Geometric Algebra: a Framework for Computing Invariants in Computer Vision

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Abstract

In this paper we present *geometric algebra* as a new framework for the theory and computation of invariants in computer vision and compare it with the currently popular Grassmann-Cayley algebra. We also discuss the formation of 3D projective invariants in terms of image coordinates.

1 Introduction

Geometric algebra (GA) is a coordinate-free approach to geometry based on the algebras of Grassmann [7] and Clifford [5]; the system we adopt here was pioneered by David Hestenes [8]. Some preliminary applications of geometric algebra in the field of computer vision have already been given [1, 10], and here we present a new methodology for the study of geometric invariance (more detail can be found in [11]). The Grassmann-Cayley (GC) algebra expresses the ideas of projective geometry very elegantly, but it lacks an inner (regressive) product and some other key concepts. We compare our methods and those of the GC algebra by considering 3D projective invariants and discussing some confusion which has occurred over this issue in the recent literature.

2 Geometric Algebra: an outline

Let $\mathcal{G}_n$ denote the geometric algebra of n-dimensions. This is a graded linear space with vector addition, scalar multiplication and a non-commutative product which is associative and distributive over addition – this is the **geometric** or **Clifford** product. Any vector squares to give a scalar. The geometric product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written $\boldsymbol{ab}$ where

$$\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}. \qquad (1)$$

The inner product of two vectors, $\boldsymbol{a}\cdot\boldsymbol{b}$, is the standard *scalar* product and produces a scalar. The outer or wedge product of two vectors, $\boldsymbol{a}\wedge\boldsymbol{b}$, is a new quantity we call a **bivector**. We think of a bivector as a directed area in the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$, formed by sweeping $\boldsymbol{a}$ along $\boldsymbol{b}$ – see Figure 1.

Thus, $\boldsymbol{b}\wedge\boldsymbol{a}$ will have the opposite orientation making the wedge product anticommutative. The outer product is immediately generalizable to higher dimensions – for example, $(\boldsymbol{a}\wedge\boldsymbol{b})\wedge\boldsymbol{c}$, a

Figure 1: The directed area, or bivector, $\boldsymbol{a}\wedge\boldsymbol{b}$.

trivector, is interpreted as the oriented volume formed by sweeping the area $\boldsymbol{a}\wedge\boldsymbol{b}$ along vector $\boldsymbol{c}$. The outer product of k vectors is a k-vector or k-blade, and is said to have *grade* k. A **multivector** (linear combination of objects of different type) is *homogeneous* if it contains terms of only a single grade. Geometric algebra allows us to manipulate multivectors in a way which keeps track of different grade objects.

In a space of 3 dimensions we can construct a trivector $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$, but no 4-vectors exist. The highest grade element in a space is called the **pseudoscalar**. The unit pseudoscalar is denoted by I and is crucial to ideas of duality.

2.1 The Geometric Algebra of 3D Space

In an n-dimensional space we can introduce an orthonormal basis of vectors $\{\sigma_i\}$ $i = 1, ..., n$, such that $\sigma_i \cdot \sigma_j = \delta_{ij}$. This leads to a basis for the entire algebra:

$$1,\ \{\sigma_i\},\ \{\sigma_i\wedge\sigma_j\},\ \ldots,\ \sigma_1\wedge\sigma_2\wedge\ldots\wedge\sigma_n. \tag{2}$$

Note that we shall not use bold symbols for these basis vectors. Any multivector can be expressed in terms of this basis which has $2^3 = 8$ elements given by:

$$\underbrace{1}_{scalar},\ \underbrace{\{\sigma_1,\sigma_2,\sigma_3\}}_{vectors},\ \underbrace{\{\sigma_1\sigma_2,\sigma_2\sigma_3,\sigma_3\sigma_1\}}_{bivectors},\ \underbrace{\{\sigma_1\sigma_2\sigma_3\}\equiv i}_{trivector}. \tag{3}$$

The trivector or pseudoscalar $\sigma_1\sigma_2\sigma_3$ squares to -1 and commutes with all multivectors in the 3D space. We therefore give it the symbol i; noting that this is not the uninterpreted commutative scalar imaginary j.

2.2 Formulation of Projective Geometry

A geometric algebra can be written as $\mathcal{G}(p,q)$ where p and q are the dimensions of the maximal subspaces with positive and negative signatures respectively. We adopt the standard Euclidean signature $\mathcal{G}(3,0)$ for ordinary space, $\mathcal{E}^3$, but are forced to adopt a signature of $\mathcal{G}(1,3)$ for the 4-dimensional space we associate with the projective space (see later).

In Euclidean spaces of 2 and 3 dimensions the unit pseudoscalar squares to -1. In $\mathcal{G}(1,3)$, if γ_i, $i = 1,2,3,4$ are our basis vectors and $\gamma_j{}^2 = -1$ for j=1,2,3 and $\gamma_4^2 = +1$, then it is easy to see that this is also the case. The sign of I^2 depends on the signature of the space. In a given space any pseudoscalar P can be written as $P = \alpha I$ where α is a scalar. If I^{-1} is the inverse of I, so that $II^{-1} = 1$, then,

$$PI^{-1} = \alpha II^{-1} = \alpha \equiv [P] \tag{4}$$

where we have defined the **bracket** of the pseudoscalar P, $[P]$, as its magnitude, arrived at by multiplication on the right by I^{-1}. This bracket is precisely the same as the bracket of the Grassmann-Cayley algebra.

We define the **dual** A^* of an r-vector A as

$$A^* = AI^{-1}, \tag{5}$$

so that the dual of an r-vector is an $(n-r)$-vector. In an n-dimensional space, if A is an r-vector and B is an s-vector (such that $r+s=n$), it can be shown that the following identity for the dual of the outer product holds

$$[A\wedge B] = (A\wedge B)I^{-1} = A\cdot B^*. \tag{6}$$

Here duality is simply multiplication by an element of the algebra and no special operator or different space is needed.

One can define the **join** $J = A \bigwedge B$ of an r-vector A and an s-vector B by

$$J = A\wedge B \quad \text{if } A \text{ and } B \text{ are linearly independent.} \tag{7}$$

If A and B are not linearly independent the join is not given simply by the wedge but by the subspace that they span. If $(r+s) \geq n$ then J will be the pseudoscalar for the space. In what follows we will use $\bigwedge$ for the join only when the blades A and B are not linearly independent, otherwise we will use the outer product, $\wedge$.

If A and B have a common factor (i.e. there exists a k-vector C such that $A = A'C$ and $B = B'C$ for some A', B') then we can define the 'intersection' or **meet** of A and B as $A \vee B$ where

$$(A \vee B)^* = A^* \wedge B^*. \quad (8)$$

That is, the dual of the meet is given by the join of the duals. In equation 8 the dual of $(A \vee B)$ is taken with respect to the *join* of A and B. If the join is the whole space the meet is given by

$$A \vee B = (A^* \wedge B^*)I = (A^* \wedge B^*)(I^{-1}I)I = \pm(A^* \cdot B) \quad (9)$$

according as $I^2 = \pm 1$. We therefore have the very simple and readily computed relation of $A \vee B = \pm(A^* \cdot B)$. Algebra in projective space (intersections of planes and planes/lines etc.) is discussed in the accompanying paper in this volume [2] and a more extended discussion of projective geometry in GA formalism is given in [9].

2.3 Linear Algebra

Consider a linear function f mapping vectors to vectors in the same space. We can extend f to act linearly on multivectors via the **outermorphism**, $\underline{f}$, such that

$$\underline{f}(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \ldots \wedge \boldsymbol{a}_r) = \underline{f}(\boldsymbol{a}_1) \wedge \underline{f}(\boldsymbol{a}_2) \wedge \ldots \wedge \underline{f}(\boldsymbol{a}_r). \quad (10)$$

$\underline{f}$ therefore preserves the grade of any r-vector it acts on. The action of $\underline{f}$ on general multivectors is then defined through linearity. Since the outermorphism preserves grade, the pseudoscalar of the space must be mapped onto some multiple of itself. The scale factor in this mapping is the **determinant** of $\underline{f}$;

$$\underline{f}(I) = \det(\underline{f})I. \quad (11)$$

This is much simpler than many definitions of the determinant.

3 Projective Space and the Projective Split

Points in real 3D space will be represented by vectors in $\mathcal{E}^3$, a 3D space with a Euclidean metric. Since any point on a line through some origin O will be mapped to a single point in the image plane, we associate a point in $\mathcal{E}^3$ with a line in a 4D space, R^4. In these two distinct spaces we define basis vectors: $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ in R^4 and $\{\sigma_1, \sigma_2, \sigma_3\}$ in $\mathcal{E}^3$. We identify R^4 and $\mathcal{E}^3$ with the geometric algebras of 4 and 3 dimensions. We require that vectors, bivectors and trivectors in R^4 will represent points, lines and planes in $\mathcal{E}^3$. Choosing γ_4 as a selected direction in R^4, we can define a mapping which associates the bivectors $\gamma_i\gamma_4$, $i = 1,2,3$, in R^4 with the vectors $\sigma_i = \gamma_i\gamma_4$, $i = 1,2,3$, in $\mathcal{E}^3$. To preserve the Euclidean structure of the spatial vectors $\{\sigma_i\}$ (i.e. $\sigma_i^2 = +1$) we are forced to assume a non-Euclidean metric for the basis vectors in R^4 (we choose $\gamma_4^2 = +1$, $\gamma_i = -1$, $i = 1,2,3$). This process of associating the quantities in the higher dimensional space with quantities in the lower dimensional space is an application of what Hestenes calls the **projective split**.

For a vector $\mathbf{X} = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + X_4\gamma_4$ in R^4 the projective split is achieved by taking the geometric product of $\mathbf{X}$ and γ_4;

$$\mathbf{X}\gamma_4 = \mathbf{X} \cdot \gamma_4 + \mathbf{X} \wedge \gamma_4 = X_4 \left(1 + \frac{\mathbf{X} \wedge \gamma_4}{X_4}\right) \equiv X_4(1 + \boldsymbol{x}). \quad (12)$$

Note that $\boldsymbol{x}$ contains terms of the form $\gamma_1\gamma_4$, $\gamma_2\gamma_4$, $\gamma_3\gamma_4$ or (via the mapping) terms in $\sigma_1, \sigma_2, \sigma_3$. We can therefore think of the vector $\boldsymbol{x}$ as a vector in $\mathcal{E}^3$ which is associated with the bivector $\mathbf{X}\wedge\gamma_4/X_4$ in R^4.

A vector $\boldsymbol{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ in $\mathcal{E}^3$, can be represented in R^4 by the vector $\mathbf{X} = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + X_4\gamma_4$ such that

$$\boldsymbol{x} = \frac{\mathbf{X}\wedge\gamma_4}{X_4} = \frac{X_1}{X_4}\sigma_1 + \frac{X_2}{X_4}\sigma_2 + \frac{X_3}{X_4}\sigma_3, \tag{13}$$

$\Rightarrow x_i = \frac{X_i}{X_4}$, for $i = 1, 2, 3$. This is therefore equivalent to using **homogeneous coordinates**, $\mathbf{X}$, for $\boldsymbol{x}$. We therefore postulate distinct spaces in which we represent ordinary 3D quantities and their 4D counterparts, together with a well-defined way of moving between these spaces.

3.1 Projective transformations

If a general point (x, y, z) in 3-D space is projected onto an image plane, the coordinates (x', y') in the image plane will be related to (x, y, z) via a transformation of the form:

$$x' = \frac{\alpha_1 x + \beta_1 y + \delta_1 z + \epsilon_1}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}, \quad y' = \frac{\alpha_2 x + \beta_2 y + \delta_2 z + \epsilon_2}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}. \tag{14}$$

To make this non-linear transformation in $\mathcal{E}^3$ into a linear transformation in R^4 we define a linear function $\underline{f}$ mapping vectors onto vectors in R^4 such that the action of $\underline{f}$ on the basis vectors $\{\gamma_i\}$ is given by

$$\begin{aligned} \underline{f}(\gamma_1) &= \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 + \tilde{\alpha}\gamma_4 \\ \underline{f}(\gamma_2) &= \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \tilde{\beta}\gamma_4 \text{ etc..} \end{aligned} \tag{15}$$

For points $\boldsymbol{x}$ and $\mathbf{X}$ in $\mathcal{E}^3$ and R^4 (as above) we can see that $\underline{f}$ maps $\mathbf{X}$ onto $\mathbf{X}'$ such that the vector $\boldsymbol{x}' = x'\sigma_1 + y'\sigma_2 + z'\sigma_3$ in $\mathcal{E}^3$ is formed from $\mathbf{X}'$ via the projective split. Similarly for y' and z'.

Note that in general we would take $\alpha_3 = f\tilde{\alpha}$, $\beta_3 = f\tilde{\beta}$ etc. so that $z' = f$ (focal length), independent of the point chosen. Via this means the non-linear transformation in $\mathcal{E}^3$ becomes a linear transformation, $\underline{f}$, in R^4. Use of the linear function $\underline{f}$ makes the invariant nature of various quantities very easy to establish.

4 Invariance using Geometric Algebra

In this section we will show how one can use the GA framework to derive standard 1-D and 2-D projective invariants from a single view.

The 1D Cross-Ratio

The '*fundamental projective invariant*' of points on a line is the so-called **cross-ratio**, ρ, defined as

$$\rho = \frac{AC}{BC}\frac{BD}{AD} = \frac{(t_3 - t_1)(t_4 - t_2)}{(t_4 - t_1)(t_3 - t_2)},$$

where $t_1 = |PA|$, $t_2 = |PB|$, $t_3 = |PC|$, $t_4 = |PD|$, for points P, A, B, C, D on some line L. We now derive this invariant as an illustration of the general approach. For this 1D case, any point q on the line L can be written as $\boldsymbol{q} = t\sigma_1$ relative to P, where σ_1 is a unit vector in the direction of L. We then move up a dimension to a 2D space, with basis vectors (γ_1, γ_2) (call this R^2) in which $\boldsymbol{q}$ is represented by the vector $\mathbf{Q} = T\gamma_1 + S\gamma_2$, where, as before, we associate $\boldsymbol{q}$ with the bivector $\frac{\mathbf{Q}\wedge\gamma_2}{\mathbf{Q}\cdot\gamma_2} = \frac{T}{S}\sigma_1$ so that $t = T/S$. When a point on line L is projected onto another line

L', the distances t and t' are related by a projective transformation of the form $t' = \frac{\alpha t + \beta}{\tilde{\alpha} t + \tilde{\beta}}$. This non-linear transformation in $\mathcal{E}^1$ can be made into a linear transformation in R^2 by defining the linear function $\underline{f}_1$ mapping vectors onto vectors in R^2;

$$\underline{f}_1(\gamma_1) = \alpha_1\gamma_1 + \tilde{\alpha}\gamma_2, \quad \underline{f}_1(\gamma_2) = \beta_1\gamma_1 + \tilde{\beta}\gamma_2. \tag{16}$$

Consider 2 vectors $\mathbf{X}_1, \mathbf{X}_2$ in R^2. Form the bivector $\mathcal{S}_1 = \mathbf{X}_1 \wedge \mathbf{X}_2 = \lambda_1 I_2$, where $I_2 = \gamma_1\gamma_2$ is the pseudoscalar for R^2. We now look at how $\mathcal{S}_1$ transforms under $\underline{f}_1$:

$$\mathcal{S}_1' = \mathbf{X}_1' \wedge \mathbf{X}_2' = \underline{f}_1(\mathbf{X}_1 \wedge \mathbf{X}_2) = (\det \underline{f}_1)(\mathbf{X}_1 \wedge \mathbf{X}_2). \tag{17}$$

We now take 4 points on the line L whose corresponding vectors in R^2 are $\{\mathbf{X}_i\}$, $i = 1,..,4$, and consider the ratio $\mathcal{R}_1$ of 2 wedge products;

$$\mathcal{R}_1 = \frac{\mathbf{X}_1 \wedge \mathbf{X}_2}{\mathbf{X}_3 \wedge \mathbf{X}_4}. \tag{18}$$

Then, under $\underline{f}_1$, $\mathcal{R}_1 \to \mathcal{R}_1'$, where

$$\mathcal{R}_1' = \frac{\mathbf{X}_1' \wedge \mathbf{X}_2'}{\mathbf{X}_3' \wedge \mathbf{X}_4'} = \frac{(\det \underline{f}_1)\mathbf{X}_1 \wedge \mathbf{X}_2}{(\det \underline{f}_1)\mathbf{X}_3 \wedge \mathbf{X}_4}. \tag{19}$$

$\mathcal{R}_1$ is therefore invariant under $\underline{f}_1$. However, we want to express our invariants in terms of distances on the 1D line; for this we must consider how the bivector $\mathcal{S}_1$ in R^2 projects down to $\mathcal{E}^1$.

$$\begin{aligned} \mathbf{X}_1 \wedge \mathbf{X}_2 = (T_1\gamma_1 + S_1\gamma_2) \wedge (T_2\gamma_1 + S_2\gamma_2) = \\ (T_1 S_2 - T_2 S_1)\gamma_1\gamma_2 = S_1 S_2 (t_1 - t_2) I_2. \end{aligned} \tag{20}$$

In order to form a projective invariant which is independent of the choice of the arbitrary scalars S_i, we must then take *ratios* of the bivectors $\mathbf{X}_i \wedge \mathbf{X}_j$ (so that $\det \underline{f}_1$ cancels) and *multiples* of such ratios so that the S_i's cancel. More precisely, consider the following expression

$$Inv_1 = \frac{(\mathbf{X}_3 \wedge \mathbf{X}_1) I_2^{-1} (\mathbf{X}_4 \wedge \mathbf{X}_2) I_2^{-1}}{(\mathbf{X}_4 \wedge \mathbf{X}_1) I_2^{-1} (\mathbf{X}_3 \wedge \mathbf{X}_2) I_2^{-1}}.$$

In terms of distances along the lines, under the projective transformation $\underline{f}_1$, Inv_1 goes to Inv_1' where

$$Inv_1' = \frac{S_3 S_1 (t_3 - t_1) S_4 S_2 (t_4 - t_2)}{S_4 S_1 (t_4 - t_1) S_3 S_2 (t_3 - t_2)} = \frac{(t_3 - t_1)(t_4 - t_2)}{(t_4 - t_1)(t_3 - t_2)}, \tag{21}$$

which is independent of the S_i's and is indeed the 1D classical projective invariant, the **cross-ratio**. We can now generalize this approach to form invariants in higher dimensions.

The 2D generalization of the Cross-Ratio

Let a point P in the plane M be described by the vector $\boldsymbol{x}$ in $\mathcal{E}^2$ where $\boldsymbol{x} = x\sigma_1 + y\sigma_2$. In R^3 this point will be represented by $\mathbf{X} = X\gamma_1 + Y\gamma_2 + Z\gamma_3$ where $x = X/Z$ and $y = Y/Z$. As before, we can define a general projective transformation via a linear function $\underline{f}_2$ mapping vectors to vectors in R^3. We can then follow through the arguments used in the 1D case to form invariant quantities by taking *multiples* of *ratios* of **trivectors**, e.g.

$$Inv_2 = \frac{(\mathbf{X}_5 \wedge \mathbf{X}_4 \wedge \mathbf{X}_3) I_3^{-1} (\mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_1) I_3^{-1}}{(\mathbf{X}_5 \wedge \mathbf{X}_1 \wedge \mathbf{X}_3) I_3^{-1} (\mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_4) I_3^{-1}}.$$

We can then interpret this ratio in $\mathcal{E}^2$ as

$$\frac{(\boldsymbol{x}_5 - \boldsymbol{x}_4) \wedge (\boldsymbol{x}_5 - \boldsymbol{x}_3) I_2^{-1} (\boldsymbol{x}_5 - \boldsymbol{x}_2) \wedge (\boldsymbol{x}_5 - \boldsymbol{x}_1) I_2^{-1}}{(\boldsymbol{x}_5 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_5 - \boldsymbol{x}_3) I_2^{-1} (\boldsymbol{x}_5 - \boldsymbol{x}_2) \wedge (\boldsymbol{x}_5 - \boldsymbol{x}_4) I_2^{-1}}.$$

This can also be written as $\{A_{543} A_{521}\}/\{A_{513} A_{524}\}$, where $\frac{1}{2} A_{ijk}$ is the area of the triangle defined by the 3 vertices $\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k$. This invariant is the 2D generalization of the 1D cross-ratio.

Figure 2: Defining points of two camera planes

5 3D Projective Invariants

When considering general points in $\mathcal{E}^3$ we have seen that we move up one dimension to work in the 4D space R^4. The point $\boldsymbol{x} = x\sigma_1 + y\sigma_2 + z\sigma_3$ in $\mathcal{E}^3$ is written as $\mathbf{X} = X\gamma_1 + Y\gamma_2 + Z\gamma_3 + W\gamma_4$, where $x = X/W$, $y = Y/W$, $z = Z/W$. As before, a non-linear projective transformation in $\mathcal{E}^3$ becomes a linear transformation, described by a linear function $\underline{f}_3$, in R^4.

Analagous to the 1D and 2D cases, we form invariant quantities by taking *multiples* of *ratios* of **4-vectors**, e.g.

$$Inv_3 = \frac{(\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4)I_4^{-1}(\mathbf{X}_4\wedge\mathbf{X}_5\wedge\mathbf{X}_2\wedge\mathbf{X}_6)I_4^{-1}}{(\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_4\wedge\mathbf{X}_5)I_4^{-1}(\mathbf{X}_3\wedge\mathbf{X}_4\wedge\mathbf{X}_2\wedge\mathbf{X}_6)I_4^{-1}}. \tag{22}$$

Using the arguments of the previous sections we can write $(\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4)I_4^{-1}$ as

$$W_1W_2W_3W_4\{(\boldsymbol{x}_2-\boldsymbol{x}_1)\wedge(\boldsymbol{x}_3-\boldsymbol{x}_1)\wedge(\boldsymbol{x}_4-\boldsymbol{x}_1)\}I_3^{-1}. \tag{23}$$

Inv_3 is the 3D equivalent of the 1D cross-ratio and consists of ratios of volumes;

$$Inv_3 = \{V_{1234}V_{4526}\}/\{V_{1245}V_{3426}\}, \tag{24}$$

where V_{ijkl} is the volume of the solid formed by the 4 vertices $\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k, \boldsymbol{x}_l$.

5.1 3D invariants in terms of image coordinates

¿From six general 3D points P_i, $i = 1, .., 6$, represented by vectors $\{\boldsymbol{x}_i, \mathbf{X}_i\}$ in $\mathcal{E}^3$ and R^4 respectively, we can form a number of 3D projective invariants. One such invariant is

$$Inv_3 = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_4\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_5][\mathbf{X}_3\mathbf{X}_4\mathbf{X}_2\mathbf{X}_6]}. \tag{25}$$

This is simply equation 22 rewritten in terms of brackets. If it is possible to express the bracket $[\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k\mathbf{X}_l]$ in terms of the image coordinates of points P_i, P_j, P_k, P_l, then we can easily compute this invariant. While it has been shown [3] that it is not possible to compute general 3D invariants from a single image, recent work has used the Grassmann-Cayley algebra [4] to compute such invariants from a pair of images using image coordinates and the fundamental matrix, $\boldsymbol{F}$. Subsequent work by Csurka and Faugeras [6] claims to have corrected some of Carlsson's expressions by including omitted scale factors. Here we translate the approach of Carlsson into the geometric algebra framework in an attempt to address this question. We will also show that the claim of Csurka *et al.* is indeed correct but that the resolution lies simply in reordering the bracket decomposition rather than finding large numbers of omitted scale factors.

Consider the scalar S_{1234} formed from the bracket of 4 points

$$S_{1234} = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] = (\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_3\wedge\mathbf{X}_4)I_4{}^{-1}. \tag{26}$$

The quantities $(\mathbf{X}_1\wedge\mathbf{X}_2)$ and $(\mathbf{X}_3\wedge\mathbf{X}_4)$ represent the lines joining points P_1 and P_2, and P_3 and P_4. We let $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ be the centres of projection of the two cameras and suppose that the two camera image planes can be defined by the two sets of vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ – see figure 2. Let the projection of points $\{P_i\}$ be given by the vectors $\{\boldsymbol{a}'_i\}$ and $\{\boldsymbol{b}'_i\}$. Note our vectors, $\boldsymbol{a}_i, \boldsymbol{b}_i$, etc. are ordinary vectors in $\mathcal{E}^3$; we then let the representations of these vectors in R^4 be $\mathbf{A}_i, \mathbf{B}_i, \mathbf{A}'_i, \mathbf{B}'_i$..., etc. Expressing $\mathbf{X}_1\wedge\mathbf{X}_2$ (resp. $\mathbf{X}_3\wedge\mathbf{X}_4$) as the meet of the planes through the image points and the centres of projection gives

$$\mathbf{X}_1 \wedge \mathbf{X}_2 = (\mathbf{A}_0 \wedge \mathbf{A}'_1 \wedge \mathbf{A}'_2) \vee (\mathbf{B}_0 \wedge \mathbf{B}'_1 \wedge \mathbf{B}'_2) \quad (27)$$

$$\mathbf{X}_3 \wedge \mathbf{X}_4 = (\mathbf{A}_0 \wedge \mathbf{A}'_3 \wedge \mathbf{A}'_4) \vee (\mathbf{B}_0 \wedge \mathbf{B}'_3 \wedge \mathbf{B}'_4). \quad (28)$$

Using the definition of the meet and some results set out in [2, 11], S_{1234} can be expanded as

$$\begin{aligned} S_{1234} &= -[\mathbf{A}_0\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_3][\mathbf{B}_0\mathbf{B}'_1\mathbf{B}'_2\mathbf{B}'_4][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_4\mathbf{B}'_3] \\ &+ [\mathbf{A}_0\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_4][\mathbf{B}_0\mathbf{B}'_1\mathbf{B}'_2\mathbf{B}'_3][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_3\mathbf{B}'_4]. \end{aligned} \quad (29)$$

Now consider a point in the first image plane which is the intersection of the lines joining points $\{\boldsymbol{a}'_1$ and $\boldsymbol{a}'_2\}$ and $\{\boldsymbol{a}'_3$ and $\boldsymbol{a}'_4\}$ – call this point $\boldsymbol{a}'_{1234}$ ($\mathbf{A}'_{1234}$ in R^4). We can expand the points $\mathbf{A}'_i$ and $\mathbf{A}'_{1234}$ in terms of the R^4 vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$: $\mathbf{A}'_i = \alpha_{ij}\mathbf{A}_j$ and $\mathbf{A}'_{1234} = \alpha_{1234,j}\mathbf{A}_j$, summed over j. Similarly, if $\mathbf{B}'_{1234}$ is the R^4 representation of the intersection point $\boldsymbol{b}'_{1234}$ in the second image plane (i.e. the intersection of the lines joining $\{\boldsymbol{b}'_1$ and $\boldsymbol{b}'_2\}$ and $\{\boldsymbol{b}'_3$ and $\boldsymbol{b}'_4\}$) then we can form similar expressions for the points $\mathbf{B}'_i$ and $\mathbf{B}'_{1234}$. Forming $\mathbf{A}'_{1234}$ from the meet of the line $\mathbf{A}'_1 \wedge \mathbf{A}'_2$ and the plane $\mathbf{A}_0 \wedge \mathbf{A}'_3 \wedge \mathbf{A}'_4$ and similarly for $\mathbf{B}'_{1234}$, the bracket $[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1234}\mathbf{B}'_{1234}]$ can be evaluated as follows:

$$[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1234}\mathbf{B}'_{1234}] = -\{\mathbf{A}_0 \wedge \mathbf{A}'_{1234}\} \wedge \{\mathbf{B}_0 \wedge \mathbf{B}'_{1234}\} I_4{}^{-1}. \quad (30)$$

The righthandside of this equation can then be expanded using previous expressions for $\mathbf{A}'_{1234}$ and $\mathbf{B}'_{1234}$ and after some reordering of terms we note that it is equal to the bracket S_{1234} given in equation 29. To summarize, we are able to write the bracket of the 4 points (in R^4) as

$$S_{1234} = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] \equiv [\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1234}\mathbf{B}'_{1234}]. \quad (31)$$

Note here that when we take ratios of brackets to cancel out scale factors and form our invariants we must ensure that the same decomposition of $\mathbf{X}_i \wedge \mathbf{X}_j$ occurs in both numerator and denominator so that these arbitrary factors cancel. For example, Inv_3 can be written as

$$\frac{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_4)\} I_4{}^{-1} \{(\mathbf{X}_4 \wedge \mathbf{X}_5) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}}{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_4 \wedge \mathbf{X}_5)\} I_4{}^{-1} \{(\mathbf{X}_3 \wedge \mathbf{X}_4) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}} \quad (32)$$

where this decomposition rule has been obeyed. In [6] it is claimed that the invariant of 6 points which can be thought of as arising from 4 points and a line, namely

$$\frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_5][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_6]}. \quad (33)$$

is not invariant when expressed in Carlsson's terms. Their solution is to include a large number of correcting scale factors. If we were to decompose the expression as given in equation 33 in the manner outlined previously, we would have

$$\frac{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_4)\} I_4{}^{-1} \{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_5 \wedge \mathbf{X}_6)\} I_4{}^{-1}}{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_5)\} I_4{}^{-1} \{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_4 \wedge \mathbf{X}_6)\} I_4{}^{-1}}. \quad (34)$$

It is clear that the same bivectors do *not* appear in both the numerator and denominator and therefore there will not be the required cancelling of scale factors. However, we are free to rearrange equation 33 in the following way;

$$\frac{[\mathbf{X}_1\mathbf{X}_4\mathbf{X}_2\mathbf{X}_3][\mathbf{X}_1\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_5\mathbf{X}_2\mathbf{X}_3][\mathbf{X}_1\mathbf{X}_4\mathbf{X}_2\mathbf{X}_6]}. \quad (35)$$

The decomposition now looks like

$$\frac{\{(\mathbf{X}_1\wedge\mathbf{X}_4)\wedge(\mathbf{X}_2\wedge\mathbf{X}_3)\}I_4{}^{-1}\{(\mathbf{X}_1\wedge\mathbf{X}_5)\wedge(\mathbf{X}_2\wedge\mathbf{X}_6)\}I_4{}^{-1}}{\{(\mathbf{X}_1\wedge\mathbf{X}_5)\wedge(\mathbf{X}_2\wedge\mathbf{X}_3)\}I_4{}^{-1}\{(\mathbf{X}_1\wedge\mathbf{X}_4)\wedge(\mathbf{X}_2\wedge\mathbf{X}_6)\}I_4{}^{-1}}. \tag{36}$$

and we see that the*same* bivectors appear in both numerator and denominator and therefore all scale factors should cancel. Writing

$$Inv_3 = \frac{[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1423}\mathbf{B}'_{1423}][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1526}\mathbf{B}'_{1526}]}{[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1523}\mathbf{B}'_{1523}][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1426}\mathbf{B}'_{1426}]} \tag{37}$$

will indeed produce an invariant and there is thus no need for the introduction of the scale factors proposed in [6].

6 Conclusions

We have shown how geometric algebra can be used in the formation and computation of invariants. Using the concept of the *projective split* and a new definition of duality we find that the standard concepts or machinery of classical projective geometry are not needed. The system is able to form 3D projective invariants from multiple views in a straightforward and geometrically intuitive manner.

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