ term. Despite this we find that (8.53) still diverges in general, and further renormalisation is necessary to render it finite.

8.4 Gravitational Backgrounds

The Lagrangian for a charged scalar field minimally coupled to a gravitational background is[22]:

$$\mathcal{L} = \sqrt{-g}\{g^{\mu\nu}\nabla_\mu\phi^*\nabla_\nu\phi - m^2\phi^*\phi\} \tag{8.54}$$

where ∇_μ denotes the covariant derivative, and $g = \det(g_{\mu\nu})$. This leads to the governing equation:

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + m^2\phi = 0 \tag{8.55}$$

and the conserved current density:

$$J_\mu = i(\phi^*\nabla_\mu\phi - (\nabla_\mu\phi^*)\phi) \equiv i\phi^* \overleftrightarrow{\nabla}_\mu \phi \tag{8.56}$$

from which we can deduce that the inner product:

$$\langle\psi|\phi\rangle = i\int_\Sigma \sqrt{-g}\psi^* \overleftrightarrow{\nabla}_\mu \phi d\Sigma^\mu \tag{8.57}$$

is independent of Σ. We now wish to describe this system in Hamiltonian form. For this purpose, suppose we have chosen an observer, with radar time $\tau(x)$, and time-translation vector field $k^\mu(x)$ defined as in Chapter 3. Introduce two scalar fields $\phi(x)$ and $\psi(x)$ satisfying the governing equation:

$$\begin{aligned}
k^\mu\nabla_\mu\phi &= \pi \\
k^\mu\nabla_\mu\pi &= -|k|^2(\nabla^2_\Sigma + m^2)\phi - |k|(\nabla\cdot k)\pi + (k^\rho\nabla_\rho k^\nu)\nabla^\Sigma_\nu\phi
\end{aligned} \tag{8.58}$$

where I have defined

$$
\begin{aligned}
|k|^2 \equiv k^\mu k_\mu = (\frac{\partial\tau}{\partial x^\mu}\frac{\partial\tau}{\partial x^\nu}g^{\mu\nu})^{-1} \qquad & n^\mu \equiv \frac{k^\mu}{|k|} \\
H^{\mu\nu} \equiv g^{\mu\nu} - n^\mu n^\nu & \\
\nabla^\Sigma_\nu \phi \equiv H^\mu{}_\nu \nabla_\mu \phi \qquad \text{and} \qquad & \nabla^2_\Sigma \phi \equiv g^{\mu\nu}\nabla^\Sigma_\mu \nabla^\Sigma_\nu \phi
\end{aligned} \tag{8.59}
$$

Is it easy to see that if $\phi(x)$ and $\pi(x)$ satisfy (8.58) then $\phi(x)$ satisfies (8.55). Hence, the system described by (8.55) and (8.57) can also be described by the governing equation (8.58) along with the inner product:

$$
\langle\Psi_1|\Psi_2\rangle = i\int_{\Sigma_\tau} \frac{\sqrt{-g}}{|k|}(\phi_1^*\pi_2 - \pi_1^*\phi_2)dV \tag{8.60}
$$

where $d\Sigma^\mu = n^\mu dV$, and we restrict ourselves to hypersurfaces Σ_τ here. We have used the symbol $|\Psi\rangle$ to represent the doublet $\begin{bmatrix} \phi(x) \\ \pi(x) \end{bmatrix}$ just as in Section 8.1.

The energy momentum tensor can be written (in the minimally coupled case) as:

$$
T_{\mu\nu} = \nabla_\mu\phi^*\nabla_\nu\phi + \nabla_\nu\phi^*\nabla_\mu\phi - \frac{g_{\mu\nu}}{\sqrt{-g}}\mathcal{L} \tag{8.61}
$$

in terms of which we may write:

$$
H_\tau(\phi) \equiv \int_{\Sigma_\tau} \sqrt{-g}T_{\mu\nu}k^\mu d\Sigma^\nu \tag{8.62}
$$

$$
= \int_{\Sigma_\tau} \frac{\sqrt{-g}}{|k|}\{-|k|^2 H^{\mu\nu}\nabla_\mu\phi^*\nabla_\nu\phi + |k|^2 m^2\phi^*\phi + k^\mu\nabla_\mu\phi^* k^\nu\nabla_\nu\phi\}
$$

$$
= \langle\Psi|\hat{H}_1(\tau)|\Psi\rangle \tag{8.63}
$$

where

$$
\hat{H}_1(\tau)|\Psi\rangle = \begin{bmatrix} i\pi \\ -i|k|^2(\nabla^2_\Sigma + m^2)\phi - \frac{i}{2}\frac{\partial|k|^2}{\partial x^\mu}g^{\mu\nu}\nabla^\Sigma_\nu\phi \end{bmatrix} = i\begin{bmatrix} 0 & 1 \\ -|k|^2(\nabla^2_\Sigma + m^2) + \frac{1}{2}\frac{\partial|k|^2}{\partial x^\mu}g^{\mu\nu}\nabla^\Sigma_\nu & 0 \end{bmatrix}|\Psi\rangle \tag{8.64}
$$

and we have used $\pi = k^\mu\nabla_\mu\phi$. This can be proved by direct substitution, although the proof does rely on the assumption that $\phi(x)$ falls off sufficiently rapidly at spacelike infinity, so that boundary terms can be ignored. We can now write (8.58) in terms of the Hamiltonian $\hat{H}_1(\tau)$ as:

$$
ik^\mu\nabla_\mu|\Psi\rangle = \hat{H}_1(\tau)|\Psi\rangle + i\begin{bmatrix} 0 \\ g^{\mu\nu}k^\rho(\nabla_\rho k_\mu + \nabla_\mu k_\rho)\nabla^\Sigma_\nu\phi - |k|(\nabla\cdot k)\pi \end{bmatrix} \tag{8.65}
$$

Now that we have a well-defined operator $\hat{H}_1(\tau)$ representing the first quantized Hamiltonian 'at time τ' (according to our chosen observer) we can use it to define $\mathcal{H}^\pm(\tau)$ in terms of the positive/negative spectrum of $\hat{H}_1(\tau)$, just as in Section 8.1. This definition is, as might be expected, equivalent to diagonalising the second quantized Hamiltonian

$\hat{H}_{naive}(\tau)$ obtained by substituting the field operator $\hat{\phi}(x)$ into (8.62). Notice however, that there is no analog of (2.29) for bosonic systems, since $\langle\Psi|\hat{H}_1(\tau)|\Psi\rangle$ is positive definite now.

Also, just like at the end of Section 8.1, the condition that $|\Psi(\tau)\rangle$ be an eigenstate of $\hat{H}_1(\tau)$ can be rewritten in terms of boundary conditions on $\phi(x)$ as:

$$\nabla^2_\Sigma \phi(x|_{\Sigma_\tau}) = ((\frac{E_\tau}{|k|})^2 - m^2)\phi(x|_{\Sigma_\tau}) \tag{8.66}$$

$$k^\mu \nabla_\mu \phi(x)|_{\Sigma_\tau} = \begin{cases} -iE_\tau \phi(x|_\Sigma) & \text{eigenstate } E_\tau > 0 \\ iE_\tau \phi(x|_\Sigma) & \text{eigenstate } -E_\tau < 0 \end{cases} \tag{8.67}$$

This allows us to define $u_{\tau_{in}}(x)$, $v_{\tau_{in}}(x)$, $u_{\tau_{out}}(x)$, $v_{\tau_{out}}(x)$ and to define Bogoliubov coefficients $\boldsymbol{\alpha}(\tau_{out}, \tau_{in}), \boldsymbol{\beta}(\tau_{out}, \tau_{in})$ etc, which can be substituted into the various formulae of Sections 8.2 and 8.3 to yield S-matrix elements and (non-renormalised) expectation values just as desired.

Now, consider the case when $k^\mu(x)$ is Killing. The extra terms in (8.65) disappear in this case, giving $ik^\mu\nabla_\mu|\Psi\rangle = \hat{H}_1(\tau)|\Psi\rangle$, while $\langle\Psi|\hat{H}_1(\tau)|\Psi\rangle$ becomes independent of τ. Hence, (8.65) takes the form of a time-independent Schrödinger equation, and no particle creation will be observed in this case. There is however, the possibility of two different observers travelling along the integral curves of two different Killing vector fields. Although neither of them will observe particles as 'being created' they will each have different definitions of particle, so that the vacuum according to one will not be the vacuum according to the other. This is the situation in the Unruh effect for instance, where the accelerating observer observes the inertial vacuum as containing a thermal distribution of particles. The study of quantum field theory in situations where the time-translation vector field is a time-like Killing vector field was first investigated by Gibbons[75] in 1975, and the particle definition presented here can be viewed as the generalisation of his work to an arbitrary observer in an arbitrary spacetime.

Finally, it is worth noting that the case of electromagnetic and gravitational backgrounds can be handled straightforwardly by replacing ∇_μ with $D_\mu = \nabla_\mu + ieA_\mu$ in the Lagrangian. We do not present this case in any detail here. Also note that in the absence of any electromagnetic background the solution space is invariant under complex conjugation *, and further, that $^* : \mathcal{H}^\pm(\tau) \to \mathcal{H}^\mp(\tau)$, so that * performs the role of Charge conjugation. Hence, we could require that $v_{i,\tau_{in}}(x) = u^*_{i,\tau_{in}}(x)$ if we desired. We could then consider Hermitian scalar fields simply by putting $b_{i,\tau_{in},h} = a_{i,\tau_{in},h}$. For the details of Hermitian scalar fields in gravitational backgrounds, the reader is referred to DeWitt[24], or to any of the standard texts[22, 23].

Chapter 9

Discussion and Outlook

In this dissertation I have presented a formulation of fermionic quantum field theory in electromagnetic or gravitational backgrounds that is precisely like the methods used in multiparticle quantum mechanics, the only difference being that the particle interpretation requires us to consider the entire Dirac Sea, as well as any particles which may be present. Rather than working with a field operator $\hat{\psi}(x)$, and an abstract Fock space constructed solely from creation and annihilation operators acting on some postulated vacuum $|0\rangle$, I work directly with the Schrödinger picture states of the system, described in terms of Slater determinants. This provides a close link between the 'first quantized' and 'second quantized' systems, providing numerous conceptual and computational advantages, such as:

- The derivation of S-Matrix elements follows trivially from the definition of inner product on $\mathcal{F}_\wedge(\mathcal{H})$.
- Unitarity of the full evolution equations follows directly from conservation of the inner product on $\mathcal{H}$.
- Charge conservation follows trivially from conservation of grade.
- The action of 'second quantized' (i.e. Hermitian extended) operators is well-defined regardless of whether or not we possess a complete set of orthonormal modes (see Section 4.3 for a discussion of this), and provides an extremely simple derivation of the general expectation value of the theory.
- Discrete symmetries can be treated in a way that does not rely on momentum eigenstates, or on the 'asymptotically free theory'.
- New approaches are suggested to problems such as quantum anomalies, vacuum fluctuations and finite volume measurements.

This close link between the 'first quantized' and 'second quantized' systems was also invaluable in the development of a new definition of particle, some advantages of which are:

- It provides a particle interpretation in an arbitrary electromagnetic or gravitational background, and for an arbitrary choice of observer.
- It depends *only* on the choice of observer (and the background present), not on their choice of coordinates or their choice of gauge.
- It not only specifies a 'Hamiltonian diagonalisation condition', but it also provides a procedure for satisfying this condition in arbitrarily difficult situations.
- It agrees with conventional results in those situations where the conventional methods are well-defined (I have considered the Unruh effect, spatially uniform electric fields, and exponentially inflating universes) but is much more general, so allows many more situations to be studied.
- Although it is necessarily non-local on small scales (no local definition could be consistent with the Unruh effect), it is 'effectively local' on scales larger than the Compton wavelength $\lambda_c = \frac{h}{mc}$ of the particle concerned.

Also, by placing the observers hypersurface of simultaneity at the heart of their interpretation of the system, and by working entirely within the Schrödinger picture, I have provided a relativistic environment within which such non-local issues as entanglement, collapse of the wave-function, and other 'EPR-motivated' issues can be investigated.

Notice that although my particle definition depends on the motion of the observer, it does not depend on the detailed working of their particle detector. This means that implicit in this definition is the conjecture that all 'suitable particle detectors' will behave similarly under acceleration (or in the presence of a gravitational field). It is clear from the literature on this subject[22, 23, 45, 33, 65] that various people are sceptical about this conjecture, despite there being as yet no experimental evidence against it. This situation seems analogous to the situation which existed in the middle of last century regarding the relevance of proper time in general relativity. As already mentioned in Section 3.2, Bondi[56] firmly believed that "how a clock reacts to acceleration is utterly dependent on how the clock is constructed". Thus, although in theory the proper time of an accelerating observer, even in curved spacetime, was perfectly well defined, Bondi believed that this was of little importance, since it could not reliably be related to the actual ticking of that observer's watch. Due to various experiments, we are now quite confident (see [59], or [60] pgs 144,145, and the references therein) that this belief was wrong, and that all 'suitable clocks' behave similarly under acceleration. As for particle detectors, we now have a mathematically well-defined definition of particle which depends only on the motion of the observer, and the question of whether or not this can reliably be related to the 'actual ticking of the observers particle counter' can only be verified or disputed by experiment. Hopefully future experiments will (as with the case of proper time) choose to agree with the mathematical definition. Should we wait until these experiments are done before we use the definition? That is a matter of opinion. However, if the theoreticians had refused to use proper time until after the experimental evidence was in, then over 50 years of valuable research time would have been lost!

This formalism (equipped with its particle definition) has been applied to Dirac quantum field theory in time varying, spatially uniform electric fields. Various expectation values have been calculated for a state that evolved from vacuum for a finite time. These results were shown to agree with other approaches[16, 18, 14], which involved 'turning off the electric field' and calculating the expectation values in the free regions. Since gravitational backgrounds cannot simply be 'turned off when making measurements', then this particle definition provides significant improvements here. I have discussed how the particle definition would apply to the case of a uniformly accelerating observer in flat space, and of a comoving observer in DeSitter space, and I have found that the definition reduces to accepted definitions in these cases.

I have also pointed out, in Section 2.4, that should one wish to study the semi-classical back-reaction problem, then it can be studied within my formalism just as easily as within any other formalism. However, I have not added any new insights regarding the back-reaction problem, so will not discuss this issue further here, except to point out some general concerns (which I tend to share) regarding attempts to associate the right hand side of the back-reaction equation with the 'expectation value of the energy momentum tensor' as measured by any particular observer. First of all, as mentioned in Section 5.1, the 4-momentum operators $\hat{p}_\mu$, the expectation value of which is the spatial integral of the energy momentum tensor (at least at the 'first quantized' level), do not commute with the position operators $\hat{x}^\mu$, so one cannot simply 'knock off the spatial integral' in order to get a measure of $T_{\mu\nu}(x)$. This is manifested at the 'second quantized' level by the fact that, even when we can define a finite quantity $\langle\hat{T}_{\mu\nu}(x)\rangle$ to be interpreted as the 'expected energy momentum tensor', we find that the fluctuations of this quantity at a point are infinite, and can only be made finite by averaging the quantity $\langle\hat{T}_{\mu\nu}(x)\hat{T}_{\mu\nu}(y)\rangle$ over a finite region about the point x. Secondly, if we allow our expectation values to be dependent upon the motion of the observer who is calculating them, so that different observers will calculate different expectation values, then we surely cannot allow the evolution of the system as a whole to depend on this. (Gibbons and Hawking[26] were also concerned about this point, although they felt that it could be solved by using an appropriate interpretation of quantum mechanics, such as the many worlds interpretation!) Thirdly, we have no evidence that expectation values should have any relevance to the results of individual measurements. Indeed, conventional quantum mechanics points strongly to the belief that the result of an individual measurement is not an expectation value but rather an eigenvalue, with the expectation value only gaining meaning in terms of the 'average result' of performing the measurement on an infinite ensemble of identical systems. Hence, although we have evidence that the back-reaction problem[18, 12, 95] provides a useful approximation to the fully interacting case, and that an appropriate source term for this equation can be obtained by 'knocking off the spatial integral' to obtain a (prenormal ordered) $\langle\hat{T}_{\mu\nu}(x)\rangle$, and renormalising this using methods which explicitly preserve its covariant conservation, there does not appear to be any reason to require that this quantity be interpreted as the 'expectation value of the energy momentum tensor, as calculated by any particular observer'. However, given that I have not actually considered the back-reaction problem in any detail in this dissertation, it would not be appropriate to discuss this in any further detail here.

Having already discussed in some detail the new particle definition presented in this dissertation, and the clear physical advantages that it lends to the study of QFT in gravitational backgrounds, I would like now to discuss a little further the advantages and disadvantages inherent in the formalism itself. Technically of course, a new formalism does not in itself constitute new physics. It can only hope to provide us with a fresh viewpoint on existing physics. In this sense we could claim that, since Heisenberg was first, then it was only his approach to quantum mechanics that constituted 'new physics', while the approaches of Schrödinger, and later of Feynman's path integrals, were simply new formalisms, providing new approaches to old physics. Although this is technically true, it is hard to deny that those 'new formalisms' were of revolutionary importance, and hence, that the 'new physics' requirement is not always immediately relevant in theoretical physics. (Of course a good test of the worth of a new formalism is provided by how much new physics the new viewpoint will lead us to, although that test can only be applied retrospectively.)

Also, we are all well aware of the connection between the three complimentary approaches to quantum mechanics, and the three main approaches to classical mechanics, as shown (in the left and middle columns) here:

Classical		**'First Quantized'**		**'Second Quantized'**
Lagrangian mechanics	$\leftrightarrow$	Path Integrals	$\leftrightarrow$	Functional Integrals
Hamiltons Equations	$\leftrightarrow$	Heisenberg Picture	$\leftrightarrow$	Canonical Formalism
Hamilton-Jacobi equation	$\leftrightarrow$	Schrödingers equation	$\leftrightarrow$	

Also, although the Heisenberg picture can be related to Schrödinger's simply by multiplying all states by $\hat{U}$ and all operators by $\hat{A} \to \hat{U}\hat{A}\hat{U}^{-1}$, we are aware that in the way in which we use these two approaches, the kind of questions we might ask in each approach, and the way we would tackle these questions, these two approaches are certainly more distinctive than is suggested by this simple unitary transform. For instance, the harmonic oscillator, due to its nice symmetries, lends itself to a Heisenberg picture approach, whereas the double-slit experiment, with its boundary conditions, and its subtleties involving 'collapse of the wave-function' is much easier to consider in the Schrödinger picture, in which the wave-function plays a more dominant role[1].

I have also included the relativistic multiparticle (i.e. 'second quantized') analogs of two of the three approaches above. The functional integral approach to QFT is quite clearly analogous to the path integral approach to quantum mechanics, while the Canonical approach to QFT is based almost entirely on Heisenberg picture ideas. The states of the system, described in terms of representations of the Poincare symmetry group, play a relatively passive role, while the dynamics of the system are described in terms of operators and commutation relations. Although we can technically transform this description into the Schrödinger picture, by multiplying these abstract Fock states by the appropriate Unitary evolution operator, (while acting on operators by $\hat{A} \to \hat{U}\hat{A}\hat{U}^{\dagger}$) this is far from

[1] The Schrödinger picture also proves more useful in situations where we only require one particular solution to the governing equation (subject for instance to one particular initial condition) rather than requiring a complete set.

the spirit of the Schrödinger picture[2] (while it also leads to some technical mathematical difficulties when applied to infinite dimensional systems[9]). It would also be extremely cumbersome to consider 'collapse of the wave function', or any other non-local issues, within such a framework. A true multiparticle analog of Schrödinger's approach should be capable of working directly in terms of the states of the system, without having to work via the field operator. At the non-relativistic multiparticle level, this Schrödinger picture is clearly provided by the multiparticle Schrödinger equation, acting on Slater determinants. However, this has never before been carried through to the relativistic multiparticle regime.

The gap in the bottom right hand corner of the table above has, I believe, been left for such a theory. As Dirac so poignantly mentioned in this respect in his 1966 lectures[9] "I don't want to assert that the Schrödinger picture will not come back; in fact, there are so many beautiful things about it that I have the feeling in the back of my mind that it ought to come back." The approach presented in this dissertation, although clearly in its infancy (and currently only capable of tackling problems at the level of zeroth-order Hartree Fock) has, in the author's opinion, the potential to fill this gap. However, whether or not it can successfully fill this gap will be dependent in no small part upon the successful solution of various issues that I hope to investigate in future work. These include:

- Developing a bosonic equivalent of the state space of Chapter 2. I have already presented the bosonic equivalent of the particle definition and vacuum subtraction scheme, but there is still much work to be done on the formalism, to find a concrete representation of the vacuum state, and of the multiparticle inner product. Although we can proceed in analogy with Section 2.1, to develop a state-space based on 'Slater permanents' of first quantized solutions, we cannot (unless we use the same trick referred to in the first note of Section 2.1.2, which jeopardises the Schrödinger picture) find within this state space a suitable representation of the vacuum. That is, there does not seem to exist any solution to the bosonic equivalent of equation (2.37). In essence this is just the statement that you can't fill the bosonic Dirac Sea! Hence new ideas are required if we are to provide a concrete representation of the vacuum of relativistic bosonic systems. Other curious facts regarding bosonic systems, as mentioned in the last Chapter, are the fact that the vacuum - vacuum S-Matrix element is an inverse determinant, and the mapping between bosonic and fermionic Bogoliubov coefficients, which preserves not just the Bogoliubov conditions, but also the vacuum - vacuum S-Matrix element. These are both interesting results, which are worth pursuing further. Also, the consideration of non-linear self interactions will be important for bosonic theories.

- Investigating a possible formulation of fully interacting systems, which continues the strong connection with standard multiparticle quantum mechanics. Such a formulation could provide a whole new approach to the understanding, and possibly the teaching of quantum field theory. An approach which, given the vast success of its

[2]To appreciate this point, consider for instance how these 'representations of the Poincare group' could manoeuvre themselves through a double slit apparatus!

non-relativistic cousin, would be reason for some excitement. Also, if such a formulation could be found, then it could facilitate a cross-fertilisation of ideas between the fields of QFT and multiparticle quantum mechanics, allowing the numerous advances in multiparticle quantum mechanics to be carried over into the relativistic theory, and providing numerous possible applications. These could include a fully relativistic approach to quantum chemistry, based on the Dirac equation, where vacuum effects such as Lamb screening would automatically be included in the formulation. It could also provide a framework within which relativistic bound states (such as the quark structure of nucleons) could be studied using similar techniques to those already employed in quantum chemistry.

Other avenues for future research include:

- Purely classical considerations. Radar time and radar distance provide us with a unique global concept of simultaneity in curved spaces, and also with a way of calculating the distance between an event and an observer (i.e. a worldline), rather than just between two events. There are many interesting calculations that can be performed to investigate the properties of these concepts, and their relation to more familiar concepts.

- Applications of the particle definition, to situations such as cosmology or black hole evaporation. My definition should allow these subjects to be put on a more solid footing, and should enhance their predictive power. An interesting aspect of such situations is the possibility that certain classes of observer may (in spacetimes containing singularities) observe a loss of unitarity, due to not being able to observe entire Cauchy surfaces. As well as being of relevance to the 'information loss problem' in black hole evaporation, this loss of unitarity may also have an important part to play in cosmological decoherence processes much like those that invoke the 'onset of classicallity' in open systems. Investigating such issues in non-relativistic systems is already a very active research area, while decoherence in a cosmological setting is a young, but rapidly developing field. The formalism in this dissertation should provide an ideal environment within which to investigate these issues. Its possible connections with the 'quantum causal histories' and 'consistent histories' approaches to quantum cosmology may lead to a new understanding of the cosmological back-reaction problem, and may also allow us to investigate more closely non-local issues such as the role of time, and of an observer, in quantum field theory and cosmology.

Although the prospects for this future research are quite speculative, one cannot help but be excited by the possibilities.

Bibliography

[1] C. E. Dolby. A New Approach To Quantum Field Theory For Arbitrary Observers In Electromagnetic Backgrounds. To be submitted to *Ann. Phys.*

[2] C. E. Dolby. A New Approach To Quantum Field Theory In Gravitational Backgrounds. To be submitted to *Ann. Phys.*

[3] C. E. Dolby. Particle Creation in Spatially Uniform Electric Fields. To be submitted to *Phys. Rev. D.*

[4] C. E. Dolby. On Radar Time, and the Twin "Paradox". To be submitted to *Am. J. Phys.*

[5] C. E. Dolby, S. F. Gull, C. J. L. Doran & A. N. Lasenby. Radar Time and the Indicial Equation in the Unruh Effect. In Preparation

[6] C. E. Dolby & S. F. Gull. Radar Time and Particle Creation in Bosonic Systems. To be submitted to *Phys. Rev. D.*

[7] W. H. Furry & J. R. Oppenheimer. On the Theory of the Electron and Positive. *Phys. Rev.* 45:245 - 262, 1934.

[8] S. Weinberg. *The Quantum Theory of Fields.* Cambridge University Press, 1995.

[9] P. A. M. Dirac. *Lectures on Quantum Field Theory.* Academic Press Inc., 1966.

[10] C. Keifer. Functional Schrödinger equation for scalar QED. *Phys. Rev. D.* 45(6) 2044 - 2056, 1992.

[11] C. Keifer & A. Wipf. Functional Schrödinger equation for Fermions in External Gauge Fields. *Annals of Phys.* 236:241 - 285, 1994.

[12] C. Keifer, T. Padmanabhan & T. P. Singh. A comparison between semiclassical gravity and semiclassical electrodynamics. *Class. Quantum Grav.* 8: L185, 1991.

[13] R. Floreanini & R. Jackiw. Functional representation for fermionic quantum fields. *Phys. Rev. D.* 37(8):2206 - 2224, 1988.

[14] J. Hallin & P. Liljenberg. Fermionic and bosonic pair creation in an external electric field at finite temperature using the functional Schrödinger representation. *Phys. Rev. D* 52(2):1150 - 1164, 1995.

[15] F. A. Berezin. *The Method of Second Quantization.* Academic Press Inc, 1966.

[16] S. P. Gavrilov & D. M. Gitman. Vacuum instability in external fields. *Phys. Rev. D* 53(12): 7162 - 7175, 1996.

[17] E.S. Fradkin, D. M. Gitman & S. M. Shvartsman. *Quantum Electrodynamics with Unstable Vacuum.* Springer-Verlag, 1991.

[18] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper & E. Mottola. Fermion pair production in a strong electric field. *Phys. Rev. D.* 45(12):4659 - 4671, 1992.

[19] F. Cooper, J. M. Eisenberg, Y. Kluger, E. Mottola & B. Svetitsky. Particle production in the central rapidity region. *Phys. Rev. D.* 48(1):190 - 208, 1993.

[20] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper & E. Mottola. Pair Production in a Strong Electric Field. *Phys. Rev. Lett.* 67(18):2427 - 2430, 1991.

[21] W. Greiner, B. Müller & J. Rafelski *Quantum Electrodynamics of Strong Fields.* Springer-Verlag, 1985.

[22] N. D. Birrell & P. C. W. Davies. *Quantum Fields in Curved Spacetime.* Cambridge University Press, 1982.

[23] S. A. Fulling. *Aspects of Quantum Field Theory in Curved Space-Time.* Cambridge University Press, 1989.

[24] B.S. DeWitt. Quantum Field Theory in Curved Spacetime. *Physics Reports* 19(6):295 - 357, 1975.

[25] S. W. Hawking. Particle Creation by Black Holes. *Commun. math. Phys.* 43:199 - 220, 1975.

[26] G. W. Gibbons & S. W. Hawking. Cosmological event horizons, thermodynamics, and particle creation. *Phys. Rev. D* 15(10):2738 - 2751, 1977.

[27] W. G. Unruh. Notes on black-hole evaporation. *Phys. Rev. D* 14:870 - 892, 1976.

[28] J. Schwinger. On Gauge Invariance and Vacuum Polarisation. *Phys. Rev.* 82(5):664 - 679, 1951.

[29] E. Schrödinger. The Proper Vibrations of the Expanding Universe. *Physica.* VI(9):899 - 912, 1939.

[30] L. Parker. Quantized Fields and Particle Creation in Expanding Universes. I *Phys. Rev.* 183(5):1057 - 1068, 1969.

[31] L. Parker. Quantized Fields and Particle Creation in Expanding Universes. II *Phys. Rev. D* 3(2):346 - 356, 1971.

[32] L. Parker. Particle Creation in Expanding Universes. *Phys. Rev. Lett.* 21(8):562 - 564, 1969.

[33] P.C. W. Davies. Particles Do Not Exist. In S. M. Christensen, editor, *Quantum Theory of Gravity.*, page 387. Adam Hilger Ltd, 1984.

[34] L. Sriramkumar & T. Padmanabhan. Does a nonzero tunneling probability imply particle production in time-independent classical electromagnetic backgrounds? *Phys. Rev. D.* 54(12):7599 - 7606, 1996.

[35] J. T. Ottesen. *Infinite Dimensional Groups and Algebras in Quantum Physics.* Springer, 1995.

[36] P. Jordan & E. Wigner. Pauli's Equivalence Prohibition. *Z. Phys.* 47:631, 1928.

[37] B. Thaller. *The Dirac Equation.* Springer-Verlag, 1992.

[38] R. M. Wald. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics.* University of Chicago Press, 1994.

[39] V. I. Arnold. *Mathematical Methods of Classical Mechanics. Second Edition.* Springer Verlag, 1989.

[40] J. Dimock. Dirac Quantum Fields on a Manifold. *Trans. Am. Math. Soc.* 269(1):133 - 147, 1982.

[41] C. Itzykson & J. Zuber. *Quantum Field Theory.* McGraw Hill, 1985.

[42] A. A. Grib & S. G. Mamaev. Contribution To The Field Theory In Friedmann Space. *Sov. Journal Of Nuc. Phys.* 10(6):722 - 725, 1970.

[43] A. A. Grib & S. G. Mamaev. Creation Of Matter In The Friedmann Model Of The Universe. *Sov. Journal Of Nuc. Phys.* 14(4):450 - 452, 1972.

[44] A. A. Grib, S. G. Mamayev & V. M. Mostepanenko. Particle Creation and Vacuum Polarisation in Isotropic Universe. *J. Phys. A: Math. Gen.* 13:2057 - 2065, 1980.

[45] S. A. Fulling. Remarks on Positive Frequency and Hamiltonians in Expanding Universes. *Gen. Rel. and Grav.* 10(10):807 - 824, 1979.

[46] A. Guth. Personal communication.

[47] A. L. Lasenby, C. Doran & S. Gull. Gravity, Gauge Theories and Geometric Algebra. *Phil. Trans. R. Soc. Lond. A.* 356(1737):487 - 582, 1998.

[48] C. A. Manogue. The Klein Paradox and Superradiance. *Ann. Phys.* 181:261 - 283, 1988.

[49] J. Ambjorn, J. Greensite & C. Peterson. The Axial Anomoly And The Lattice Dirac Sea. *Nuclear Physics B* 221:381 - 408, 1983.

[50] J. Schwinger. The Theory of Quantized Fields. V *Phys. Rev.* 93(3):615 - 628, 1953.

[51] B.S. DeWitt. Quantum Theory of Gravity. II. The Manifestly Covariant Theory. *Phys. Rev.* 162(5):1195 - 1239, 1967.

[52] F. Cooper, S. Habib, Y. Kluger, E. Mottola, J. P. Paz & P. R. Anderson. Nonequilibrium quantum fields in the large-N expansion. *Phys. Rev. D.* 50(4):2848 - 2869, 1994.

[53] C. W. Misner, K. S. Thorne & J. A. Wheeler. *Gravitation.* W. H. Freeman and company, 1973.

[54] M. Kaku. *Quantum Field Theory. A Modern Introduction.* Oxford University Press, 1993.

[55] F. W. Hehl, P. von der Heyde & G. D. Kerlick. General relativity with spin and torsion:Foundations and prospects. *Rev. Mod. Phys.* 48(3):393 - 416, 1976.

[56] H. Bondi. *Assumption and Myth in Physical Theory.* Cambridge University Press, 1967.

[57] D. Bohm. *The Special Theory of Relativity.* W. A. Benjamin, 1965.

[58] R. D'Inverno. *Introducing Einsteins Relativity.* OUP, 1992.

[59] L. Marder. *Time and the Space-Traveller.* George Allen & Unwin Ltd., 1971.

[60] P. C. W. Davies. *About Time.* Orion Productions, 1995.

[61] T. A. Debs & M. L. G. Redhead. The twin "paradox" and the conventionality of simultaneity. *Am. J. Phys.* 64(4):384 - 392, 1995.

[62] W. G. Unruh. Parallax distance, time, and the twin "paradox". *Am. J. Phys.* 49(6):589 - 592, 1981.

[63] W. Pauli. *Theory of Relativity.* Pergamon Press, 1958.

[64] M. Soffel, B. Muller & W. Greiner. Dirac particles in Rindler space. *Phys. Rev. D* 22(8):1935 - 1937, 1980.

[65] R. Brout, S. Massar, R. Parentani & Ph. Spindel. A Primer For Black Hole Physics. *Phys. Rep.* 260:329 - 446, 1995.

[66] T. Damour. Klein paradox and vacuum polarization. *Proceedings of The First Marcel Grossmann Meeting on General Relativity.* North-Holland, Amsterdam, 1975.

[67] R. A. Horn & C. A. Johnson. *Matrix Analysis.* Cambridge University Press, 1985.

[68] A. Szabo & N. S. Ostlund. *Modern Quantum Chemistry.* McGraw Hill, 1989.

[69] N. H. March, W. H. Young & S. Sampanther. *The Many-Body Problem In Quantum Mechanics.* Dover, 1967.

[70] T. Imamura. Quantized Meson Field in a Classical Gravitational Field. *Phys. Rev.* 118(5):1430 - 1434, 1960.

[71] N. N. Bogoliubov. On The Theory of Superfluidity. *J. Phys. U.S.S.R.* 11:23 - 32, 1947.

[72] N. N. Bogoliubov. A New Method in The Theory of Superconductivity. *Sov. Phys. JETP* 34(7):41 - 55, 1958.

[73] S. G. Mamaev, V. M. Mostepanenko, & A. A. Starobinskii. Particle creation from the vacuum near a homogeneous isotropic singularity. *Sov. Phys. JETP* 43(5):823 - 830, 1976.

[74] A. A. Grib, S. G. Mamaev & V. M. Mostepanenko. *Gen Rel. Grav.* 7:535 - ?, 1976.

[75] G. W. Gibbons. Vacuum Polarisation and the Spontaneous Loss of Charge by Black Holes. *Comm. Math. Phys.* 44:245 - 264, 1975.

[76] D. J. H. Chung, E. W. Kolb, A. Riotto, & I. I. Tkachev. Probing Plankian physics: resonant production of particles during inflation and features in the primordial power spectrum. hep-ph/9910437.

[77] D. Wick. *The Infamous Boundary.* Springer Verlag, 1995.

[78] R. Penrose. *The Emperors New Mind.* Oxford University Press, 1989.

[79] S. P. Gavrilov, D. M. Gitman & Sh. M. Shvartsman. Greens functions in an external electric field. *Sov. J. Nucl. Phys.* 29(4):567 - 572, 1979.

[80] N. D. Birrell & J. G. Taylor. Analysis of interacting quantum field theory in curved spacetime. *J. Math. Phys.* 21(7):1740 - 1760, 1980.

[81] R. P. Feynman. The Theory of Positrons. *Phys. Rev.* 76(6):749 - 760, 1949.

[82] C. Kiefer & T. P. Singh. Quantum gravitational corrections to the functional Schrödinger equation. *Phys. Rev. D* 44(4):1067 - 1076, 1991.

[83] J. D. Bjorken & S. D. Drell. *Relativistic Quantum Mechanics.* McGraw-Hill, 1964.

[84] P. W. Milloni. *The Quantum Vacuum. An Introduction to Quantum Electrodynamics.* Academic Press, Inc, 1994.

[85] A. I. Nikishov. Pair Production by a Constant External Field. *Soviet Physics JETP* 30(4):660 - 662, 1970.

[86] V. S. Beskin, Ya. N. Istomin & V. I. Par'ev. The mechanism of filling the magnetosphere of a supermassive black hole with a plasma. *Astronomicheskij Zhurnal* 69(6):1258 - 1274, 1992.

[87] Y. Kluger, E. Mottola & J. M. Eisenberg. Quantum Vlasov equation and its Markov limit. *Phys. Rev. D* 58, 125015-1, 1998.

[88] A. Erdelyi et al. *Higher Transendental Functions, Vols 1 and 2.* McGraw-Hill, 1953.

[89] R. Blandford & R. Znajek. Electromagnetic extraction of energy from Kerr black holes. *Monthly Not. of the Roy. Ast. Soc.* 179:433 - 456, 1977.

[90] S. Habib, Y. Kluger, E. Mottola & J. P. Paz. Dissipation and Decoherence in Mean Field Theory. *Phys. Rev. Lett.* 76(25):4660 - 4663, 1996.

[91] W. H. Zurek. Decoherence and the Transition From Quantum to Classical. *Physics Today* 44: 36 - 44, Oct 1991 (see also the letters in Physics Today, 46, April 1993).

[92] L. H. Ryder. *Quantum Field Theory. Second Edition.* Cambridge University Press, 1996.

[93] R. A. Brualdi & H. J. Ryser. *Combinatorial Matrix Theory.* Cambridge University Press, 1991.

[94] H. Minc. *Permanents.* Addison-Wesley Publishing Company, 1978.

[95] S. P. Kim. New asymptotic expansion method for the Wheeler-DeWitt equation. *Phys. Rev. D.* 52(6):3382 - 3391, 1995.