Cylindrically Symmetric Systems in Gauge Theory Gravity

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Reason had harnessed the tame,
Holding the sky in their arms.
Gravity pulls me down.

-REM

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Chapter 1

Introduction

The mathematical simplicity of systems possessing cylindrical symmetry in General Relativity has made them a fruitful area of investigation both of abstract ideas, such as Mach's Principle and causality violation, and of potentially realistic astrophysical phenomena, such as cosmic strings. Indeed, since the early work of Weyl [1], Levi-Civita [2], Lewis [3], and van Stockum [4], there has been much study of such systems. Yet, as pointed out in [5], we still lack a solid physical understanding of such systems.

This thesis is an application of the gauge theory of gravitation developed by Lasenby, Doran, and Gull [6] to cylindrically symmetric systems. This approach, based on the mathematics of geometric algebra, deals with gauge fields in a flat spacetime, and keeps the physical relevance of quantities unambiguous: physically meaningful quantities must be gauge invariant. Furthermore, the gauge theory formalism allows for a new way of solving the field equations, called the 'intrinsic' method, which reduces the problem to a set of first-order differential equations. These equations are generally much more manageable than the field equations in the standard General Relativity approach, which deals with second-order equations in the metric components. For example, spherically symmetric systems in the gauge theory reduce to a quite simple set of equations which reveal a striking similarity to Newtonian theory. The theory has also provided a great deal of insight into axisymmetric systems (principally the Kerr solution) [7] and offers a fairly radical vision of cosmology [8]. In [9], the gauge theoretic approach was applied to cylindrically symmetric systems in (2+1)-dimensions with enlightening results. Here, this analysis is extended to a full (3+1)-dimensional treatment of both stationary systems and systems with general time dependence.

Chapter 1 is a brief outline of geometric algebra and gauge theory gravity, presenting the general formalism used in the rest of the thesis. Chapter 2 is devoted to the analysis of stationary cylindrically symmetric systems. Gauge theory gravity is used to provide quick derivation of the van Stockum solution describing an infinitely long rigidly rotating dust cylinder and the vacuum region surrounding it. The vacuum solution is known to fall into three distinct categories depending on the value of a parameter which is related to the mass per unit length of the string. These three cases shall be referred to as the low-mass, critical, and high-mass regimes. In the low-mass regime, Bonnor [10] has demonstrated that the vacuum solution may be brought into static form, but only at the expense of the introduction of a periodic time coordinate. Tipler [11], has noted that such a manipulation may be interpreted as describing a spacetime with nontrivial topology, but the physical relevance of the static coordinates is then quite unclear. In any case, such an interpretation is at odds with the gauge theoretic approach, which requires globally valid solutions in a flat background space. Bonnor himself notes that there is no smooth match of the static vacuum onto the solution for the interior, and ends discussion of the problem by declaring the vacuum to be "locally static," but not globally so. This is a useful term, but in itself, it sheds little light on the physical significance of the solution. The static nature of the fields is of great importance for Mach's Principle, for Machian arguments would dictate that the rotating cylinder would drag the vacuum spacetime around with it, in which case it could not be brought into static form. Study of geodesic motion and the structure of the Weyl tensor, made relatively simple in gauge theory gravity, is shown to reveal much of the physical significance of locally static fields.

The high-mass regime of the van Stockum exterior brings along with it its own unique problems. A real valued transformation into static coordinates is no longer possible, even in General Relativity, which at first glance would seem to be good news for Mach's principle. However, it is quite remarkable that Machian effects would only come into play for certain values of the mass of the cylinder. Furthermore, the high-mass regime necessarily admits causality violation in the form of closed timelike loops (CTL), and has prompted much discussion on the possibility of building a time machine from some system of rotating cylinders. One must be careful in drawing conclusions from cylindrically symmetric systems, as they involve infinitely long objects and thus an infinite mass. Even Newtonian infinite cylinders have bizarre long-range properties. Nevertheless, it seems reasonable that such solutions would still be valid locally near very long, but finite cylinders. Therefore, the possibility of creating a time machine from such systems should be taken seriously. One could simply declare CTL as pathological, and dismiss solutions which contain them as unphysical; but this would really just be begging the question, and would also leave the status of solutions like those for Kerr black holes unclear. (Although the CTL in the Kerr solution are confined within an event horizon and so perhaps are less worrisome than those appearing in rotating cylinder solutions.)

The question is: can CTL appear as a result of a physically sensible matter configuration? Within the confines of van Stockum type solutions, the answer seems to be no. Once again, this result is motivated by an analysis of the Weyl tensor. One may define a fluid velocity to a rotating cylinder relative to a frame in which the Weyl tensor is diagonal which is intrinsic to the physical system. That this definition is sensible is bolstered by the consideration of counterrotating cylinders. It turns out that for systems for which this intrinsic velocity is less than the speed of light, CTL cannot form. If one believes this definition of the fluid velocity, then the critical and high-mass regimes of the van Stockum solution are simply not physically possible, for they would seem to require matter moving at luminal or superluminal velocities. This analysis is quite satisfying in that it implies that the appearance of CTL is a manifestation of the mistaken ascription of a timelike vector (the eigenvector of the stress-energy tensor) to a null or spacelike direction. In this light, it should hardly be surprising that CTL appear; closed spacelike loops are quite unremarkable, but when spacelike and timelike directions are confused, the situation becomes quite bizarre indeed. It is interesting that such an ascription may be locally innocuous in that the causal structure of spacetime at the point where the velocity becomes null is still well-behaved, but ultimately, spacetime has been contorted in such a way as to produce pathological behaviour elsewhere.

Chapter 3 extends the gauge theoretic techniques to time-dependent cylindrically symmetric systems. Although the equations governing such systems are substantially more complicated than those for stationary systems, a good deal of progress can be made towards a general solution, using the results obtained for stationary systems as a guide towards making sensible simplifications. The time dependent equations are also seen to exhibit an unusual type of symmetry which does not appear in General Relativity. Restricting to diagonal line elements is proposed as a possible route towards obtaining wave solutions.

1.1 Geometric Algebra

The gauge theory of gravity employed here is best presented in the mathematical language of geometric algebra, the fundamentals of which are described rigorously in [12]. Here we present a brief outline of geometric algebra, citing rather than deriving important results. A geometric algebra is a graded vector space with an associative product, called the geometric product, which is distributive over addition. The geometric product has the property that the square of any vector (grade-1) is a scalar (grade-0). It then follows from the identity

$$(a+b)^2 = a^2 + (ab+ba) + b^2 \tag{1.1}$$

that the symmetrised product of two vectors is a scalar, and we may thus define an inner product, denoted by a dot, as

$$a \cdot b = \tfrac{1}{2}(ab+ba). \tag{1.2}$$

The remaining antisymmetric part of the geometric product of two vectors is the outer, or wedge product,

$$a \wedge b = \tfrac{1}{2}(ab-ba). \tag{1.3}$$

The quantity $a \wedge b$ is a new type of object, a grade-2 element, or *bivector*, which represents a directed area. The full geometric product of two vectors may then be written as

$$ab = a \cdot b + a \wedge b. \tag{1.4}$$

In expressions involving different types of products, we adopt the convention that the dot and wedge products take precedence and are to be evaluated before any geometric products; for example, $ab \wedge c$ is to be understood as $a(b \wedge c)$.

Forming further geometric products leads to quantities containing higher grade terms which are generally called *multivectors*. A principal advantage of geometric algebra is that it allows the addition of objects of differing grades. This is to be understood in much the same way that real and imaginary numbers are added together in complex algebra. This ability to form multi-grade objects makes geometric algebra an extremely compact and computationally efficient language for physics. For example, Maxwell's equations can be combined into a single multivector equation with an invertible differential operator [13].

In this thesis, we are primarily interested in the geometric algebra of Minkowski spacetime, or the "spacetime algebra" (STA). The STA is generated by four orthonormal basis vectors $\{\gamma_\mu\}, \mu = 0 \ldots 3$, such that

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+ - - -). \tag{1.5}$$

The full STA is then spanned by the quantities

$$1, \qquad \{\gamma_\mu\}, \qquad \{\sigma_k, i\sigma_k\}, \qquad \{i\gamma_\mu\}, \qquad i, \tag{1.6}$$

where

$$\sigma_k \equiv \gamma_k \gamma_0 \tag{1.7}$$

and i is the spacetime *pseudoscalar*, a grade-4 object defined as

$$i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3. \tag{1.8}$$

It is important not to confuse i with the unit imaginary in complex analysis, j; both i and j square to -1, but i is a geometrically significant quantity which anticommutes with odd-grade elements and commutes only with even-grade elements. All quantities encountered in gauge theory gravity are real valued.

A spacetime vector a has a corresponding spatial vector $\mathbf{a}$ relative to the inertial frame determined by γ_0 given by

$$a\gamma_0 = a_0 + \mathbf{a}, \tag{1.9}$$

where

$$a_0 = a \cdot \gamma_0, \qquad \mathbf{a} = a \wedge \gamma_0. \tag{1.10}$$

The spatial vector $\mathbf{a}$ has components in the orthonormal frame spanned by the spacetime *bi*vectors $\{\sigma_k\}$. The spatial vector relative to any other frame may be be determined similarly by forming $a \wedge v$, where v is the timelike vector which determines the frame.

We shall also need some results from geometric calculus. The derivative with respect to a vector a, denoted ∂_a, is defined in terms of its directional derivatives $b \cdot \partial_a$ via

$$b \cdot \partial_a F(a) \equiv \lim_{\epsilon \to 0} \frac{F(a + \epsilon b) - F(a)}{\epsilon}. \tag{1.11}$$

If we now introduce an arbitrary basis $\{e_k\}$ with a reciprocal basis $\{e^k\}$ defined so that $e_j \cdot e^k = \delta_j^k$, then the full derivative may be written as

$$\partial_a \equiv e^k e_k \cdot \partial_a. \tag{1.12}$$

The derivative ∂_a has the algebraic properties of a vector, and is useful for performing operations like contractions in a frame-free manner. The derivative with respect to the spacetime position vector x is particularly important and is denoted by ∇.

Geometric algebra is also convenient for handling linear functions. A linear function which maps vectors to vectors is written with an underbar $\underline{f}(a)$. The adjoint of $\underline{f}(a)$ is written with an overbar $\overline{f}(a)$, and satisfies

$$a \cdot \underline{f}(b) = \overline{f}(a) \cdot b. \tag{1.13}$$

Using the result $\partial_b b \cdot c = c$, the adjoint may then be written in a frame-free manner as

$$\overline{f}(a) = \partial_b (a \cdot \underline{f}(b)). \tag{1.14}$$

Linear functions can be extended to operate on objects of arbitrary grade via

$$\underline{f}(a \wedge b \wedge \ldots \wedge c) \equiv \underline{f}(a) \wedge \underline{f}(b) \ldots \wedge \underline{f}(c). \tag{1.15}$$

In particular, since the pseudoscalar I of a space is unique up to a scale factor, we can define the determinant of $\underline{f}(a)$ by

$$\det(\underline{f}) = \underline{f}(I) I^{-1}. \tag{1.16}$$

The inverse functions are obtained from the formulae

$$\underline{f}^{-1}(A) = \det(\underline{f})^{-1} \overline{f}(AI) I^{-1} \tag{1.17}$$

and

$$\overline{f}^{-1}(A) = \det(\underline{f})^{-1} I^{-1} \underline{f}(IA). \tag{1.18}$$

1.2 Gauge Theory Gravity

A complete development of gauge theoretic gravitation is set out in [6]. In this section, only a brief overview of the theory is presented. As in the preceding outline of geometric algebra, important results are cited rather than derived. For the purposes of this thesis, it is enough to consider gravity in the absence of torsion.

The fundamental claim of gauge theory gravity is that physical relations, which are generally relations between field quantities at certain spacetime points, should be independent of arbitrary (*i.e.*, local) displacements and rotations of these fields. Thus the theory employs two gauge fields. The first, denoted $\overline{h}(a,x)$, is the *position-gauge field* which ensures covariance under displacements. It is a vector-valued linear function in the vector argument a and generally has nonlinear dependence on the spacetime position vector x. To ensure covariance under arbitrary Lorentz rotations of the fields, we also introduce the *rotation-gauge field*, $\omega(a,x)$, which is a bivector-valued function, also linear in the vector a and with general spacetime dependence. In referring to these fields, the spacetime dependence is often assumed to be implicit, and explicit reference to x is dropped, writing simply $\overline{h}(a)$ and $\omega(a)$.

Under spacetime displacements, which may be written as

$$x \mapsto x' = f(x) \tag{1.19}$$

the gauge fields transform as

$$\overline{h}(a,x) \mapsto \overline{h}'(a,x) = \overline{h}(\overline{f}^{-1}(a), f(x)) \tag{1.20}$$

and

$$\omega(a,x) \mapsto \omega'(a,x) = \omega(a, f(x)), \tag{1.21}$$

where $\overline{f}^{-1}(a)$ is the inverse adjoint of the function $\underline{f}(a)$ defined by

$$\underline{f}(a) \equiv a \cdot \nabla f(x). \tag{1.22}$$

One of the advantages of geometric algebra is the ease with which spacetime rotations are described in it. Rotations of objects of any grade (vector, bivector, *etc.*) are represented by the double sided action of a single rotor. A rotor, R is a multivector of the form (scalar + bivector) which satisfies the relation $R\tilde{R} = 1$. Here the tilde denotes the operation of reversion, whereby the order of vectors in any geometric product is reversed. Having defined a rotor, a spacetime vector a is rotated by

$$a \mapsto a' = Ra\tilde{R}. \tag{1.23}$$

Under the action of such a rotor, the gauge fields are defined to transform as

$$\overline{h}(a) \mapsto \overline{h}'(a) = R\overline{h}(a)\tilde{R} \tag{1.24}$$

and

$$\omega(a) \mapsto \omega'(a) = R\omega(\tilde{R}aR)\tilde{R} - 2L_{\tilde{R}aR}R\tilde{R}, \tag{1.25}$$

where

$$L_a \equiv a \cdot \overline{h}(\nabla) \tag{1.26}$$

is the covariant derivative under displacements. We may define a fully covariant directional derivative of a multivector M in the direction of the vector a via

$$a \cdot \mathcal{D}M \equiv L_a M + \omega(a) \times M, \tag{1.27}$$

where the commutator product ($\times$) is defined on arbitrary multivectors A and B via $A \times B \equiv \frac{1}{2}(AB - BA)$. This directional derivative may be used to construct a covariant vector derivative $\mathcal{D}$, which acts as

$$\mathcal{D}M = \mathcal{D} \cdot M + \mathcal{D} \wedge M, \tag{1.28}$$

where

$$\begin{aligned} \mathcal{D} \cdot M &\equiv \partial_b \cdot (b \cdot \mathcal{D}M) \\ \mathcal{D} \wedge M &\equiv \partial_b \wedge (b \cdot \mathcal{D}M). \end{aligned} \tag{1.29}$$

The gauge theoretic equivalent of the Riemann tensor is then given by

$$\mathcal{R}(a \wedge b) = L_a\omega(b) - L_b\omega(a) + \omega(a) \times \omega(b) - \omega(c), \tag{1.30}$$

where

$$c = a \cdot \mathcal{D}b - b \cdot \mathcal{D}a. \tag{1.31}$$

The Riemann tensor is a linear mapping of bivectors to bivectors, and like the individual gauge fields, has general spacetime dependence. Again, we shall often simply write $\mathcal{R}(B)$ for the Riemann tensor, suppressing the implicit spacetime dependence. $\mathcal{R}(B)$ transforms under local displacements (1.19) and rotations (1.23) as

$$\mathcal{R}(B, x) \mapsto \mathcal{R}'(B, x) = \mathcal{R}(B, f(x)) \tag{1.32}$$

and

$$\mathcal{R}(B, x) \mapsto \mathcal{R}'(B, x) = R\mathcal{R}(\tilde{R}BR, x)\tilde{R} \tag{1.33}$$

respectively. This general behavior under displacements and rotations is referred to as *covariant*, and we shall often denote covariant quantities by script-style letters, so that the covariance of expressions may easily be seen. Various contractions of $\mathcal{R}(B)$ yield the Ricci Tensor,

$$\mathcal{R}(a) = \partial_b \cdot \mathcal{R}(b \wedge a) \tag{1.34}$$

the Ricci Scalar,

$$\mathcal{R} = \partial_b \cdot \mathcal{R}(b) \tag{1.35}$$

and the Einstein tensor

$$\mathcal{G}(a) = \mathcal{R}(a) - \frac{1}{2}\mathcal{R}a. \tag{1.36}$$

Also of importance is the Weyl tensor, given by

$$\mathcal{W}(a \wedge b) = \mathcal{R}(a \wedge b) - 4\pi[a \wedge \mathcal{T}(b) + \mathcal{T}(a) \wedge b - \tfrac{2}{3}\mathcal{T}\, a \wedge b], \tag{1.37}$$

where $\mathcal{T} = \partial_a \cdot \mathcal{T}(a)$ is the trace of the matter stress-energy tensor $\mathcal{T}(a)$. $\mathcal{W}(B)$ has a number of useful symmetries. It is self dual

$$\mathcal{W}(iB) = i\mathcal{W}(B), \tag{1.38}$$

and symmetric and trace-free

$$\sigma_k \mathcal{W}(\sigma_k) = 0. \tag{1.39}$$

It should be stressed that the above definitions are *not* to be interpreted as describing the curvature of spacetime, they are simply field constructions in a flat background space.

The field equations governing $\overline{h}(a)$ and $\omega(a)$ are obtained from an action principle on the invariant action

$$S = \int |d^4x| \det(\underline{h})^{-1}(\tfrac{1}{2}\mathcal{R} - 8\pi G\mathcal{L}_m), \tag{1.40}$$

where $\mathcal{L}_m$ is the matter Lagrangian (in which the gauge fields are assumed to appear undifferentiated.) In the absence of spin and cosmological constant, the field equations obtained from varying this action with respect to the two gauge fields may be written as

$$\mathcal{G}(a) = 8\pi\mathcal{T}(a) \tag{1.41}$$

and

$$\mathcal{D} \wedge \overline{h}(a) = \overline{h}(\nabla \wedge a). \tag{1.42}$$

The connection between the gauge theory and General Relativity may be made by introducing a coordinate frame $\{e_\mu\}$ and constructing the metric tensor via

$$g_{\mu\nu} = \underline{h}^{-1}(e_\mu) \cdot \underline{h}^{-1}(e_\nu). \tag{1.43}$$

Further exposition of the connections between gauge theory gravity and General Relativity may be found in Appendix A.

At this point, one could use the gauge theory in a manner mimicking General Relativity, solving (1.41) and (1.42) for the various metric components. This would generally result in difficult set of second-order differential equations. An alternative approach, adopted in this thesis, is to follow the so-called "intrinsic" method [6]. The metric components solved for in the relativistic approach are invariant under arbitrary rotations (1.23), thus many degrees of freedom are eliminated from the problem by never dealing explicitly with the rotation gauge. The intrinsic method, on the other hand, keeps the rotation gauge field explicit, and thus works with a larger set of much simpler first-order equations. Systematically, one begins by positing a general abstract form for $\overline{h}(a)$ suited to the symmetries of the system under consideration. An abstract form of $\omega(a)$ is then generated by demanding consistency with equation (1.42). The Riemann and Einstein tensors may then be written down in terms of the abstract functions in $\omega(a)$ and their L-derivatives. Only then is the rotation gauge fixed by a choice convenient to the problem at hand. For example, the rotation gauge is often chosen by specifying the form which the stress-energy or the Weyl tensor is to take.

Crucial to the success of the intrinsic method is the bracket structure implied by equation (1.42). Namely,

$$[L_a, L_b] = L_c \tag{1.44}$$

where c is as defined in (1.31). These relations provide a set of first order differential equations relating $\overline{h}(a)$ and $\omega(a)$ that is much easier to work with than the original relation (1.42). As we shall see in the study of time-dependent systems, the bracket structure can also be applied to the field equations to generate further relations until a complete or nearly complete set of equations is obtained. After choosing a specific coordinatisation for the solution, one may then solve for the abstract functions in $\omega(a)$ and then use the relations obtained from (1.44) to solve for $\overline{h}(a)$.

Chapter 2

Stationary Systems

In 1937, van Stockum [4] found the complete solution for an infinitely long cylinder of rigidly rotating dust. In this chapter, the van Stockum solution is derived within the gauge theory framework. That this derivation is substantially shorter and less involved than the corresponding calculation in General Relativity is a testament to the calculational virtues of the theory as a means of solving the Einstein equations. This solution will also provide a gateway into the issues of Mach's Principle and causality violation as a specific (and well studied) case of general ideas.

2.1 Cylindrical Symmetry

A natural choice of coordinates for cylindrically symmetric systems is

$$\begin{array}{rclrcl} t & \equiv & x\cdot\gamma_0 & r & \equiv & \sqrt{-(x\wedge\sigma_3)^2} \\ \tan\phi & \equiv & (x\cdot\gamma^2)/(x\cdot\gamma^1) & z & \equiv & x\cdot\gamma^3. \end{array} \tag{2.1}$$

These coordinates have an associated coordinate frame $\{e_t, e_r, e_\phi, e_z\}$ given by

$$\begin{array}{rclrcl} e_t & \equiv & \gamma_0 & e_r & \equiv & \cos\phi\gamma_1 + \sin\phi\gamma_2 \\ e_\phi & \equiv & r(-\sin\phi\gamma_1 + \cos\phi\gamma_2) & e_z & \equiv & \gamma_3\ . \end{array} \tag{2.2}$$

The associated reciprocal frame is denoted by $\{e^t, e^r, e^\phi, e^z\}$. We shall also make use of the unit vector $\hat{\phi}$ defined by

$$\hat{\phi} \equiv -\sin\phi\gamma_1 + \cos\phi\gamma_2 \tag{2.3}$$

and the unit spatial bivectors

$$\sigma_r \equiv e_r e_t, \qquad \sigma_\phi \equiv \hat{\phi} e_t. \tag{2.4}$$

Finally, we shall occasionally make use of the vectors

$$\begin{array}{ll} g^t \equiv \overline{h}(e^t) & g_t \equiv \underline{h}^{-1}(e_t) \\ g^\phi \equiv \overline{h}(e^\phi) & g_\phi \equiv \underline{h}^{-1}(e_\phi). \end{array} \tag{2.5}$$

Cylindrical symmetry of the fields is imposed by demanding that our solution be invariant under translations in and reversals of the e_z direction and invariant under the combined effect of the translation

$$x' = f(x) = \tilde{R} x R, \qquad R = \exp\left(\frac{1}{2}\beta i\sigma_3\right) \tag{2.6}$$

and the rotation given by the rotor R. For stationary systems, we also demand that the solution be symmetric under the simultaneous reversal of t and ϕ. Finally, the spacetime dependence of $\overline{h}(a)$ should be restricted to dependence on the coordinate r alone. The most general form of the $\overline{h}$-function satisfying these conditions is

$$\begin{array}{rclrcl} \overline{h}(e^t) & = & f_1(r)e^t + rf_2(r)e^\phi & \overline{h}(e^r) & = & g_1(r)e^r \\ \overline{h}(e^\phi) & = & h_2(r)e^t + rh_1(r)e^\phi & \overline{h}(e^z) & = & i_1(r)e^z. \end{array} \tag{2.7}$$

An abstract form for $\omega(a)$ consistent with this $\overline{h}$-function and equation (1.42) is given by

$$\begin{array}{rclrcl} \omega(e_t) & = & -T(r)\sigma_r + (K(r)+h_2(r))i\sigma_3 & \omega(e_r) & = & \bar{K}(r)\sigma_\phi \\ \omega(\hat{\phi}) & = & K(r)\sigma_r + (h_1(r)-G(r))i\sigma_3 & \omega(e_z) & = & \bar{G}(r)i\sigma_\phi. \end{array} \tag{2.8}$$

The functions h_1 and h_2 appear explicitly in the form of $\omega(a)$ to simplify later formulae, namely to ensure that the expressions for the components of the Riemann tensor involve only the functions G, $\bar{G}$, T, K, $\bar{K}$, and their L-derivatives.

The gauge fields in the form above still retain a single degree of rotational freedom, namely that of a boost in the σ_ϕ plane. Under the action of the rotor

$$R = \exp\left(\frac{1}{2}\lambda(r)\sigma_\phi\right), \tag{2.9}$$

the functions in $\omega(a)$ transform according to (1.25) as

$$\begin{aligned} 2K &\mapsto 2K\cosh 2\lambda + (G+T)\sinh 2\lambda \\ G+T &\mapsto (G+T)\cosh 2\lambda + 2K\sinh 2\lambda \\ G-T &\mapsto G-T \\ \bar{G} &\mapsto \bar{G} \\ \bar{K} &\mapsto \bar{K} - L_r\lambda. \end{aligned} \tag{2.10}$$

These transformations have been written in the above form to highlight the gauge invariant quantities $\bar{G}$ and $G-T$. The Lorentzian structure on the quantities $2K$ and $G+T$ shall also prove noteworthy.

Also important is the radial coordinatisation of the solutions. Under a change of radial coordinate $r \mapsto r'(r)$, all of the functions in $\overline{h}(a)$ and $\omega(a)$ simply change their functional dependence from r to $r'(r)$ with the exception of g_1, which transforms according to (1.20) as

$$g_1(r) \mapsto \frac{g_1(r'(r))}{\partial_r r'(r)}. \tag{2.11}$$

In General Relativity, physical solutions are constrained by the elementary flatness criteria, a set of conditions imposed at the metric level. As pointed out in [9], solutions which satisfy the conditions of elementary flatness may still be ill-defined in the gauge theoretic formulation. In gauge theory gravity, elementary flatness is replaced by the requirement that the gauge fields be everywhere well-defined. For cylindrically symmetric systems, this principally involves requiring the values of the gauge fields on the z-axis to be independent of the direction of approach. The necessary constraints may easily be seen by writing $\overline{h}(a)$ and $\omega(a)$ in Cartesian coordinates and demanding that all ϕ dependence vanish as $r \to 0$. Explicitly, these conditions are

$$\begin{aligned} \lim_{r\to 0} f_2 &= 0 & \lim_{r\to 0} T &= 0 \\ \lim_{r\to 0} rh_1 &= g_1 & \lim_{r\to 0} G &= h_1 \\ \lim_{r\to 0} h_2 &= 0 & \lim_{r\to 0} \bar{G} &= 0 \end{aligned} \tag{2.12}$$
$$\lim_{r\to 0}(K+\bar{K}) = 0.$$

In terms of the posited $\overline{h}$-function, the L-derivatives with respect to the vectors $\{e_t, e_r, \hat{\phi}, e_z\}$ may be written explicitly as

$$\begin{aligned} L_t &= f_1\partial_t + h_2\partial_\phi & L_r &= g_1\partial_r \\ L_\phi &= f_2\partial_t + h_1\partial_\phi & L_z &= i_1\partial_z. \end{aligned} \tag{2.13}$$

Equations (1.44) and (1.31) give the bracket structure on the L-derivatives as

$$\begin{aligned} [L_r, L_t] &= TL_t + (K + \bar{K})L_\phi \\ [L_r, L_\phi] &= -(K - \bar{K})L_t - GL_\phi \\ [L_r, L_z] &= -\bar{G}L_z, \end{aligned} \tag{2.14}$$

with all remaining commutators vanishing. By applying these expressions to an arbitrary function $X(t, r, \phi, z)$ and equating the terms in the various partial derivatives of X, one readily obtains the equations

$$L_r f_1 = Tf_1 + (K + \bar{K})f_2 \tag{2.15}$$

$$L_r h_2 = Th_2 + (K + \bar{K})h_1 \tag{2.16}$$

$$L_r h_1 = -Gh_1 - (K - \bar{K})h_2 \tag{2.17}$$

$$L_r f_2 = -Gf_2 - (K - \bar{K})f_1 \tag{2.18}$$

$$L_r i_1 = -\bar{G}i_1 \tag{2.19}$$

relating $\overline{h}(a)$ and $\omega(a)$.

The components of the Riemann tensor may be computed in terms of the abstract functions in $\omega(a)$ via (1.30) as

$$\begin{aligned} \mathcal{R}(\sigma_r) &= \alpha_1\sigma_r + \beta\, i\sigma_3 \\ \mathcal{R}(i\sigma_3) &= \alpha_2 i\sigma_3 - \beta\sigma_r \\ \mathcal{R}(\sigma_\phi) &= \alpha_3\sigma_\phi \\ \mathcal{R}(\sigma_3) &= -\bar{G}T\sigma_3 - \bar{G}Ki\sigma_r \\ \mathcal{R}(i\sigma_r) &= K\bar{G}\sigma_3 + G\bar{G}i\sigma_r \\ \mathcal{R}(i\sigma_\phi) &= (L_r\bar{G} + \bar{G}^2)i\sigma_\phi \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} \alpha_1 &\equiv -L_rT + T^2 - K(K + 2\bar{K}) \\ \alpha_2 &\equiv L_rG + G^2 - K(K - 2\bar{K}) \\ \alpha_3 &\equiv K^2 - GT \\ \alpha_4 &\equiv L_r\bar{G} + \bar{G}^2 + \bar{G}(G - T) \\ \beta &\equiv L_rK + G(K + \bar{K}) - T(K - \bar{K}). \end{aligned} \tag{2.21}$$

The Ricci tensor, Ricci scalar, and Einstein tensor are given by

$$\begin{aligned} \mathcal{R}(e_t) &= (\alpha_1 + \alpha_3 - \bar{G}T)e_t - (\beta + \bar{G}K)\hat{\phi} \\ \mathcal{R}(e_r) &= (\alpha_1 + \alpha_2 + L_r\bar{G} + \bar{G}^2)e_r \\ \mathcal{R}(\hat{\phi}) &= (\alpha_2 + \alpha_3 + G\bar{G})\hat{\phi} + (\beta + K\bar{G})e_t \\ \mathcal{R}(e_z) &= \alpha_4 e_z, \end{aligned} \tag{2.22}$$

$$\mathcal{R} = 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \tag{2.23}$$

and

$$\begin{aligned} \mathcal{G}(e_t) &= -(\alpha_2 + \alpha_4 + \bar{G}T)e_t - (\beta + \bar{G}K)\hat{\phi} \\ \mathcal{G}(e_r) &= -(\alpha_3 + \alpha_4 - L_r\bar{G} - \bar{G}^2)e_r \\ \mathcal{G}(\hat{\phi}) &= -(\alpha_1 + \alpha_4 - G\bar{G})\hat{\phi} + (\beta + \bar{G}K)e_t \\ \mathcal{G}(e_z) &= -(\alpha_1 + \alpha_2 + \alpha_3)e_z \end{aligned} \tag{2.24}$$

respectively. We may now proceed to reproduce the van Stockum interior solution by imposing physical constraints appropriate to a rigidly rotating dust cylinder.

2.2 The van Stockum Interior

We take the cylinder to be centered on the z-axis and to extend out to radial coordinate $r = R$. The general stress-energy tensor for dust is

$$\mathcal{T}(a) = \rho a \cdot vv, \tag{2.25}$$

where ρ is the density distribution of the cylinder, and v is its covariant velocity. As mentioned earlier, the $\overline{h}$-function (2.7) retains the rotational degree of freedom of a boost in the σ_ϕ plane. We make the assumption that v is a timelike vector, whereupon this freedom may be used to set $v = e_t$. It is worth repeating that positing the existence of a timelike v is a *physical assumption*, namely the assumption that the matter comprising the cylinder is moving at a subluminal velocity. The conditions under which this assumption can safely and consistently be made will be of critical importance to the issue of possible causality violation. For the purposes of this section, we will take it for granted that a timelike v is always reasonable and consider possible caveats later.

To obtain the dust solution appropriate to rigid rotation, certain physical properties of the cylinder must be expressed in terms of the gauge fields. Specifically, the vorticity bivector and the shear tensor are given by

$$\varpi = e^\mu \wedge (e_\mu \cdot \mathcal{D}v) + (v \cdot \mathcal{D}v) \wedge v \tag{2.26}$$

and

$$\sigma(a) = \frac{1}{2}(H(a) \cdot \mathcal{D}v + H(e^\mu)(e_\mu \cdot \mathcal{D}v) \cdot a) - \frac{1}{3}e^\mu \cdot (e_\mu \cdot \mathcal{D}v)H(a), \tag{2.27}$$

respectively [9]. Here $H(a)$ is the projection operator

$$H(a) \equiv a - a \cdot v v. \tag{2.28}$$

With $v = e_t$, and our abstract form for $\omega(a)$, these expressions take the simple forms

$$\varpi = -(K - \bar{K}) i\sigma_3 \tag{2.29}$$

and

$$\sigma(a) = -\frac{1}{2}(K + \bar{K})(a \cdot e_r \hat{\phi} + a \cdot \hat{\phi} e_r). \tag{2.30}$$

A rigidly rotating system is shear-free, and we therefore restrict our attention to the case in which $\bar{K} = -K$. Finally, we impose the contracted Bianchi identity

$$\dot{\mathcal{T}}(\dot{\mathcal{D}}) = 0, \tag{2.31}$$

where the overdot denotes the scope of the differentiation. This yields the constraints

$$T = 0 \tag{2.32}$$

and

$$L_t \rho = 0. \tag{2.33}$$

The first constraint is an expression of the fact that the pressure free matter constituting the cylinder moves only under the force of gravitation, so that its acceleration $v \cdot \mathcal{D} v = -T e_r$ vanishes. The latter constraint is to be expected, as the density ρ should be a function only of r for stationary cylindrically symmetric systems. Using (2.24) and the above constraints, the Einstein equations take the reduced form

$$\begin{aligned}
L_r G + G^2 - 3K^2 + L_r \bar{G} + \bar{G}^2 + G\bar{G} &= -8\pi\rho & (2.34)\\
K^2 + L_r \bar{G} + \bar{G}^2 &= 0 & (2.35)\\
K^2 + G\bar{G} &= 0 & (2.36)\\
L_r G + G^2 - K^2 &= 0 & (2.37)\\
L_r K + \bar{G} K &= 0. & (2.38)
\end{aligned}$$

The full Bianchi identities are here satisfied automatically by the field equations. Now, relations (2.15) and (2.16) imply that h_2 and f_1 are both constant, which means that they encode a global rotation, which may be removed to set $h_2 = 0$. We may then scale the time direction so that $f_1 = 1$. To make further progress, we must now make the position-gauge choice of a radial coordinate. At first sight, the most natural choice is to choose r so that $g_1 = 1$, and indeed this choice is used later in deriving the vacuum solutions. For the case at hand, however, it turns out that a more prudent choice is to take a coordinatisation in which

$$h_1 = 1/r, \tag{2.39}$$

so that $\overline{h}(e^\phi) = e^\phi$. Having made this choice, it follows from (2.17) that

$$G = g_1/r. \tag{2.40}$$

Equation (2.37) now becomes

$$\partial_r(g_1^2) = 2K^2 r, \tag{2.41}$$

and using (2.36), equation (2.38) may be written as

$$g_1^2 - \frac{2r(K^2)^2}{\partial_r(K^2)} = 0. \tag{2.42}$$

Taking ∂_r of the above expression and parameterising K^2 as

$$K^2 = e^{f(r)}, \tag{2.43}$$

we end up with the equation

$$r\partial_r^2 f - \partial_r f = 0, \tag{2.44}$$

which, although second-order, is readily solved by inspection to give

$$K^2 = be^{cr^2} \tag{2.45}$$

where b and c are integration constants. We may now use equation (2.41) to read off

$$g_1 = \sqrt{\frac{b}{c}} e^{\frac{1}{2}cr^2}. \tag{2.46}$$

The axis condition (2.12) on rh_1 then requires that $b = c$. Furthermore, for K to be real, we must have $b > 0$, so we shall take $a^2 \equiv b = c$. Equations

(2.40) and (2.36) along with the rigid rotation condition then give the final forms

$$
\begin{aligned}
g_1 &= e^{\frac{1}{2}a^2r^2} && (2.47)\\
G &= \frac{1}{r}e^{\frac{1}{2}a^2r^2} && (2.48)\\
\bar{G} &= -a^2re^{\frac{1}{2}a^2r^2} && (2.49)\\
K^2 = \bar{K}^2 &= a^2e^{a^2r^2}. && (2.50)
\end{aligned}
$$

The field equations imply that $K^2 = 2\pi\rho$, so the density profile of the cylinder is given uniquely by

$$\rho = \frac{a^2}{2\pi}e^{a^2r^2}. \qquad (2.51)$$

The remaining functions in $\overline{h}(a)$ are obtained from the bracket relations. Equation (2.19) immediately integrates to

$$i_1 = e^{\frac{1}{2}a^2r^2} = g_1, \qquad (2.52)$$

where the z direction has been scaled so that the integration constant is unity. All that remains now is the solution of (2.18) to obtain f_2. We have

$$\partial_r f_2 = g_1^{-1}\left(-Gf_2 - 2K\right) = -\frac{f_2}{r} + 2a, \qquad (2.53)$$

where the sign chosen for K (or equivalently the sign of a) represents a choice of the direction of rotation of the cylinder. Equation (2.53) integrates to give

$$f_2 = ar + \frac{c'}{r}, \qquad (2.54)$$

where c' is an integration constant. However, to keep $\overline{h}(a)$ well-defined on the axis, we must have $f_2 \to 0$ as $r \to 0$, so we must have $c' = 0$.

The complete solution for the $\overline{h}$-function for the interior of the cylinder ($r < R$) is then

$$
\begin{array}{lllllll}
\overline{h}(e^t) &=& e^t + ar^2e^\phi & \overline{h}(e^r) &=& e^{\frac{1}{2}a^2r^2}e^r \\
\overline{h}(e^\phi) &=& e^\phi & \overline{h}(e^z) &=& e^{\frac{1}{2}a^2r^2}e^z,
\end{array} \qquad (2.55)
$$

which has a corresponding line element of

$$ds^2 = dt^2 - 2ar^2d\phi dt - r^2(1 - a^2r^2)d\phi^2 - e^{-a^2r^2}(dz^2 + dr^2), \qquad (2.56)$$

reproducing van Stockum's original interior line element.

2.3 The van Stockum Exterior

We now proceed to look for a vacuum solution to the field equations which we can match onto the cylinder. For the interior solution, the available rotation gauge freedom was used to fix the form of the stress-energy tensor, which indirectly led to the constraint $T = 0$ through the contracted Bianchi identities. For the vacuum solution, this freedom is employed to set $T = 0$ directly. The field equations resulting from setting each component of the Einstein tensor to zero are thus

$$\begin{aligned} L_r G + G^2 - K^2 + 2K\bar{K} + L_r\bar{G} + \bar{G}^2 + G\bar{G} &= 0 && (2.57)\\ -K^2 - 2K\bar{K} + L_r\bar{G} + \bar{G}^2 &= 0 && (2.58)\\ K^2 + G\bar{G} &= 0 && (2.59)\\ L_r G + G^2 - K^2 &= 0 && (2.60)\\ L_r K + G(K + \bar{K}) + \bar{G}K &= 0, && (2.61) \end{aligned}$$

from which we may obtain

$$K\bar{K} = 0. \tag{2.62}$$

So either K or $\bar{K}$ vanishes in the vacuum. Inside the cylinder, however, we had $K = -\bar{K}$ and non-vanishing, so either K or $\bar{K}$ will be discontinuous at the boundary. Choosing K to be the vanishing function would be problematic, as the field equations and Bianchi identities contain radial derivatives of K, which would be ill-defined at the discontinuity. Letting $\bar{K}$ vanish causes no such difficulties, as it always appears undifferentiated in all equations. This discontinuity may be thought of as the outcome of a limiting procedure in which we perform a boost of the form $R = \exp(\frac{1}{2}A(r)\sigma_\phi)$ where $A(r)$ is an arbitrary 'bump' function satisfying $L_r A|_{r=R} = -\bar{K}(R)$ and $A(R) = 0$. We then take $A(r)$ to quickly match analytically back onto zero after a short radial distance and take the limit of this distance going to zero. In this limit we obtain a solution in which $\bar{K}$ becomes discontinuous and the remaining functions in $\omega(a)$ are negligibly disturbed. So, with $\bar{K} = 0$ the field equations may be combined as

$$L_r(G + \bar{G}) + (G + \bar{G})^2 = 0 \tag{2.63}$$

$$L_r(G - \bar{G}) + (G - \bar{G})(G + \bar{G}) = 0 \tag{2.64}$$

and

$$L_r K + K(G + \bar{G}) = 0. \tag{2.65}$$

These are readily solved in the $g_1 = 1$ gauge to give

$$G = \frac{1}{2}\frac{1+\kappa}{r+r_o} \tag{2.66}$$

$$\bar{G} = \frac{1}{2}\frac{1-\kappa}{r+r_o} \tag{2.67}$$

$$K = \frac{\delta}{r+r_o} \tag{2.68}$$

where r_o, κ, and δ are constants of integration fixed by matching the values of G, $\bar{G}$, and K at the boundary. To match the vacuum solution obtained in the previous section *smoothly* onto the interior solution, we must make a change of radial coordinate so that g_1 is continuous and differentiable at the boundary $r = R$. The interior solution had the boundary values

$$g_1^{in}(R) = e^{\frac{1}{2}a^2R^2} \quad \text{and} \quad \partial_r g_1^{in}(R) = a^2 R e^{\frac{1}{2}a^2R^2}. \tag{2.69}$$

An exterior function matching these values is

$$g_1^{out}(r) = e^{\frac{1}{2}a^2R^2}\left(\frac{r}{R}\right)^{a^2R^2}. \tag{2.70}$$

This choice of g_1 is by no means unique and is chosen to give the van Stockum form for the vacuum. It should be noted that g_1 need not even be differentiable at the boundary, so long at it is continuous there. The jump in $\bar{K}$ at the boundary already leads to kinks in the $\overline{h}$-function, so there should be no requirement that g_1 be differentiable. For example, more simple forms for the vacuum fields result from the choice $g_1 = \exp(\frac{1}{2}a^2R^2)$.

Applying equation (2.11) we see that the new radial coordinate must satisfy

$$\frac{1}{\partial_r r'} = e^{\frac{1}{2}a^2R^2}\left(\frac{r}{R}\right)^{a^2R^2}, \tag{2.71}$$

which integrates to give

$$r'(r) = e^{-\frac{1}{2}a^2R^2}(1-a^2R^2)^{-1} r \left(\frac{R}{r}\right)^{a^2R^2}. \tag{2.72}$$

Here we have taken the integration constant to be zero so that the z-axes of both coordinates coincide, and we have also assumed that $a^2R^2 \neq 1$. In this new gauge, $\bar{G}$ for the exterior of the cylinder is given by

$$\bar{G}(r) = \frac{1-\kappa}{2} \frac{(1-a^2R^2)e^{\frac{1}{2}a^2R^2}}{r\left(\frac{R}{r}\right)^{a^2R^2} + r'_o} \tag{2.73}$$

where r'_o is a new constant multiple of r_o. Matching (2.73) with (2.49) at $r = R$ gives the condition

$$\frac{1}{2}(1-\kappa)(1-a^2R^2) = -a^2R^2(1 + r'_o/R). \tag{2.74}$$

Similarly, matching G at the boundary gives

$$\frac{1}{2}(1+\kappa)(1-a^2R^2) = 1 + r'_o/R, \tag{2.75}$$

and substituting this into (2.74) yields

$$\kappa = \frac{1+a^2R^2}{1-a^2R^2}. \tag{2.76}$$

and hence

$$r'_o = 0. \tag{2.77}$$

The matching of K at the boundary is straightforward, and we may now write down $\omega(a)$ for the exterior as

$$G(r) = e^{\frac{1}{2}a^2R^2}\left(\frac{1}{r}\right)\left(\frac{r}{R}\right)^{a^2R^2} \tag{2.78}$$

$$\bar{G}(r) = -a^2R^2e^{\frac{1}{2}a^2R^2}\left(\frac{1}{r}\right)\left(\frac{r}{R}\right)^{a^2R^2} \tag{2.79}$$

$$K(r) = -aRe^{\frac{1}{2}a^2R^2}\left(\frac{1}{r}\right)\left(\frac{r}{R}\right)^{a^2R^2} \tag{2.80}$$

$$\bar{K}(r) = 0 \tag{2.81}$$

$$T(r) = 0. \tag{2.82}$$

To obtain the $\overline{h}$-function, we once again rely on the bracket structure. Equations (2.15)-(2.19) yield a set of coupled first order equations which may be solved for the functions in $\overline{h}(a)$. We have

$$\partial_r f_1 = -\frac{aR}{r}f_2 \tag{2.83}$$

$$\partial_r h_2 = -\frac{aR}{r} h_1 \tag{2.84}$$

$$\partial_r h_1 = -\frac{1}{r} h_1 + \frac{aR}{r} h_2 \tag{2.85}$$

$$\partial_r f_2 = -\frac{1}{r} f_2 + \frac{aR}{r} f_1 \tag{2.86}$$

$$\partial_r i_1 = -\frac{a^2 R^2}{r} i_1. \tag{2.87}$$

Equation (2.87) is easily solved and matched to the interior solution to give

$$i_1 = e^{\frac{1}{2}a^2R^2} \left(\frac{r}{R}\right)^{a^2R^2}. \tag{2.88}$$

Equation (2.83) may be written as

$$f_2 = -\frac{r}{aR} \partial_r f_1, \tag{2.89}$$

and this may be substituted into (2.84) to give the second-order equation

$$\frac{r}{aR} \partial_r^2 f_1 + \frac{2}{aR} \partial_r f_1 + \frac{aR}{r} f_1 = 0, \tag{2.90}$$

which has solutions of the form r^ξ for constants ξ. It proves convenient to put the general solutions to the above two equations into trigonometric form as

$$f_1 = \sqrt{\frac{R}{r}} \left[(\alpha + \beta) \cosh\frac{\theta}{2} + (\alpha - \beta) \sinh\frac{\theta}{2} \right] \tag{2.91}$$

and

$$f_2 = \frac{1}{2a\sqrt{rR}} \left[(\alpha(1-\chi) + \beta(1+\chi)) \cosh\frac{\theta}{2} + (\alpha(1-\chi) - \beta(1+\chi)) \sinh\frac{\theta}{2} \right], \tag{2.92}$$

where

$$\chi \equiv \sqrt{1 - 4a^2R^2} \qquad \text{and} \qquad \theta \equiv \chi \ln\left(\tfrac{r}{R}\right) \tag{2.93}$$

and α and β are constants of integration. The functions h_2 and h_1 satisfy differential equations identical to those for the f_1, f_2 pair, and thus have the same general solutions with different constants of integration. Matching these functions at the boundary, we may write the complete $\overline{h}$-function in trigonometric form as

$$
\begin{aligned}
f_1 &= \sqrt{\frac{R}{r}}\left[\cosh\frac{\theta}{2} + \coth 2\epsilon \sinh\frac{\theta}{2}\right] && (2.94)\\
f_2 &= \frac{1}{2}\sqrt{\frac{R}{r}}\left[\operatorname{sech}\epsilon \cosh\frac{\theta}{2} + \operatorname{csch}\epsilon \sinh\frac{\theta}{2}\right] && (2.95)\\
h_1 &= \frac{1}{\sqrt{rR}}\left[\cosh\frac{\theta}{2} - \coth\epsilon \sinh\frac{\theta}{2}\right] && (2.96)\\
h_2 &= -\frac{1}{\sqrt{rR}}\operatorname{csch}\epsilon \sinh\frac{\theta}{2}, && (2.97)
\end{aligned}
$$

where we have defined

$$\epsilon \equiv \tanh^{-1}\chi. \tag{2.98}$$

Although the quantity χ is only real for $aR < \frac{1}{2}$, the $\overline{h}$-function remains real for all values $0 < aR < 1$. The case $aR > \frac{1}{2}$ may be obtained from the above solution by taking $j\theta' = \theta$ and $j\epsilon' = \epsilon$ and writing the solution in terms of the (real) primed quantities as

$$
\begin{aligned}
f_1 &= \sqrt{\frac{R}{r}}\left[\cos\frac{\theta'}{2} + \cot 2\epsilon' \sin\frac{\theta'}{2}\right] && (2.99)\\
f_2 &= \frac{1}{2}\sqrt{\frac{R}{r}}\left[\sec\epsilon' \cos\frac{\theta'}{2} + \csc\epsilon' \sin\frac{\theta'}{2}\right] && (2.100)\\
h_1 &= \frac{1}{\sqrt{rR}}\left[\cos\frac{\theta'}{2} - \cot\epsilon' \sin\frac{\theta'}{2}\right] && (2.101)\\
h_2 &= -\frac{1}{\sqrt{rR}}\csc\epsilon' \sin\frac{\theta'}{2}. && (2.102)
\end{aligned}
$$

The critical case $aR = \frac{1}{2}$ may be worked out from the original differential equations (2.83)-(2.86) and is given by

$$
\begin{aligned}
f_1 &= \sqrt{\frac{R}{r}}\left[1 + \frac{1}{4}\ln\left(\frac{r}{R}\right)\right] && (2.103)\\
f_2 &= \sqrt{\frac{R}{r}}\left[\frac{1}{2} + \frac{1}{4}\ln\left(\frac{r}{R}\right)\right] && (2.104)
\end{aligned}
$$

$$h_1 = -\frac{1}{2}\frac{1}{\sqrt{rR}}\left[\ln\left(\frac{r}{R}\right) - 2\right] \quad (2.105)$$

$$h_2 = -\frac{1}{2}\frac{1}{\sqrt{rR}}\ln\left(\frac{r}{R}\right). \quad (2.106)$$

The line element derived from our $\overline{h}$-function for $aR < \frac{1}{2}$ is

$$ds^2 = Fdt^2 - 2Md\phi dt - Ld\phi^2 - H(dz^2 + dr^2), \quad (2.107)$$

where

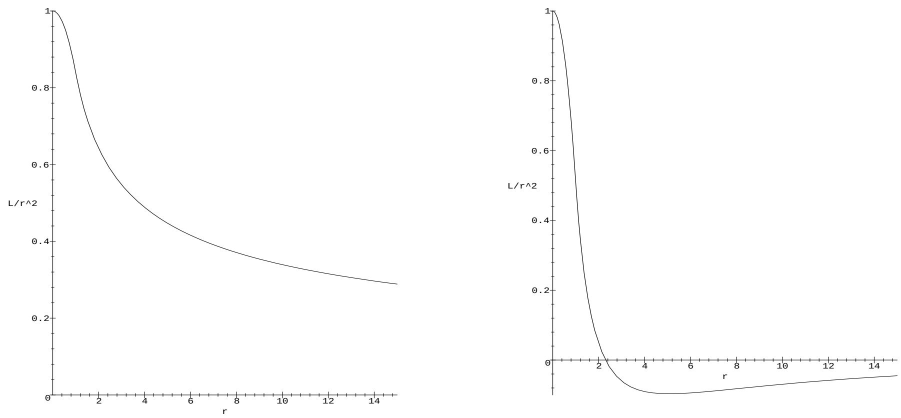
$$\begin{aligned} F &= \tfrac{h_1^2 - h_2^2}{(f_1h_1 - f_2h_2)^2} &&= (r/R)\sinh(\epsilon - \theta)\operatorname{csch}\epsilon \\ M &= \tfrac{f_2h_1 - f_1h_2}{(f_1h_1 - f_2h_2)^2} &&= r\sinh(\epsilon + \theta)\operatorname{csch}2\epsilon \\ L &= \tfrac{f_1^2 - f_2^2}{(f_1h_1 - f_2h_2)^2} &&= (rR/2)\sinh(3\epsilon + \theta)\operatorname{csch}(2\epsilon)\operatorname{sech}\epsilon \\ H &= g_1^{-2} = i_1^{-2} &&= e^{-a^2R^2}(R/r)^{2a^2R^2}. \end{aligned} \quad (2.108)$$

The solution for $aR > \frac{1}{2}$ can easily be obtained from the above by replacing ϵ and θ by their corresponding primed quantities and replacing the hyperbolic trigonometric functions by their corresponding regular trigonometric counterparts. The line element for $aR = \frac{1}{2}$ is

$$\begin{aligned} F &= (r/R)(1 - \ln(r/R)) \\ M &= (r/2)(1 + \ln(r/R)) \\ L &= (rR/4)(3 + \ln(r/R)) \\ H &= e^{-\frac{1}{4}}(R/r)^{\frac{1}{2}}. \end{aligned} \quad (2.109)$$

Figure 2.1 shows plots of the coefficients of $d\phi^2$ in the line elements for both the low and high mass regimes, illustrating the appearance of causality violating regions in the latter.

Thus we have completely reproduced the solutions for the interior and the three vacuum sectors of the van Stockum solution. Once the general framework for stationary cylindrical systems has been set up, getting to the above solutions is a fairly straightforward matter, especially when compared to the lengthy derivation in van Stockum's original presentation [4].



(a) $aR = 0.4$ (b) $aR = 0.7$

Figure 2.1: The coefficients of $d\phi^2$ in the van Stockum line element divided by r^2. Both plots are for a cylinder of radius $R = 1$. When L/r^2 becomes negative in (b), lines of constant t, r, and z become closed timelike loops.

2.4 Properties of the Solution

As in General Relativity, the existence of Killing vectors allows the construction of conserved quantities. If V_K is a Killing vector, then the vector $\mathcal{T}(V_K)$ is covariantly conserved, *i.e.*, $\mathcal{D} \cdot \mathcal{T}(V_K) = 0$ (See Appendix A.) Now, for a vector $\mathcal{J}$ [6],

$$\mathcal{D} \cdot \mathcal{J} = \det(\underline{h})\nabla \cdot [\underline{h}(\mathcal{J}) \det(\underline{h})^{-1}]. \tag{2.110}$$

Therefore, the current

$$J_K = \underline{h}(\mathcal{T}(V_K)) \det(\underline{h})^{-1} \tag{2.111}$$

satisfies $\nabla \cdot J_K = 0$. We may then define

$$Q_K = \int dS \cdot J_K, \tag{2.112}$$

where dS is the normal to an open spacelike hypersurface. Since J_K is divergenceless with respect to ∇, the value of Q_K is independent of the choice of hypersurface. It is convenient to choose $dS = e^t|d^3x|$. Since the fields are cylindrically symmetric, the integrand generally has no z or ϕ dependence, and we may thus define

$$q_K = 2\pi \int_0^R r dr g^t \cdot \mathcal{T}(V_K) \det(\underline{h})^{-1}, \tag{2.113}$$

which is a conserved "charge" per unit length in z as measured along the axis of the cylinder.

It is instructive to examine the physical properties of the van Stockum solution *vis à vis* its Newtonian equivalent. The mass per unit length of the cylinder may be obtained by applying (2.113) to the Killing vector g_t, giving

$$\mu \equiv q_{g_t} = \frac{1}{2} a^2 R^2. \tag{2.114}$$

The same expression may be derived from the covariant integral of the density over a spacetime region of unit length in t and z. A cylinder rigidly rotating under its own gravitation with angular velocity a in Newtonian theory has a constant density given by

$$\rho_N = \frac{a^2}{2\pi}. \tag{2.115}$$

This density agrees with (2.51) in the limit of small aR. Furthermore, the Newtonian mass per unit length agrees exactly with the gauge theory expression (2.114) for any value of aR. The connection with the Newtonian solution becomes even closer when one considers the angular momentum per unit length of the cylinder, which is derived from (2.113) using the Killing vector g_ϕ. For the van Stockum solution, the integral evaluates to give

$$J \equiv q_{g_\phi} = \frac{1}{4} a^3 R^4, \tag{2.116}$$

once again in full agreement with the Newtonian expression for all values of aR. In [10], Bonnor obtains these Newtonian values for μ and J only in the approximation of small aR, so the correspondence with Newtonian theory that he finds is not particularly surprising. More interesting is the fact that the Newtonian results appear in our gauge theory even when no approximations are made. If one considers these results along side those obtained in [6] for spherically symmetric systems in a 'Newtonian gauge,' one sees how comfortably Newtonian gravity fits into a full relativistic treatment in gauge theory gravity.

In light of the Newtonian correspondence described above, it is tempting to interpret the quantity ar as the tangential velocity of the rotating cylinder, even in the full relativistic theory. In the context of General Relativity, one can even show that a may be interpreted as the as the limiting value of the angular velocity as the z-axis is approached [4, 10]. Indeed, this is the path usually taken when discussing the van Stockum solution. The quantity aR is thought of as the tangential velocity of the cylinder at its perimeter, at least in some loose sense, and attention is restricted to the regime $aR < 1$ with the idea in mind being that this 'velocity' should be less than the speed of light. The analysis of the behavior of the Riemann invariants as $r \to \infty$ in [10] lends further weight to this interpretation. However, this interpretation can be taken too far, and the looseness of aR as a velocity cannot be overstressed. The coordinate r is not a proper radial coordinate, and the proper circumference of a circle of radius r centered on the z-axis is not $2\pi r$, nor is it directly related to the proper radius. Furthermore, the marked change in the character of the solutions at $aR = \frac{1}{2}$ is left quite mysterious in this interpretation. That things should change so dramatically when a velocity becomes half the speed of light seems almost arbitrary. Nonetheless, an important issue is at stake. It is one of the fundamental principles of relativistic physics that matter cannot travel at or above the speed of light, so clearly a rigidly rotating cylinder should be constrained by this condition in some fashion. This problem is especially difficult when examining the spacetime of an isolated rotating object, for there is no absolute frame against which to measure its speed of rotation. Naturally, the status of Mach's Principle, which dictates that inertial frames are determined by the total distribution of matter in all of spacetime, is intimately tied into this issue.

If the cylinder is not necessarily 'rotating at light-speed' when $aR = 1$, then the question becomes, when exactly is it? The loose interpretation of aR as a velocity is simply insufficient, but the strange changes that occur when $aR = \frac{1}{2}$ should immediately raise suspicion. Further analysis will point towards the interpretation that $aR = \frac{1}{2}$ should be taken as the true physical upper limit to the van Stockum class of solutions, but before continuing this line of argument, it is useful to examine the nature of geodesic motion in the context of our gauge theory.

2.5 Geodesic Motion

Geodesic motion provides a useful tool for studying the properties of spacetime, and is of obvious direct physical relevance. Opher, Santos, and Wang [5] have presented a thorough analysis of geodesic motion for the interior van Stockum solution. In this section we present a framework for the study of geodesic motion in general for cylindrically symmetric systems. Gauge theory gravity has the advantage of leading quite naturally to a set of first-order differential equations describing geodesic trajectories.

We wish to find solutions to the point particle geodesic equation

$$v \cdot \mathcal{D} v = 0, \tag{2.117}$$

where v is the covariant velocity of the particle. If the particle trajectory is described by a parametrised path in spacetime, $x(\tau)$, then v is given by the expression

$$v = \underline{h}^{-1}(\dot{x}), \tag{2.118}$$

where here the overdot denotes differentiation with respect to τ. The geodesic equation may then be written as

$$\dot{v} + \omega(v) \cdot v = 0. \tag{2.119}$$

Restricting to motion in the $i\sigma_3$ plane, in which most of the interesting dynamics takes place, v may be written explicitly in terms of $\overline{h}(a)$ and the coordinate derivatives as

$$v = (f_1 h_1 - f_2 h_2)^{-1}[(\dot{t}h_1 - \dot{\phi} f_2)e_t + (\dot{\phi} f_1 - \dot{t} h_2)\hat{\phi}] + \dot{r} g_1^{-1} e_r. \tag{2.120}$$

We first consider null geodesics, writing an arbitrary null ($v^2 = 0$) velocity vector abstractly as

$$v = AR(\gamma_0 + \gamma_1)\tilde{R}, \tag{2.121}$$

where the rotor R is given by

$$R = \exp\left(\frac{1}{2}\alpha i \sigma_3\right) \tag{2.122}$$

and A and α are arbitrary functions of r and τ. Expanding (2.121) gives

$$v = Ae_t + A\cos\psi e_r - A\sin\psi\hat{\phi}, \tag{2.123}$$

where we have defined $\psi \equiv \phi + \alpha$. We may now write the coordinate derivatives in terms of the abstract functions as

$$\begin{aligned} \dot{r} &= A g_1 \cos\psi \\ \dot{\phi} &= A(h_2 - h_1 \sin\psi) \\ \dot{t} &= A(f_1 - f_2 \sin\psi). \end{aligned} \tag{2.124}$$

Using the abstract form of v (2.123) in the geodesic equation (2.119) we obtain the differential equations

$$\begin{aligned} \partial_\tau A &= A^2[T\cos\psi + (K + \bar{K})\sin\psi\cos\psi] \\ \partial_\tau \cos\psi &= A\sin\psi[(G+T)\sin\psi + K(1+\sin^2\psi) - \bar{K}\cos^2\psi] \\ \partial_\tau \sin\psi &= -A\cos\psi[(G+T)\sin\psi + K(1+\sin^2\psi) - \bar{K}\cos^2\psi] \end{aligned} \tag{2.125}$$

which together with (2.124) gives a complete set of first-order evolution equations for null geodesics. Given an initial position and initial values for A and ψ, a numerical integration of these equations may easily be implemented on a computer to generate geodesic trajectories. Null geodesics for various values of the parameter aR are shown in Figure 2.2.

The Killing vectors g_t and g_ϕ also generate the conserved quantities (see Appendix A) Q_1 and Q_2 given by

$$Q_1 = g_t \cdot v = A\frac{h_1 - h_2\sin\psi}{f_1 h_1 - f_2 h_2} \tag{2.126}$$

and

$$Q_2 = g_\phi \cdot v = A\frac{f_1\sin\psi - f_2}{f_1 h_1 - f_2 h_2}. \tag{2.127}$$

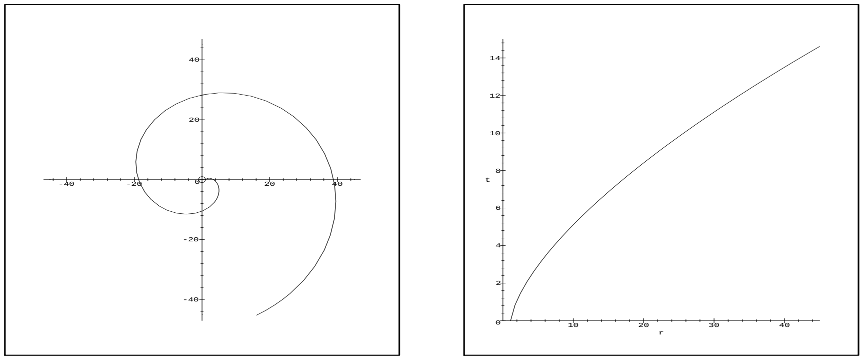
These two quantities are apparently related to a conserved energy and angular momentum.

A similar analysis may be used to obtain a system of equations for the timelike geodesics describing the motion of massive particles. We begin by writing a general normalized ($v^2 = 1$) timelike velocity vector as

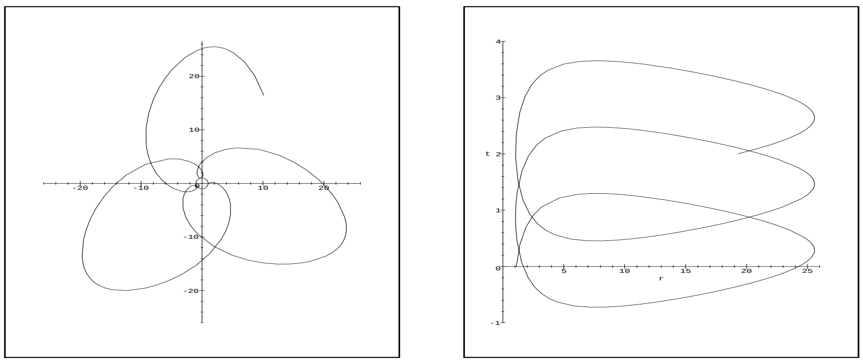
$$v = R\gamma_0\tilde{R} \tag{2.128}$$

with the rotor R here being given by

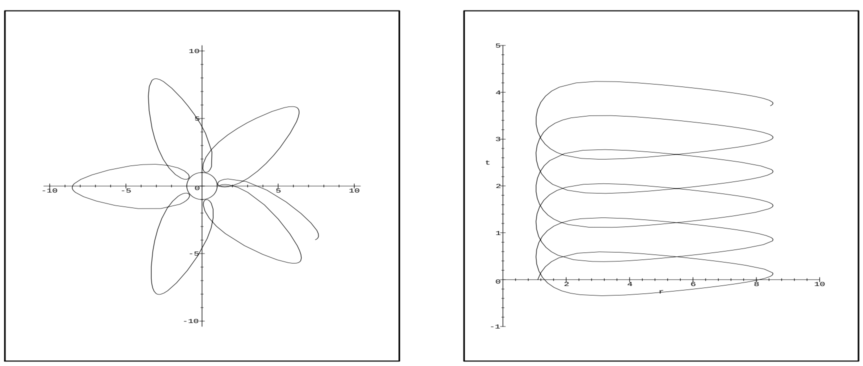
$$R = \exp\left(\frac{1}{2}\alpha\sigma_r\right)\exp\left(\frac{1}{2}\beta\sigma_\phi\right), \tag{2.129}$$



(a) $aR = 0.4$



(b) $aR = 0.7$



(c) $aR = 0.9$

Figure 2.2: Null geodesics in the van Stockum exterior. The left hand plots show r versus ϕ and the right hand plots show t versus r. All three trajectories have $R = 1$ and begin at $r = 1.1$ with $A = 1$, $\cos\psi = 1/2$ and $\sin\psi = -\sqrt{3}/2$. Photons in the low mass regime (a) escape to infinity, but those in the high mass regime (b) and (c) remain in bound orbits and traverse causality violating regions. In the coordinates shown, the cylinder is at rest.

where α and β are arbitrary functions of r and τ. Expanding the rotor in (2.128) and comparing with (2.120), we may write

$$\begin{aligned} \dot{r} &= \tfrac{1}{2} g_1 s \\ \dot{\phi} &= h_1 \bar{s} + \tfrac{1}{2} h_2 c \\ \dot{t} &= f_2 \bar{s} + \tfrac{1}{2} f_1 c, \end{aligned} \tag{2.130}$$

where

$$\begin{aligned} s &\equiv \sinh(\alpha+\beta) + \sinh(\alpha-\beta) \\ c &\equiv \cosh(\alpha+\beta) + \cosh(\alpha-\beta) \\ \bar{s} &\equiv \sinh\beta. \end{aligned} \tag{2.131}$$

As for the null geodesics, substituting the abstract form of v into (2.119) gives the differential equations

$$\begin{aligned} \dot{c} &= -(K+\bar{K}) s\bar{s} + \tfrac{1}{2} T c s \\ \dot{s} &= 2G\bar{s}^2 - 2Kc\bar{s} + \tfrac{1}{2} T c^2 \\ \dot{\bar{s}} &= \tfrac{1}{4}(K-\bar{K}) c s - \tfrac{1}{2} G s \bar{s}, \end{aligned} \tag{2.132}$$

which together with (2.130) gives a complete set of equations for timelike geodesics. The two conserved quantities for timelike geodesics corresponding to the Killing vectors g_t and g_ϕ are

$$Q_1 = \frac{c h_1 + 2\bar{s} h_2}{f_1 h_1 - f_2 h_2} \tag{2.133}$$

and

$$Q_2 = \frac{c f_2 + 2\bar{s} f_1}{f_1 h_1 - f_2 h_2}. \tag{2.134}$$

2.6 Mach's Principle and Locally Static Fields

Frehland [14] and Som *et al.* [15] have shown that the van Stockum exterior solution may be brought into static form in the low-mass regime by a transformation of the form

$$\begin{aligned} t' &= At + B\phi \\ \phi' &= Ct + D\phi \end{aligned} \tag{2.135}$$

for some real constants A, B, C, and D. Bonnor's careful analysis of hypersurface orthogonal Killing vectors in [10] shows that this cannot be done for the critical and high-mass regimes. In the low-mass case, this transformation introduces a periodic time coordinate, so that the points (t', ϕ') and $(t' + 2\pi B, \phi' + 2\pi D)$ are to be identified. Interpreted globally [11], the new coordinates cover a manifold with topology $S^2 \times$ (half plane). The new coordinates do not match smoothly onto the interior solution, and their physical significance is unclear. In any event, such transformations as (2.135) have no place in gauge theory gravity, in which solutions must be global in a flat Minkowski spacetime.

These considerations are of crucial importance for Mach's Principle. If the van Stockum vacuum is in fact static, then there cannot be any frame-dragging effects due to the rotation of the cylinder— clearly an anti-Machian conclusion. Bonnor calls solutions like the low-mass van Stockum case locally static. They are static because they possess hypersurface orthogonal Killing vectors with respect to some timelike coordinate, but they cannot be considered globally static as this time coordinate is periodic. This will prove to be a useful distinction, but by itself, it still leaves the status of Mach's Principle unresolved. According to Bonnor [16], locally static fields are in some sense static enough to lie in contradiction with Mach's Principle, but no concrete evidence is given to support this. Our gauge theoretic analysis will show that Bonnor is, in fact, correct on this point, and perhaps will elucidate the notion of a locally static field.

Instead of trying to deduce the status of Mach's Principle from the form of the metric, we shall take a much more physical approach and examine geodesic motion in a cylindrically symmetric vacuum. Consideration of motion in the $i\sigma_3$ plane is sufficient for the argument, so the formalism of the preceding section is applicable here. We begin by examining the form of timelike geodesics which describe circular orbits around the cylinder. These generally have a covariant velocity of the form

$$v = e_t \cosh\beta + \hat{\phi}\sinh\beta. \tag{2.136}$$

The geodesic equation (2.117) then implies

$$2K\sinh\beta\cosh\beta - T\cosh^2\beta - G\sinh^2\beta = 0, \tag{2.137}$$

which generally has two solutions, β_+ and β_-, given by

$$\tanh\beta_\pm = \frac{K \pm (K^2 - GT)^{\frac{1}{2}}}{G}. \tag{2.138}$$

For the time being, we shall assume that both solutions exist. Ignoring any Machian effects, we expect that an observer in a non-rotating frame would observe particles traveling along these geodesics as moving with equal and opposite velocities. In the coordinates of the solution, which are generally rotating in some non-uniform manner, such an observer would have a covariant velocity v_o of a form similar to (2.136), namely

$$v_o = e_t \cosh\lambda + \hat{\phi}\sinh\lambda. \tag{2.139}$$

The condition that the observer sees the geodesics moving with equal and opposite relative velocities is

$$v_+ \wedge v_o + v_- \wedge v_o = 0, \tag{2.140}$$

which implies

$$\cosh\lambda(\sinh\beta_+ + \sinh\beta_-) - \sinh\lambda(\cosh\beta_+ + \cosh\beta_-) = 0, \tag{2.141}$$

and is solved by

$$\tanh 2\lambda = \frac{2K}{G+T}. \tag{2.142}$$

Now, to see whether or not such an observer feels any rotation due to Machian frame dragging, we must consider the form of the Weyl tensor, denoted $\mathcal{W}(B)$. From the form of the Riemann tensor (2.20), we see that the Weyl tensor may generally be written as

$$\begin{aligned} \mathcal{W}(\sigma_r) &= \Psi\sigma_r + \Upsilon i\sigma_3 \\ \mathcal{W}(i\sigma_3) &= \Phi i\sigma_3 - \Upsilon\sigma_r \\ \mathcal{W}(\sigma_\phi) &= \Xi\sigma_\phi, \end{aligned} \tag{2.143}$$

where self-duality $\mathcal{W}(iB) = i\mathcal{W}(B)$, determines the remaining components. Now, the forces experienced on a small rod held out by our observer in the radial direction are described by

$$\mathcal{W}(e_r \wedge v) = \cosh\lambda\mathcal{W}(\sigma_r) - \sinh\lambda\mathcal{W}(i\sigma_3) \tag{2.144}$$

as a consequence of the gauge theory analogue of the geodesic deviation equation. The rod will not feel any forces in the $\hat{\phi}$ direction arising from frame dragging effects if $e_r \wedge v$ is an eigenbivector of the Weyl tensor. This turns out to be the case if

$$\tanh 2\lambda = \frac{2\Upsilon}{\Phi - \Psi}. \tag{2.145}$$

In the vacuum, $\mathcal{W}(B)$ is identical to $\mathcal{R}(B)$ so that $\Psi = \bar{G}G$, $\Phi = -\bar{G}T$, and $\Upsilon = -\bar{G}K$, and (2.145) becomes identical to (2.142). Thus we see that the observer truly does not feel any frame dragging effects. Physics in the observer's local frame looks perfectly static, and hence the name locally static is appropriate. The preceding argument never invoked any specific form of the vacuum solution apart from that required by cylindrical symmetry, thus we may say that Mach's principle fails quite generally for stationary cylindrically symmetric systems, which are always locally static.

An alternative, and slightly more general way of looking at this is to consider the action of the globally applied rotation-gauge transformation given by the rotor $R = \exp(-\frac{1}{2}\lambda\sigma_\phi)$ where λ satisfies (2.142). In the new gauge, $v_o = e_t$ and $\mathcal{W}(B)$ is diagonal. The fact that one cannot globally make the corresponding position-gauge change $x \mapsto Rx\tilde{R}$ without introducing a periodic time coordinate illustrates why we cannot claim the fields to be globally static. Moreover, although the observer will see circularly orbiting particles moving with equal but opposite velocities, the effects of the rotation of the cylinder *do* show up in global experiments. In the new Weyl-diagonal gauge, $K = 0$, so the particle geodesics are given by the boost parameters

$$\beta_\pm = \pm\beta = \pm \tanh^{-1}\sqrt{-\frac{T}{G}}. \tag{2.146}$$

The angular velocities of the particles with respect to coordinate time are given by

$$\phi'_\pm(r) = \frac{d\phi_\pm}{dt} = \frac{\dot{\phi}_\pm}{\dot{t}_\pm} = \frac{\pm h_1 \sinh\beta + h_2 \cosh\beta}{\pm f_2 \sinh\beta + f_1 \cosh\beta}, \tag{2.147}$$

and the angular velocity of the observer is simply

$$\phi'_o(r) = \frac{h_2}{f_1}. \tag{2.148}$$

Two particles fired simultaneously at $t = 0$ from the observer along oppositely directed circular orbits would travel around the cylinder meeting at regular intervals $t_n = 2\pi n/(\phi'_+ - \phi'_-)$. In a globally static vacuum, the first meeting occurs directly opposite from the observer (which is also generally moving with nonzero coordinate speed) and the second meeting will occur at the observer's location. Superimposing a rotation of angular velocity $-\phi'_o$ so that the observer is at rest, it is clear that for the geodesics above, the second meeting will only occur at the observer if

$$\Delta\phi' = \phi'_o - \frac{1}{2}(\phi'_+ + \phi'_-) = -\frac{f_2(f_1 h_1 - f_2 h_2)}{f_1(f_1^2 \frac{G}{T} + f_2^2)} = 0, \tag{2.149}$$

which is satisfied (for solutions with $\det(\underline{h}) \neq 0$) only when $f_2 = 0$. In general, the second meeting will occur at an angle

$$\Delta\phi = \Delta\phi' t_2 = 2\pi \frac{f_2}{f_1}\sqrt{-\frac{T}{G}} \tag{2.150}$$

from the observer. The two particles will arrive back at the observer separated by a coordinate time

$$\Delta t = 4\pi \frac{f_1 f_2}{f_1 h_1 - f_2 h_2}, \tag{2.151}$$

which is equivalent to a proper time lapse for the observer of

$$\Delta\tau = 4\pi \frac{f_2}{f_1 h_1 - f_2 h_2}. \tag{2.152}$$

Thus it would appear to the observer as if a trip around the cylinder is shorter in one direction than the other. It should be remembered that these explicit expressions are to be evaluated in the gauge which diagonalizes the Weyl tensor. It will be seen later that T/G is a constant in the Weyl diagonal vacuum. The important point here is that although a local frame can be found in which the physics is that of a static system, experiments of a global nature can be conducted which reveal the non-static nature of the fields. Hence the name locally static is quite appropriate to such systems.

If we consider our original vacuum solution as one matched to a matter distribution in which the fluid velocity is e_t, then equation (2.142) describes the relative boost between the gauge in which the matter stress-energy tensor and the vacuum Weyl tensor is diagonal. This relative boost is an intrinsic property of the physical system, and from it we may define an intrinsic 'static velocity of rotation'

$$\Sigma = -\tanh\lambda = -\tanh\left[\frac{1}{2}\tanh^{-1}\left(\frac{2K}{G+T}\right)\right] \tag{2.153}$$

of the cylinder. (Here the functions are understood to be evaluated in the gauge in which $\mathcal{T}(a)$ is diagonal.) A similar procedure for defining an intrinsic fluid velocity for axisymmetric systems was suggested in [17].

In virtue of (2.142), this velocity will only be well defined if

$$\left|\frac{2K}{G+T}\right| < 1, \tag{2.154}$$

in which case $|\Sigma| < 1$ also holds; the static velocity must be subluminal if it is to exist at all. It should be noted that although $2K/(G+T)$ is not itself a gauge invariant quantity, (2.154) is a gauge invariant inequality in virtue of the transformation properties of $\omega(a)$ (2.10). In any case, the specific value of Σ describes a relation between two definite gauges, and is a well-defined quantity.

Let us relate our general argument to the particular case of the van Stockum solution. For the interior, we have

$$\frac{2K}{G+T} = -2ar. \tag{2.155}$$

In the limit of small ar, for example, as one approaches the z-axis, we do indeed recover the Newtonian result $\Sigma = ar$. We also see that the entire solution is locally static only for $|aR| < \frac{1}{2}$, as Bonnor claims. However, if we demand that Σ must have a well-defined subluminal value, then the critical and high-mass regimes $aR \geq \frac{1}{2}$ are more than simply not locally static, they must be thrown out as unphysical.

2.7 Counterrotating Shells and the Static Velocity

At this point, it is reasonable to ask just how justified our definition of the static velocity of the cylinder really is — in what sense does it actually represent the velocity of the cylinder? It has been suggested [6] that one can generally put $\mathcal{W}(B)$ into a diagonal form by appropriate choice of rotation gauge, but this has never been advanced as a general principle, and the physical meaning of this condition has never been explicitly addressed. The issue is made even more complicated by the fact that $\mathcal{W}(B)$ is generally discontinuous at a matter/vacuum interface, so that in regions occupied by matter, Σ relates the diagonalization of the stress-energy tensor to that of the full Riemann tensor, not just its Weyl tensor component.

There is no absolute frame against which the velocity of rotation of the cylinder might be measured, but the special properties of cylindrical symmetry may be exploited to do the next best thing, namely to compare the speed of the cylinder to a copy of itself spinning in the opposite direction. Suppose we have found a solution to the field equations, and have expressed this solution in a gauge in which the fluid velocity of the matter is given by e_t. The rotation-gauge field $\omega(a)$ is then essentially given by the five functions G, $\bar{G}$, T, K, and $\bar{K}$. We may construct a 'counterrotating' solution, which we shall label with the subscript 'c,' by taking

$$K_c = -K \quad \text{and} \quad \bar{K}_c = -\bar{K}, \tag{2.156}$$

whilst leaving G_c, $\bar{G}_c$ and T_c equal to the original functions. As for $\overline{h}_c(a)$, all of its component functions are the same as those in the original solution, except for

$$f_{2c} = -f_2 \quad \text{and} \quad h_{2c} = -h_2. \tag{2.157}$$

It may easily be checked that the counterrotating solution does indeed solve the field equations, and is consistent with the bracket relations. This solution is indeed a counterrotating copy of the original solution; it has the same mass per unit length μ as the original solution, and its vorticity ϖ, shear tensor $\sigma(a)$, and angular momentum J are all equal in magnitude but opposite in sign to those for the original solution. At the metric level, the only change is the sign of the $d\phi dt$ term of the line element.

Now, consider a configuration in which a 'core' of the original solution in the region $r < S$ is matched onto a 'shell' of the counterrotating solution for $r > S$. As one might expect at this point, the problem of matching the two regions lies in finding an appropriate rotation gauge in which to express the shell. Here we parameterise our rotor as $R = \exp(\frac{1}{2}\beta\sigma_\phi)$, and attempt to apply a gauge transformation to the shell so that $\omega(a)$ matches at the boundary $r = S$. As in the matching of the van Stockum interior to the vacuum, we do not need to worry about a discontinuity in $\bar{K}$, and by the definition of our counterrotating shell, the matching of $\bar{G}$ and $G - T$ is automatically taken care of, as these quantities are gauge invariant. So the problem reduces to simultaneously matching the first two expressions in (2.10) onto the core values. At fist glance, this seems impossible, as we wish somehow to flip the sign of K_c so that $K_c' = -K_c = K$ whilst leaving $G_c' + T_c' = G_c + T_c = G + T$. However, this can be done if

$$\tanh\beta = \frac{2K}{G+T}. \qquad (2.158)$$

Thus we see that

$$\beta = 2\lambda \qquad (2.159)$$

where λ is as defined in (2.142). (We could also have accomplished the matching in a more symmetric fashion by choosing a rotation gauge parameterised by ξ_c for the shell, and one parameterised by $\xi = -\xi_c$ for the shell. In this case, we would have $\xi = \lambda$.) In the matched gauge, the fluid velocity of the counterrotating shell is given by

$$v_c = e_t \cosh\beta + \hat{\phi}\sinh\beta. \qquad (2.160)$$

Considering the situation at the boundary between the core and the shell, $\tanh\beta$ represents the velocity of the shell relative to an observer comoving with the core (who would still have covariant velocity $v = e_t$), and this velocity could in principle be measured, say, by examining photon redshifts. By (2.159), the observer sees the particles constituting the shell moving with a rapidity twice that of the rapidity corresponding to Σ, which is exactly the expected relation for two particles moving with speed Σ in opposite directions. Now if we consider various possible boundary positions S, and examine the case as $|2K/(G+T)|$ approaches unity at the boundary as we vary S, we see that this velocity tends towards the speed of light (and v_c becomes null). When it actually becomes unity, no finite boost can perform the matching of the two solutions and the whole scheme breaks down. One could simply say that the matching is simply no longer possible, and leave it at that, but this would leave the physics behind the failure as something of a mystery. The physical reason for this breakdown seems to be that counterrotating shell at this radius would have to be rotating at the speed of light, and this is simply impossible. Now, we have all along been dealing solely with gauge transformations, which in themselves have no physical significance whatsoever. The shell is rotating at the same speed regardless of the gauge in which we choose to express it. Moreover, by construction, the shell and the core are rotating at the same speed, only in opposite directions, so if the shell is rotating at the speed of light at some critical radius, the core must be also. (One could just interchange the shell and the core throughout, and the argument remains the same.) This argument, although not rigorous, suggests that we supplement our theory with the constraint (2.154), and consider only those solutions for which the vacuum Weyl tensor is diagonalisable.

2.8 The Static Velocity and Causality Violation

The high-mass van Stockum vacuum exhibits causality violation in the form of CTL. This happens when the coefficient of $d\phi^2$ in the line element

$$L = \frac{f_1^2 - f_2^2}{(f_1 h_1 - f_2 h_2)^2} \tag{2.161}$$

becomes negative. As this happens when the magnitude of f_2 exceeds that of f_1 it is useful to consider the behavior of the quantity

$$\eta = \frac{f_1}{f_2}. \tag{2.162}$$

It is then clear that CTL exist when $|\eta| < 1$. Without loss of generality, we may take $g_1 = 1$, so that a simple application of the bracket relations yields

$$\partial_r \eta = K(1+\eta^2) + \bar{K}(1-\eta^2) + (G+T)\eta. \tag{2.163}$$

So, at any point at which

$$\eta = \pm 1, \tag{2.164}$$

we have

$$\partial_r \eta = 2K \pm (G+T). \tag{2.165}$$

Note that η transforms under rotation gauge transformations parameterised by β as

$$\eta \mapsto \frac{\eta - \tanh\beta}{1 - \eta\tanh\beta} \tag{2.166}$$

so that points where (2.164) holds and thus the regions in which CTL exist are, as expected, gauge invariant.

Now, if the constraint equation (2.154) is satisfied, then the sign of $G+T$ is fixed for all r for dust solutions. This is because the fields at any point at which $G+T=0$ would horribly violate the constraint equation unless perhaps $K = 0$ at that point as well. If both K and $G + T$ vanish, say at some point $r = s$, then they do so in any choice of rotation gauge, so we may study the situation in which the stress-energy tensor is diagonal. We have

$$L_r(G + T) + (G + T)(G + \bar{G} - T) + 4K\bar{K} = -8\pi(\rho + P_\phi) \qquad (2.167)$$

and

$$L_rK + K(G + \bar{G} - T) + \bar{K}(G + T) = 0. \qquad (2.168)$$

A simple application of l'Hôpital's rule gives

$$\lim_{r\to s} \frac{2K}{G + T} = \lim_{r\to s} \frac{2L_rK}{L_r(G + T)} = \frac{0}{-8\pi(\rho + P_\phi)} = 0 \qquad (2.169)$$

where we note that $\rho + P_\phi \geq 0$ by the strong energy condition. Thus (2.154) by itself is not enough to rule out $G + T = 0$ in every case. However, at $r = s$, the null paths described by

$$v = e_t \pm \hat{\phi} \qquad (2.170)$$

become geodesics, so that light will orbit in circles at this radius. Clearly, then, such points cannot exist for any dust solution, since the particles which constitute a dust cylinder must travel along timelike circular geodesics.

At points like $r = s$, $K = 0$ and $G + T = 0$ regardless of rotation gauge, and the values of $\bar{G}$ and $G - T$ are also invariant under a gauge change, so that the only freedom in $\omega(a)$ at this point is in $\bar{K}$. In fact, by applying a boost in the σ_ϕ plane characterized by a parameter which is finite everywhere, but whose derivative diverges at $r = s$, $\bar{K}$ can be made to diverge without altering the values of any of the other functions at that point. The velocity Σ in fact vanishes as well, so that the cylinder is in some sense not rotating at that point. So points like $r = s$ would display some curious properties if some (non-dust) solutions which contained them could be found.

So, apart from points like $r = s$, any system which contained a region in which $G + T$ changed sign (which would necessarily imply a point at which $G+T$ vanished) would also contain a region in which the constraint is violated (in any gauge). Now, the axis conditions (2.12) require that $G \to 1/r$ and $T \to 0$ as $r \to 0$, so that $G + T$ is clearly positive in some neighborhood of the z-axis. It follows that $G + T > 0$ everywhere. It also follows from the constraint equation that

$$2K + (G + T) > 0 \qquad (2.171)$$

and

$$2K - (G + T) < 0. \tag{2.172}$$

So, if $\eta = 1$ then $\partial_r \eta > 0$ and if $\eta = -1$ then $\partial_r \eta < 0$. Thus the radial streamlines of η tend to take it out of the causality violating range. Namely, if $|\eta| > 1$ for some value of r, then $|\eta| > 1$ for any larger value of r. To pin down η, we again appeal to the axis conditions. Generally, we expect f_1 to tend towards some finite nonzero value on the axis which determines the scale of the time coordinate, and well-definedness of $\overline{h}(a)$ requires that $f_2 \to 0$. Thus η diverges as we approach the axis, so that

$$|\eta| > 1 \tag{2.173}$$

must hold in some neighborhood of $r = 0$. Thus (2.173) must hold for all r and hence no CTL can exist for a dust solution obeying the constraint equation (2.154). Recall that the metric components are rotation-gauge invariant, so that like the constraint equation, (2.173) is a gauge invariant inequality.

For non-dust solutions, equation (2.167) and the strong energy condtion imply that a solution can have at most one point like $r = s$, and that $G + T$ is positive for $r < s$ and negative for $r > s$. For $r > s$, we see that CTL are possible, but by no means necessary. Since $G + T$ is negative in this region, the streamlines of η at $\eta = \pm 1$ tend to take η *into* the causality violating range with increasing r. So, if CTL exist, then they exist for all values of r greater than some $r_{min} > s$, right out to infinity. Thus, the long range properties of such solutions are pathological, and one does not expect such behaviour from any realistic cylinder of finite length. In this light, the CTL which can arise under the constraint (2.154) are not quite so worrisome.

2.9 Conclusions

Let us now take stock of what we have shown here. Analysis of the Weyl tensor has led us to a definition of an intrinsic velocity Σ, and this definition was justified by further analysis of counterrotating cylinders. The condition that this velocity be subluminal led us to a constraint equation, a gauge invariant inequality which the rotation gauge field must satisfy. Under this constraint, *the fields of any stationary cylindrically symmetric system may always be brought into a locally static form by appropriate choice of rotation gauge.* This suggests that the metric diagonalization process of Som *et al.* [15]

should always be possible in the context of General Relativity. However, we reject this diagonalization process in gauge theory as it cannot be interpreted globally without a nontrivial change in topology (except in cases in which the solution truly is globally static.)

This constraint also restricts our attention to solutions for which $\mathcal{W}(B)$ can be brought into diagonal form. With this additional restriction, stationary cylindrically symmetric systems consisting only of dust cannot contain closed timelike loops. Whether or not closed timelike loops can be excluded rigorously for other forms of matter is unclear, and depends on the nature and consequences of points like $r = s$ in the previous section. The character of such points seems closer to that of a static Schwarzschild-type solution, in which photons will orbit in circles at $r = 3M$, than to a solution which would contort spacetime in such a way as to produce CTL. Moreover, Σ vanishes at such points, and the existence of closed timelike loops seems tied to rotational aspects of the solutions. A solution which is non-rotating everywhere ($K = \bar{K} = 0$), for example, could conceivably sustain points like $r = s$, but will also necessarily have $f_2 = 0$, so that no CTL appear. In any case, the CTL that could appear in general non-dust solutions are not representative of the behaviour one expects from a finite cylinder, the fields for which one expects to asymptotically approach those of the Kerr solution at large radii.

In [8], somewhat parallel results were obtained in the context of spherically symmetric systems. Birkhoff's theorem states that a spherically symmetric vacuum solution is necessarily static. This issue is not under contention; however, the gauge theory approach denies that one can always bring the line element of such a solution into diagonal (Schwarzschild) form. The fields outside of a stationary cylindrically symmetric matter configuration are necessarily locally static, but can have global properties which distinguish them from a fully static configuration.

Horský and Štefaník [18] have claimed that since the Schwarzschild solution after a transformation of the form (2.135) (where ϕ is to be interpreted as a spherical coordinate) does not produce the Kerr solution, then similarly, the van Stockum vacuum solution does not truly describe the field around a rotating cylinder, since it may be obtained from the solution for a static line mass by a similar transformation. This leads them to conclude that the van Stockum exterior is only appropriate in the regime $\frac{1}{2} < aR < 1$, where the transformation cannot be carried out. Here we have come to precisely the opposite conclusion, that $aR < \frac{1}{2}$ is the only physically sensible solution, but for quite different reasons. The fields about a rotating cylinder do not exhibit the local frame dragging effects seen about axisymmetric sources, but they are not simply static, and there is no reason to think that such solutions are not correct. Gauge theory gravity comfortably accommodates the Kerr solution, but does not admit the Schwarzschild line element in the presence of horizons, as such a solution cannot be made global. Rather, spherically symmetric solutions with horizons are best described in a 'Newtonian' gauge in which the equations describing the fields are quite simple, but in which the line element is not in static form. Similarly, rotating cylindrically symmetric systems are generally best treated using non-diagonal line elements. Because gauge theory gravity works with quantities that are more fundamental than the metric, the constraints which it imposes are often tighter than those of General Relativity. However, the physical meaning of these quantities and constraints is generally much more clear.

A potential qualification to the above analysis of Mach's Principle is that it began by examining the properties of timelike geodesics which represent circular orbits, operating under the assumption that such orbits do, in fact, exist. For nonsingular matter fields, we certainly expect that they would at least exist in some neighborhood of the z-axis, but we cannot expect them to occur throughout all space for any given matter configuration. Nevertheless, the definition of Σ is not necessarily tied to timelike geodesics, and may be justified in terms of the diagonalization of the stress-energy and Weyl tensors, or by an analysis of counterrotating shells.

Chapter 3

Time-dependent Systems

In this chapter gauge theory gravity is extended to cylindrically symmetric systems with explicit time dependence. It has long been known that the field equations of General Relativity allow exact cylindrical wave solutions. The radiative properties of such solutions were studied in detail by Marder [19]. It is hoped that gauge theory gravity will ultimately yield further insights into the physical nature of these solutions.

3.1 Stationary Vacuum Solutions

Before continuing with a derivation of the time-dependent vacuum equations, it is instructive to examine the stationary vacuum solutions in a more general fashion than in Chapter 2, where the focus was simply on obtaining a solution for the exterior of a van Stockum cylinder. Our later discussion suggested that it should always be possible to diagonalize the Weyl tensor for physical systems by a suitable choice of rotation gauge, so we shall examine the general types of solution that can arise in this gauge. In the vacuum, the Riemann and Weyl tensors coincide, so all the necessary information may be read off from (2.20). The off-diagonal term $\bar{G}K$ must vanish, and since $\bar{G} \neq 0$ generally for (3+1)-dimensional systems, we must have $K = 0$. The other off-diagonal term $\beta = \bar{K}(G + T)$ must also vanish. A generating matter configuration typically will not have $G + T = 0$, and since discontinuities in $\bar{K}$ do not seem to cause problems in the stationary setup, it is most reasonable to take

$$K = 0 = \bar{K}. \tag{3.1}$$

Considering (2.29) and (2.30), we see that the vacuum in the Weyl-diagonal gauge is rigidly non-rotating in the sense that a set of observers with covariant velocity e_t experience no vorticity or shear. Since the Weyl tensor is trace free, it follows that

$$GT - \bar{G}(G - T) = 0, \tag{3.2}$$

and since it is self-dual,

$$\begin{aligned} L_r G + G^2 + \bar{G}T &= 0 & (3.3)\\ -L_r T + T^2 - \bar{G}G &= 0 & (3.4)\\ L_r \bar{G} + \bar{G}^2 + GT &= 0. & (3.5) \end{aligned}$$

The above three equations may be rearranged as

$$\begin{aligned} L_r(G + \bar{G} - T) + (G + \bar{G} - T)^2 &= 0 & (3.6)\\ L_r(G + T) + (G + T)(G + \bar{G} - T) &= 0 & (3.7)\\ L_r(G - T) + (G - T)(G + \bar{G} - T) &= 0. & (3.8) \end{aligned}$$

In the $g_1 = 1$ gauge, these are easily solved to give

$$\begin{aligned} G &= \frac{1}{2}(a + b)(r + r_o)^{-1} & (3.9)\\ T &= \frac{1}{2}(a - b)(r + r_o)^{-1} & (3.10)\\ \bar{G} &= (1 - b)(r + r_o)^{-1}, & (3.11) \end{aligned}$$

where a, b, and r_o are integration constants which must match onto the generating matter fields and satisfy the trace-free condition (3.2)

$$a^2 + 3b^2 - 4b = 0. \tag{3.12}$$

Note that for $b = 1$, the two solutions $a = \pm 1$ yield the two vacuum sectors for (2+1)-dimensional systems described in [9], where $\bar{G} = 0$ and either T or G vanishes as well. Note that we do not see the three sectors of the van Stockum solution here, since the critical and high-mass regimes are excluded by demanding that $\mathcal{W}(B)$ be diagonalised. The bracket relations (2.15)-(2.19) are also quite easily solved to give

$$\begin{aligned} f_2, h_1 &\propto (r + r_o)^{-\frac{1}{2}(a+b)} & (3.13)\\ f_1, h_2 &\propto (r + r_o)^{\frac{1}{2}(a-b)} & (3.14)\\ i_i &\propto (r + r_o)^{b-1}, & (3.15) \end{aligned}$$

where the constants of proportionality are obtained from matching onto the generating fields. The stationary cylindrically symmetric vacuum is remarkably simple, indeed, almost trivial, in this gauge. Unfortunately, reducing the symmetry by allowing general time dependence leads to a substantially more complicated framework.

3.2 Time-dependent Systems: Basic Formalism

Cylindrically symmetric systems with general time dependence no longer have symmetry under the simultaneous reversal of t and ϕ. The $\overline{h}$-function for such systems can generally have coupling between the t, r, and ϕ directions, and is of the form

$$\begin{aligned}\overline{h}(e^t) &= f_1 e^t + r f_2 e^\phi + f_3 e^r \\ \overline{h}(e^r) &= g_2 e^t + r g_3 e^\phi + g_1 e^r \\ \overline{h}(e^\phi) &= h_2 e^t + r h_1 e^\phi + h_3 e^r \\ \overline{h}(e^z) &= i_1 e^z,\end{aligned} \tag{3.16}$$

where f_1, f_2 *etc.*, are taken to be functions of both t and r. The form (3.16) retains three degrees of rotation-gauge freedom, being invariant under boosts in the σ_ϕ and σ_r planes, as well as rotations in the $i\sigma_3$ plane. The form of $\omega(a)$ implied by this $\overline{h}$-function is

$$\begin{aligned}\omega(e_t) &= -T\sigma_r + A\sigma_\phi + (K + h_2)i\sigma_3 \\ \omega(\hat{\phi}) &= M\sigma_r + B\sigma_\phi + (h_1 - G)i\sigma_3 \\ \omega(e_r) &= C\sigma_r + \bar{K}\sigma_\phi + (D + h_3)i\sigma_3 \\ \omega(e_z) &= E\sigma_3 + Fi\sigma_r + \bar{G}i\sigma_\phi,\end{aligned} \tag{3.17}$$

where again, A, B, *etc.*, are functions of t and r.

The L-derivatives are given in terms of the $\overline{h}$-function as

$$\begin{aligned} L_t &= f_1\partial_t + g_2\partial_r + h_2\partial_\phi \\ L_r &= f_3\partial_t + g_1\partial_r + h_3\partial_\phi \\ L_\phi &= f_2\partial_t + g_3\partial_r + h_1\partial_\phi \\ L_z &= i_1\partial_z. \end{aligned} \tag{3.18}$$

Note that when acting on functions with no ϕ-dependence, the derivatives L_t, L_r, and L_ϕ satisfy the relation

$$L_\phi = m(t,r)L_t + n(t,r)L_r, \tag{3.19}$$

where

$$m(t,r) \equiv \frac{f_2 g_1 - f_3 g_3}{f_1 g_1 - f_3 g_2} \tag{3.20}$$

and

$$n(t,r) \equiv \frac{f_1 g_3 - f_2 g_2}{f_1 g_1 - f_3 g_2}. \tag{3.21}$$

This linear relation amongst the derivatives will prove crucial in obtaining a complete set of field equations. The bracket structure implied by equations (1.44) and (1.31) is given by

$$\begin{aligned} [L_r, L_t] &= TL_t + CL_r + (K+\bar{K})L_\phi \\ [L_r, L_\phi] &= (\bar{K}-M)L_t + DL_r - GL_\phi \\ [L_t, L_\phi] &= AL_t + (K-M)L_r - BL_\phi \\ [L_r, L_z] &= -\bar{G}L_z \\ [L_t, L_z] &= -EL_z \\ [L_\phi, L_z] &= FL_z. \end{aligned} \tag{3.22}$$

Following a similar procedure to that used to derive equations (2.15)-(2.19) we may write

$$L_r f_1 - L_t f_3 = Tf_1 + (K+\bar{K})f_2 + Cf_3 \tag{3.23}$$
$$L_r g_2 - L_t g_1 = Tg_2 + (K+\bar{K})g_3 + Cg_1 \tag{3.24}$$

$$
\begin{aligned}
L_r h_2 - L_t h_3 &= Th_2 + (K + \bar{K})h_1 + Ch_3 && (3.25)\\
L_r f_2 - L_\phi f_3 &= -Gf_2 + (\bar{K} - M)f_1 + Df_3 && (3.26)\\
L_r g_3 - L_\phi g_1 &= -Gg_3 + (\bar{K} - M)g_2 + Dg_1 && (3.27)\\
L_r h_1 - L_\phi h_3 &= -Gh_1 + (\bar{K} - M)h_2 + Dh_3 && (3.28)\\
L_t f_2 - L_\phi f_1 &= Af_1 + (K - M)f_3 - Bf_2 && (3.29)\\
L_t g_3 - L_\phi g_2 &= Ag_2 + (K - M)g_1 - Bg_3 && (3.30)\\
L_t h_1 - L_\phi h_2 &= Ah_2 + (K - M)h_3 - Bh_1 && (3.31)\\
L_t i_1 &= -Ei_1 && (3.32)\\
L_r i_1 &= -\bar{G}i_1 && (3.33)\\
L_\phi i_1 &= Fi_1. && (3.34)
\end{aligned}
$$

The components of the Riemann tensor are of the form

$$
\begin{aligned}
\mathcal{R}(\sigma_r) &= \alpha_1\sigma_r + \beta_1 i\sigma_3 + \gamma_1\sigma_\phi && (3.35)\\
\mathcal{R}(\sigma_\phi) &= \alpha_2\sigma_r + \beta_2 i\sigma_3 + \gamma_2\sigma_\phi && (3.36)\\
\mathcal{R}(i\sigma_3) &= \alpha_3\sigma_r + \beta_3 i\sigma_3 + \gamma_3\sigma_\phi && (3.37)\\
\mathcal{R}(i\sigma_r) &= \alpha_4 i\sigma_r + \beta_4\sigma_3 + \gamma_4 i\sigma_\phi && (3.38)\\
\mathcal{R}(i\sigma_\phi) &= \alpha_5 i\sigma_r + \beta_5\sigma_3 + \gamma_5 i\sigma_\phi && (3.39)\\
\mathcal{R}(\sigma_3) &= \alpha_6 i\sigma_r + \beta_6\sigma_3 + \gamma_6 i\sigma_\phi, && (3.40)
\end{aligned}
$$

where the "greek variables" α_1, α_2, *etc.*, are defined explicitly in terms of

$\omega(a)$ as

$$\begin{array}{rcl}
\alpha_1 & = & -L_rT - L_tC + T^2 - K(M+\bar{K}) - C^2 - \bar{K}M + AD \\
\alpha_2 & = & -L_tM - L_\phi T - A(G+T) + C(K-M) - B(K+M) \\
\alpha_3 & = & -L_rM + L_\phi C - G(M+\bar{K}) + T(M-\bar{K}) - D(B-C) \\
\alpha_4 & = & -L_\phi F + F^2 - EB + G\bar{G} \\
\alpha_5 & = & L_rF + \bar{G}(D+F) + E\bar{K} \\
\alpha_6 & = & -L_tF - E(A+F) - \bar{G}K \\
\beta_1 & = & L_rK - L_tD + G(K+\bar{K}) - T(K-\bar{K}) + C(A-D) \\
\beta_2 & = & L_tG - L_\phi K + D(K-M) + B(G+T) + A(K+M) \\
\beta_3 & = & L_rG + L_\phi D + G^2 + D^2 - K(M-\bar{K}) + \bar{K}M - BC \\
\beta_4 & = & -L_\phi E + F(E-B) + \bar{G}M \\
\beta_5 & = & L_rE - \bar{G}(C-E) + F\bar{K} \\
\beta_6 & = & -L_tE - E^2 - AF - \bar{G}T \\
\gamma_1 & = & L_rA - L_t\bar{K} - B(K+\bar{K}) + C(K-\bar{K}) - T(A-D) \\
\gamma_2 & = & L_\phi A - L_tB + A^2 - B^2 + KM - GT + \bar{K}(K-M) \\
\gamma_3 & = & -L_rB + L_\phi\bar{K} + D(M+\bar{K}) - A(M-\bar{K}) - G(B-C) \\
\gamma_4 & = & -L_\phi\bar{G} - F(G-\bar{G}) + EM \\
\gamma_5 & = & L_r\bar{G} + \bar{G}^2 - EC - DF \\
\gamma_6 & = & -L_t\bar{G} - E(\bar{G}+T) + FK.
\end{array} \tag{3.41}$$

We may write the Einstein tensor in terms of these quantities as

$$\begin{array}{rcl}
\mathcal{G}(e_t) & = & -(\alpha_4+\beta_3+\gamma_5)e_t + (\beta_2-\gamma_6)e_r + (\alpha_6-\beta_1)\hat{\phi} \\
\mathcal{G}(e_r) & = & (\gamma_3-\beta_5)e_t - (\alpha_4+\beta_6+\gamma_2)e_r + (\gamma_1-\alpha_5)\hat{\phi} \\
\mathcal{G}(\hat{\phi}) & = & (\beta_4-\alpha_3)e_t + (\alpha_2-\gamma_4)e_r - (\alpha_1+\beta_6+\gamma_5)\hat{\phi} \\
\mathcal{G}(e_z) & = & -(\alpha_1+\beta_3+\gamma_2)e_z.
\end{array} \tag{3.42}$$

Defining the quantity

$$\delta \equiv \frac{4\pi}{3}\rho \tag{3.43}$$

where ρ is the density of the (dust) cylinder in the e_t frame, we may write the Weyl tensor as

$$\mathcal{W}(\sigma_r) = (\alpha_1-\delta)\sigma_r + \beta_1 i\sigma_3 + \gamma_1\sigma_\phi \tag{3.44}$$

$$\mathcal{W}(\sigma_\phi) = \alpha_2\sigma_r + \beta_2 i\sigma_3 + (\gamma_2-\delta)\sigma_\phi \tag{3.45}$$

$$\mathcal{W}(i\sigma_3) = \alpha_3\sigma_r + (\beta_3+2\delta)i\sigma_3 + \gamma_3\sigma_\phi \tag{3.46}$$

$$\mathcal{W}(i\sigma_r) = (\alpha_4 + 2\delta) i\sigma_r + \beta_4 \sigma_3 + \gamma_4 i\sigma_\phi \tag{3.47}$$
$$\mathcal{W}(i\sigma_\phi) = \alpha_5 i\sigma_r + \beta_5 \sigma_3 + (\gamma_5 + 2\delta) i\sigma_\phi \tag{3.48}$$
$$\mathcal{W}(\sigma_3) = \alpha_6 i\sigma_r + (\beta_6 - \delta)\sigma_3 + \gamma_6 i\sigma_\phi. \tag{3.49}$$

The self-duality of the Weyl tensor implies the equations

$$\alpha_1 - \alpha_4 = \beta_6 - \beta_3 = \gamma_2 - \gamma_5 = 3\delta \tag{3.50}$$

and

$$\begin{array}{lll} \alpha_2 = \alpha_5 & \beta_2 = -\beta_5 & \gamma_1 = \gamma_4 \\ \alpha_3 = -\alpha_6 & \beta_1 = -\beta_4 & \gamma_3 = -\gamma_6. \end{array} \tag{3.51}$$

That the Weyl tensor is symmetric implies that

$$\alpha_2 = \gamma_1 \qquad \alpha_3 = -\beta_1 \qquad \gamma_3 = -\beta_2 \tag{3.52}$$

and that it is trace-free gives

$$\alpha_1 + \gamma_2 + \beta_6 = 3\delta. \tag{3.53}$$

3.3 Time-dependent Vacuum Solutions

In a vacuum, $\delta = 0$, so that $\alpha_1 = \alpha_4$, $\beta_3 = \beta_6$, $\gamma_2 = \gamma_5$, and $\alpha_1 + \gamma_2 + \beta_6 = 0$. The equations resulting from the vanishing of the Einstein tensor are

$$\begin{array}{ll} \beta_2 = \gamma_6 & \alpha_5 = \gamma_1 \\ \beta_1 = \alpha_6 & \alpha_3 = \beta_4 \\ \beta_5 = \gamma_3 & \alpha_2 = \gamma_4. \end{array} \tag{3.54}$$

The Bianchi identities are automatically satisfied by fields satisfying the above equations, and thus yield no new information. To make further progress, we employ our rotation-gauge freedom to diagonalize the Weyl tensor. As stated earlier, experience with stationary systems suggests that this should always be possible for physically reasonable systems. Moreover, because of the symmetries of the Weyl tensor, diagonalisation requires elimination of three degrees of freedom, and there are three degrees of rotation-gauge freedom available. So, assuming that diagonalisation can be achieved, we have

$$\begin{aligned} \alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = 0 \\ \beta_1 = \beta_2 = \beta_4 = \beta_5 = 0 \\ \gamma_1 = \gamma_3 = \gamma_4 = \gamma_6 = 0. \end{aligned} \tag{3.55}$$

The bracket structure may now be exploited to extract new information from the expressions (3.41). For example, we may write

$$\begin{aligned} -L_r\alpha_6 - L_t\alpha_5 &= 0 \\ L_r\beta_4 + L_\phi\beta_5 &= 0 \\ L_t\gamma_4 - L_\phi\gamma_6 &= 0. \end{aligned} \tag{3.56}$$

Expanding each term out explicitly, applying the known bracket relations (3.22) and then substituting the expressions for the greek variables back in, we obtain

$$\begin{aligned} (\beta_6 - \alpha_4)\bar{K} + (\gamma_5 - \alpha_4)K &= 0 \\ (\beta_6 - \alpha_4)\bar{K} + (\gamma_5 - \beta_6)M &= 0 \\ (\alpha_4 - \gamma_5)K + (\gamma_5 - \beta_6)M &= 0. \end{aligned} \tag{3.57}$$

The third of these equations is an algebraic consequence of the first two, but upon combining the first two equations with the trace-free condition (3.53), we obtain the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ -(K+\bar{K}) & K & \bar{K} \\ -\bar{K} & M & \bar{K} - M \end{pmatrix} \begin{pmatrix} \alpha_4 \\ \gamma_5 \\ \beta_6 \end{pmatrix} = 0 \tag{3.58}$$

For this equation to have a non-trivial solution, the matrix must have a vanishing determinant, which implies the simple equation

$$MK + M\bar{K} - K\bar{K} = 0. \tag{3.59}$$

This equation may be seen as the time-dependent analog of (3.1).

A similar technique may be applied to the non-vanishing greek variables by constructing equations like

$$L_\phi \alpha_1 = L_\phi \alpha_1 - L_r \alpha_2 + L_t \alpha_3 \tag{3.60}$$

and

$$L_r \alpha_4 = L_r \alpha_4 + L_\phi \alpha_5 \tag{3.61}$$

and applying the bracket structure to the right-hand sides. At this point the subscripts on the nonvanishing greek variables may be dropped, writing

$$\begin{aligned} \alpha &\equiv \alpha_1 = \alpha_4 \\ \beta &\equiv \beta_3 = \beta_6 \\ \gamma &\equiv \gamma_2 = \gamma_5. \end{aligned} \tag{3.62}$$

This procedure yields the remarkable set of equations

$$L_t \alpha = B(\beta - \alpha) + E(\gamma - \alpha) \tag{3.63}$$
$$L_r \alpha = \bar{G}(\beta - \alpha) + G(\gamma - \alpha) \tag{3.64}$$
$$L_\phi \alpha = A(\beta - \alpha) + D(\gamma - \alpha) \tag{3.65}$$
$$L_t \beta = B(\alpha - \beta) + C(\gamma - \beta) \tag{3.66}$$
$$L_r \beta = \bar{G}(\alpha - \beta) - T(\gamma - \beta) \tag{3.67}$$
$$L_\phi \beta = A(\alpha - \beta) - F(\gamma - \beta) \tag{3.68}$$
$$L_t \gamma = E(\alpha - \gamma) + C(\beta - \gamma) \tag{3.69}$$
$$L_r \gamma = G(\alpha - \gamma) - T(\beta - \gamma) \tag{3.70}$$
$$L_\phi \gamma = D(\alpha - \gamma) - F(\beta - \gamma). \tag{3.71}$$

It is in virtue of these equations that the Bianchi identities are satisfied. These equations exhibit a "triality" symmetry whereby the set is invariant under simultaneous cyclic substitutions

$$\begin{array}{ccc} \alpha & \longrightarrow & \beta \\ \nwarrow & & \swarrow \\ & \gamma & \end{array} \qquad \begin{array}{ccc} B & \longrightarrow & C \\ \nwarrow & & \swarrow \\ & E & \end{array} \qquad \begin{array}{ccc} \bar{G} & \longrightarrow & -T \\ \nwarrow & & \swarrow \\ & G & \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & -F \\ \nwarrow & & \swarrow \\ & D & \end{array} \quad . \tag{3.72}$$

We may proceed further still, and apply the bracket structure to the triality equations (3.63)-(3.71). Applying the three commutators $[L_r, L_t]$, $[L_r, L_\phi]$, and $[L_t, L_\phi]$ to α, β, and γ in turn yields the nine new "second order" bracket equations

$$\begin{aligned} &\alpha[F(K+\bar{K}) - L_\phi(K+\bar{K})] \\ +\ &\beta[L_\phi\bar{K} - A(K+M) + D(\bar{K}+M) - FK] \\ +\ &\gamma[L_\phi K + A(K+M) - D(\bar{K}+M) - F\bar{K}] = 0 \end{aligned} \tag{3.73}$$

$$\begin{aligned} &\alpha[L_\phi(G+\bar{G}) - L_r(A+D) + (\bar{K}-M)(B+E) - AG + D\bar{G}] \\ +\ &\beta[L_rA - L_\phi\bar{G} + F(G-\bar{G}) - B(\bar{K}-M) + A(2G+T) - D(2\bar{G}+T)] \\ +\ &\gamma[L_rD - L_\phi G - F(G-\bar{G}) - E(\bar{K}-M) - A(G+T) + D(\bar{G}+T)] = 0 \end{aligned} \tag{3.74}$$

$$\begin{aligned} &\alpha[L_\phi(B+E) - L_t(A+D) + (K-M)(G+\bar{G}) + AE - DB] \\ +\ &\beta[L_tA - L_\phi B + F(E-B) - D(B-C) + A(E-C) - \bar{G}(K-M)] \\ +\ &\gamma[L_tD - L\phi E - F(E-B) + D(2B-C) - A(2E-C) - G(K-M)] = 0 \end{aligned} \tag{3.75}$$

$$\begin{aligned} &\alpha[L_rB - L_t\bar{G} - A(K+\bar{K}) - B(2T+G) - C(2\bar{G}-G) + E(\bar{G}+T)] \\ +\ &\beta[L_t(\bar{G}-T) - L_r(B+C) + (K+\bar{K})(A-F) + BT + C\bar{G}] \\ +\ &\gamma[L_rC + L_tT + F(K+\bar{K}) + B(G+T) - C(G-\bar{G}) - E(\bar{G}+T)] = 0 \end{aligned} \tag{3.76}$$

$$\begin{aligned} &\alpha[L_t\bar{K} + B(M+K) + C(\bar{K}-K) - EM] \\ +\ &\beta[-L_t\bar{K} - L_\phi T - B(M+K) - C(M-K) - A(G+T) + E(M-\bar{K})] \\ +\ &\gamma[L_\phi T + A(G+T) + C(M-\bar{K}) + E\bar{K}] = 0 \end{aligned} \tag{3.77}$$

$$\begin{aligned} &\alpha[L_tA - L_\phi B + \bar{G}(M-K) + D(B-C) - E(A+F) + AC + BF] \\ +\ &\beta[L_\phi(B+C) - L_t(A-F) + (T-\bar{G})(M-K) + BF + AC] \\ +\ &\gamma[-L_tF - L_\phi C - T(M-K) - D(B-C) + E(A+F) - 2AC - 2BF] = 0 \end{aligned} \tag{3.78}$$

$$\begin{aligned} &\alpha[L_rE - L_tG - D(K+\bar{K}) - E(\bar{G}+2T) - C(2G-\bar{G}) + B(G+T)] \\ +\ &\beta[L_tT + L_rC + F(K+\bar{K}) + E(\bar{G}+T) + C(G-\bar{G}) - B(G+T)] \\ +\ &\gamma[L_t(G-T) - L_r(C+E) + (D-F)(K+\bar{K}) + GC + ET] = 0 \end{aligned} \tag{3.79}$$

$$\begin{aligned} & \alpha[L_r D - L_\phi G - E(\bar{K} - M) - D(\bar{G} + T) - F(\bar{G} - G) + A(G + T)] \\ + \; & \beta[L_\phi T - L_r F - C(\bar{K} - M) + D(2T + \bar{G}) - F(2G - \bar{G}) - A(G + T)] \\ + \; & \gamma[L_\phi (G - T) - L_r (D - F) + (C + E)(\bar{K} - M) + FG - DT] = 0 \end{aligned} \tag{3.80}$$

$$\begin{aligned} & \alpha[L_t D + G(M - K) - C(A - D) - \bar{G}M] \\ + \; & \beta[-L_\phi C - T(M - K) + D(B - C) + \bar{G}K] \\ + \; & \gamma[L_\phi C - L_t D - (G - \bar{G} - T)(M - K) + AC - BD] = 0. \end{aligned} \tag{3.81}$$

Although these equations are themselves somewhat unwieldy, they exhibit an interesting and useful structure. Suppose α, β, and γ are thought of as a vector $\mathbf{G}$ in an abstract three-dimensional space spanned by the orthonormal basis $\{\sigma_x, \sigma_y, \sigma_z\}$ with $\sigma_i \cdot \sigma_j = \delta_{ij}$, as in

$$\mathbf{G} \equiv \alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z. \tag{3.82}$$

The above nine equations may then be written as

$$\mathbf{X}_i \cdot \mathbf{G} = 0, \quad i = 1, \ldots, 9 \tag{3.83}$$

where the $\mathbf{X}_i \equiv x_i\sigma_x + y_i\sigma_y + z_i\sigma_z$ are the vectors of the coefficients on the greek variables. Now, algebraically, $x_i + y_i + z_i = 0$ for all i, since for each equation, one coefficient is opposite in sign to the sum of the other two. This may be written compactly in vector form as

$$\mathbf{n} \cdot \mathbf{X}_i = 0, \tag{3.84}$$

where $\mathbf{n} \equiv \sigma_x + \sigma_y + \sigma_z$. Similarly, the trace-free condition $\alpha + \beta + \gamma = 0$ may be written as

$$\mathbf{n} \cdot \mathbf{G} = 0. \tag{3.85}$$

Together, (3.83), (3.84), and (3.85) imply that the vectors $\mathbf{n}$, $\mathbf{G}$, and any one of the $\mathbf{X}_i$ form an orthogonal set. In particular, the $\mathbf{X}_i$ are all in the same direction, being orthogonal to the plane defined by the bivector $\mathbf{n} \wedge \mathbf{G}$. Thus, for some constants k_i,

$$\mathbf{X}_i = k_i \, \mathbf{I}\mathbf{n} \wedge \mathbf{G} = k_i[(\beta - \gamma)\sigma_x + (\gamma - \alpha)\sigma_y + (\alpha - \beta)\sigma_z], \tag{3.86}$$

where $\mathbf{I} \equiv \sigma_x\sigma_y\sigma_z$ is the pseudoscalar of the abstract space. Thus, if the k_i are known, the nine second order equations will in general break up into 18 smaller equations. It is unclear how one might continue to apply the bracket structure to yield further equations.

3.4 A Physical Ansätz

In order to obtain a fully determined set of equations, it is necessary to make some simplifications to the full time-dependent scheme developed above. These simplifications are generally speaking *not* gauge choices and have physical significance. The approach taken here is to try to stick to the stationary scheme as much as possible. Consider the expressions for the vorticity and shear for a covariant velocity e_t in the general time-dependent setup,

$$\varpi = -(M - \bar{K}) i\sigma_3 \tag{3.87}$$

and

$$\begin{aligned} \sigma(e_r) &= \tfrac{1}{2}(M + \bar{K})\hat{\phi} + \tfrac{1}{3}(2C - B - E)e_r \\ \sigma(\hat{\phi}) &= \tfrac{1}{2}(M + \bar{K})e_r + \tfrac{1}{3}(2B - C - E)\hat{\phi} \\ \sigma(e_z) &= \tfrac{1}{3}(2E - B - C)e_z \\ \sigma(e_t) &= 0. \end{aligned} \tag{3.88}$$

Now, an easy way to solve (3.59) is to set two of the three variables M, K, and $\bar{K}$ to zero. So, in keeping with the observation that the stationary Weyl-diagonal vacuum is shear and vorticity free, the natural choice is to take

$$M = 0 = \bar{K}. \tag{3.89}$$

Since the set of equations (3.57) can then only be solved if $K(\gamma - \alpha) = 0$, this choice leads to two possible cases, when $\alpha = \gamma$ and when $K = 0$. In the first case, the trace free condition implies

$$\beta = -2\alpha. \tag{3.90}$$

It is worth noting that in this case, with two of the eigenvalues of the Weyl tensor degenerate, we recover the rotation-gauge freedom to boost in the $i\sigma_3$ plane. Substituting these relations into the "triality" equations (3.63)-(3.71) then yields

$$B = C \qquad \bar{G} = -T \qquad A = -F \tag{3.91}$$

and

$$L_t\alpha = -3B\alpha \qquad L_r\alpha = -3\bar{G}\alpha \qquad L_\phi\alpha = -3A\alpha. \tag{3.92}$$

The greek variable equations (3.55) also imply that

$$L_\phi B = 0, \;\; L_r B = 0 \tag{3.93}$$

so that B (and thus C) is a constant. The nonvanishing components along the diagonal of the Weyl tensor have the simplified forms

$$\begin{aligned}
\alpha &= L_r\bar{G} + \bar{G}^2 - B^2 + AD = L_\phi A + A^2 - EB + G\bar{G} && (3.94)\\
\gamma &= L_\phi A + A^2 - B^2 + G\bar{G} = L_r\bar{G} + \bar{G}^2 - EB + AD && (3.95)\\
\beta &= L_r G + G^2 + L_\phi D + D^2 - B^2 = -L_t E - E^2 + A^2 + \bar{G}^2. && (3.96)
\end{aligned}$$

Subtracting the first two of these equations gives

$$B^2 - EB = 0 \tag{3.97}$$

which implies that either $B = 0$ or $B = E$. In the latter case, (3.96) yields

$$-2\alpha = -B^2 + A^2 + \bar{G}^2, \tag{3.98}$$

and applying L_t to this equation gives

$$\begin{aligned}
L_t\alpha &= -(AL_tA + \bar{G}L_t\bar{G})\\
&= (A\bar{G}K - \bar{G}AK)\\
&= 0 = -3B\alpha
\end{aligned} \tag{3.99}$$

so that once again $B = 0$. The fact that B vanishes has the remarkable consequence of reducing the entire $\alpha = \gamma$ case to a stationary configuration, for when $B = 0$, the vector

$$V = \alpha^{-\frac{1}{3}} e_t \tag{3.100}$$

satisfies Killing's equation

$$a \cdot (b \cdot \mathcal{D}V) + b \cdot (a \cdot \mathcal{D}V) = 0. \tag{3.101}$$

The existence of a timelike Killing vector implies that the fields must be stationary.

Turning to the second case, in which $M = K = \bar{K} = 0$, the vanishing greek variable equations give the set of compact equations

$$\begin{array}{lll}
L_\phi T = -A(G+T) & L_t D = C(A-D) & L_r A = T(A-D)\\
L_\phi C = D(B-C) & L_t G = -B(G+T) & L_r B = -G(B-C)\\
L_\phi E = F(E-B) & L_t F = -E(A+F) & L_r E = \bar{G}(C-E)\\
L_\phi \bar{G} = -F(G-\bar{G}) & L_t\bar{G} = -E(\bar{G}+T) & L_r F = -\bar{G}(D+F).
\end{array} \tag{3.102}$$

The nonvanishing greek variables are

$$
\begin{aligned}
\alpha &= -L_r T - L_t C + T^2 - C^2 + AD = -L_\phi F + F^2 - EB + G\bar{G} \\
\beta &= L_r G + L_\phi D + G^2 + D^2 - BC = -L_t E - E^2 - AF - \bar{G}T \\
\gamma &= L_\phi A - L_t B + A^2 - B^2 - GT = L_r \bar{G} + \bar{G}^2 - EC - DF.
\end{aligned} \tag{3.103}
$$

We would now like to find quadratic forms for α, β, and γ which do not involve L-derivatives of any functions. This is done by exploiting the linear relation (3.19) amongst L_t, L_r, and L_ϕ. If all three derivatives of three different functions (of only t and r) ψ_1, ψ_2, and ψ_3 are known, then we may write the matrix equation

$$
\begin{pmatrix} L_t\psi_1 & L_r\psi_1 & L_\phi\psi_1 \\ L_t\psi_2 & L_r\psi_2 & L_\phi\psi_2 \\ L_t\psi_3 & L_r\psi_3 & L_\phi\psi_3 \end{pmatrix} \begin{pmatrix} -m \\ -n \\ 1 \end{pmatrix} = 0. \tag{3.104}
$$

Thus, the matrix of derivatives in the above equation must have vanishing determinant for any three such functions ψ_i. This fact can yield useful algebraic relations amongst the functions which do not involve any derivatives. The three derivatives of i_1 are known from (3.32), (3.33), and (3.34), so they can be used as a row in a matrix. The derivatives of i_1 are also useful as integrating factors, since we may write

$$
\begin{array}{lll}
L_t\left(\frac{F}{i_1}\right) = -\frac{EA}{i_1} & L_r\left(\frac{F}{i_1}\right) = -\frac{\bar{G}D}{i_1} & L_\phi\left(\frac{F}{i_1}\right) = \frac{Fx_1}{i_1} \\
L_t\left(\frac{E}{i_1}\right) = \frac{Ex_2}{i_1} & L_r\left(\frac{E}{i_1}\right) = \frac{\bar{G}C}{i_1} & L_\phi\left(\frac{E}{i_1}\right) = -\frac{FB}{i_1} \\
L_t\left(\frac{\bar{G}}{i_1}\right) = -\frac{ET}{i_1} & L_r\left(\frac{\bar{G}}{i_1}\right) = \frac{\bar{G}x_3}{i_1} & L_\phi\left(\frac{\bar{G}}{i_1}\right) = -\frac{FG}{i_1},
\end{array} \tag{3.105}
$$

where for notational convenience, we have defined

$$
\begin{aligned}
x_1 &\equiv \tfrac{1}{F}(-\alpha - EB + G\bar{G}) \\
x_2 &\equiv \tfrac{1}{E}(-\beta - AF - \bar{G}T) \\
x_3 &\equiv \tfrac{1}{\bar{G}}(\gamma + EC + DF).
\end{aligned} \tag{3.106}
$$

Two independent equations may then be obtained by taking a matrix with rows containing the derivatives of i_1 and any two of F/i_1, E/i_1, and $\bar{G}/i_1$.

Since the determinant of these matrices must vanish, the common factors of i_1 in each row may be cancelled, as may the common factors of E, $\bar{G}$, and F in columns one, two and three. The matrices for, say, the triples $(i_1, E/i_1, \bar{G}/i_1)$ and $(i_1, F/i_1, \bar{G}/i_1)$ then take the quite simple forms

$$\begin{pmatrix} 1 & 1 & 1 \\ -x_2 & -C & -B \\ T & -x_3 & -G \end{pmatrix} \tag{3.107}$$

and

$$\begin{pmatrix} 1 & 1 & 1 \\ A & D & x_1 \\ T & -x_3 & -G \end{pmatrix} \tag{3.108}$$

respectively. The two equations for the vanishing determinants are thus

$$x_2x_3 - Gx_2 - Bx_3 - TB + TC + CG = 0 \tag{3.109}$$
$$x_1x_3 + Tx_1 - Ax_3 + AG - DG - DT = 0. \tag{3.110}$$

These two equations may be solved for x_1 and x_2 in terms of x_3 and substituted into the trace free condition

$$-Fx_1 - Ex_2 + \bar{G}x_3 - F(A + D) - E(B + C) + \bar{G}(G - T) = 0 \tag{3.111}$$

to yield the cubic equation

$$\begin{aligned} &\bar{G}x_3^3 + (-2AF - DF - 2EB - EC)x_3^2 + (-G^2\bar{G} - \bar{G}T^2 - 3ETB \\ &-2FTD + EBG - AFT + 3FAG + G\bar{G}T + 2ECG)x_3 \\ &+2FTDG + FDG^2 - FAG^2 + ET^2C = 0. \end{aligned} \tag{3.112}$$

The solutions to this equation are extremely complicated and generally complex (in the mathematical sense.) However, they do suggest that we try the real solution

$$\bar{G}x_3 = \frac{1}{3}(2AF + 2EB + DF + EC) \tag{3.113}$$

supplemented by additional constraints required for the vanishing of the imaginary parts of the full solution. A similar procedure may be repeated placing the derivatives of other pairs of the functions F/i_1, E/i_1, and G/i_1 into rows of a matrix with the derivatives of i_1 alone as the first row. Cubic equations for the other x_i are then obtained and the suggested real solutions may be read off as

$$Ex_2 = -\frac{1}{3}(2DF - 2G\bar{G} + AF + \bar{G}T) \tag{3.114}$$

and

$$Fx_1 = -\frac{1}{3}(2\bar{G}T + 2EC - \bar{G}G + EB). \tag{3.115}$$

These real solutions taken together satisfy the trace free condition (3.111) automatically, but only solve the equations for the vanishing determinants if supplemented by the aforementioned constraints which are fourth-order polynomials in the functions A, B, C, *etc.* Assuming that these constraints hold, we have the quite simple quadratic forms for the diagonal elements of the Weyl tensor

$$\alpha = \frac{2}{3}[\bar{G}(G+T) - E(B-C)] \tag{3.116}$$

$$\beta = -\frac{2}{3}[F(A-D) + \bar{G}(G+T)] \tag{3.117}$$

$$\gamma = \frac{2}{3}[E(B-C) + F(A-D)]. \tag{3.118}$$

We are now very near to a fully determined set of equations. The final step is to make an ansätz about the constants k_i appearing in (3.86). Considering the quite complicated form of the coefficients of the greek variables in the second order bracket relations, it would seem quite remarkable if they all shared the common ratios implied by (3.86). The most symmetric (and perhaps the most natural) case would be for all the k_i to vanish. It should be stressed once again here that this choice is not simply a gauge choice and has some, albeit unknown, physical significance. It is made in the name of simplicity and in the hope of obtaining a set of solvable equations. If this choice is made, then the second order relations yield the three purely algebraic relations

$$G(A+F) - \bar{G}(D+F) + T(A-D) = 0 \tag{3.119}$$

$$A(C-E) + D(B-C) + F(B-E) = 0 \tag{3.120}$$

$$-B(G+T) + C(G-\bar{G}) + E(\bar{G}+T) = 0 \tag{3.121}$$

which are curious convolutions of the three-cycles appearing in (3.72). We also obtain three equations relating derivatives

$$
\begin{aligned}
L_r D &= L_\phi G && (3.122)\\
L_t A &= L_\phi B && (3.123)\\
L_r C &= -L_t T. && (3.124)
\end{aligned}
$$

We can use these new relations to construct explicit forms for the missing derivatives not in (3.102). Consider the set of equations

$$
\begin{aligned}
L_\phi A &= m L_t A + n L_r A\\
L_\phi B &= m L_t B + n L_r B\\
L_\phi A &= L_t B - A^2 + B^2 + GT + \gamma\\
L_\phi B &= L_t A.
\end{aligned} \qquad (3.125)
$$

Using the known derivatives $L_r A$ and $L_r B$, these may be solved to give

$$
L_t B = \frac{-mnG(B-C) + nT(A-D) + A^2 - B^2 - GT - \gamma}{1-m^2}. \qquad (3.126)
$$

Now, γ is known from (3.118), and m and n may be obtained by solving (3.19) for any two functions of which all three derivatives are known, for example i_1 and E/i_1. With a form for $L_t B$ in hand, we can obtain forms for any of the other unkown derivatives $L_\phi A$, $L_t A$, and $L_\phi B$. A similar procedure may be repeated for the pairs of functions (D, G) and (C, T) to arrive at a full set of equations for all the capital letter functions A, B, C, *etc.*

Another useful consequence of setting the k_i to zero is that the equations (3.119)-(3.121) open the way for a natural coordinatisation for our solution. Combining these equations with the derivatives (3.102) yields the integrating factors

$$
\begin{array}{ll}
L_t(G-\bar{G}) = -C(G-\bar{G}) & L_t(D+F) = -B(D+F)\\
L_r(B-E) = T(B-E) & L_r(A+F) = -G(A+F)\\
L_\phi(C-E) = -A(C-E) & L_\phi(\bar{G}+T) = -D(\bar{G}+T).
\end{array} \qquad (3.127)
$$

Applying (3.22), we may then construct the commuting derivatives

$$
\begin{aligned}
[(G-\bar{G})^{-1}L_r, (B-E)^{-1}L_t] &= 0 && (3.128)\\
[(\bar{G}+T)^{-1}L_r, (A+F)^{-1}L_\phi] &= 0 && (3.129)\\
[(C-E)^{-1}L_t, (D+F)^{-1}L_\phi] &= 0. && (3.130)
\end{aligned}
$$

It then becomes convenient to work in a position gauge which is suited to one of these pairs of derivatives. For example, we may choose a gauge such that

$$(G - \bar{G})^{-1} L_r = g(r)\partial_r, \qquad (B - E)^{-1} L_t = f(t)\partial_t. \tag{3.131}$$

It should be expected that one cannot arrange for all three derivatives L_t, L_r, and L_ϕ to simultaneously commute in virtue of the linear relation (3.19) among them.

3.5 Conclusions

The computational simplicity of gauge theory gravity seen when dealing with stationary cylindrically symmetric systems is not nearly as apparent here in the general time-dependent case. However, the gauge theoretic approach is quite systematic and thus much progress can be made with the use of symbolic mathematics software. Gauge theory gravity also offers a much more general route into time-dependent solutions. There are an infinite number of possible vacuum solutions in the time-dependent setup, and some choices ultimately have to be made to obtain a specific solution, but these choices need not be made until after a substantial amount of progress has been made with the general case.

With the formalism developed in this chapter, we have all the necessary tools to solve for the time-dependent vacuum fields, at least in principle. However, time-dependent systems still represent 'work in progress,' and there is plenty of further work to be done in advancing the general approach. General forms such as (3.126) are exceedingly complicated and are not solvable by any standard methods. Also problematic is the issue untouched here of solving for the functions in $\overline{h}(a)$ once the capital letter functions in $\omega(a)$ are obtained. Furthermore, despite their seeming naturalness, the choices made that M, K, $\bar{K}$, and all the k_i vanish are not rigorously justifiable. It is also not entirely clear that any solution to the differential equations for the capital letter functions will solve both the algebraic constraints (3.119)-(3.121) and the fourth order polynomial constraints implied by our ansätz for α, β, and γ.

A more specific area of future work which may prove fruitful is the study of wave solutions like those described in [19], which appear in General Relativity with diagonal line elements. Making the restriction to a diagonal

$\overline{h}$-function in our gauge theory framework considerably simplifies the equations, and may lead to a quick solution. This restriction requires that the functions A, D, and F all vanish, while automatically putting us in the $K = \bar{K} = M = 0$ case as well. All L_ϕ-derivatives automatically vanish, so that the determinant structure described above is no longer relevant. Finally, the differential equations relating the functions in $\overline{h}(a)$ to those in $\omega(a)$ all decouple from one another, and should be easy to solve once the remaining capital letter functions B, C, E, G, $\bar{G}$, and T are obtained. Even with all these simplifications, however, it is not yet clear that the resulting set of equations can be solved without making some further choices. It should also be kept in mind that the majority of possible time-dependent solutions are likely not to have diagonalisable line elements. Another potential area of future work would be the examination of whether or not CTL might form in a time-dependent system starting from an initial configuration which did not contain any.

Appendix A

Connecting Gauge Theory Gravity and General Relativity

This appendix follows the discussions given in Chapter 5 and Appendix C of [6]. Index-free expressions in gauge theory gravity may be cast into the traditional tensor calculus formalism of General Relativity by fixing all gauge freedom and choosing a set of coordinates $\{x^\mu\}$. A set of basis vectors $\{e_\mu\}$ and its reciprocal basis $\{e^\mu\}$ are then given by

$$e_\mu \equiv \frac{\partial x}{\partial x^\mu} \quad \text{and} \quad e^\mu \equiv \nabla x^\mu. \tag{A.1}$$

Upon defining $g_\mu \equiv \underline{h}^{-1}(e_\mu)$ and $g^\mu \equiv \overline{h}(e^\mu)$, the metric is defined by $g_{\mu\nu} \equiv g_\mu \cdot g_\nu$ as in equation (1.43) and the connection is given by

$$\Gamma^\alpha_{\mu\nu} = g^\alpha \cdot (g_\mu \cdot \mathcal{D} g_\nu) = g^\alpha \cdot (\partial_\mu g_\nu + \omega(g_\mu) \times g_\nu). \tag{A.2}$$

In the absence of torsion, $\Gamma^\alpha_{\mu\nu}$ reduces to the standard Christoffel symbol $\{^{\alpha}_{\mu\nu}\}$.

Decomposing gauge theory tensors into their indexed counterparts is a straightforward process. Tensor components are obtained by taking the dot product of the covariant gauge theory tensor of interest with the appropriate combination of the $\{g_\mu\}$ and $\{g^\mu\}$ wedged together to yield the desired combination of covariant and contravariant indices. For example, the tensor claculus components of the Riemann tensor are given by

$$R^\mu{}_{\nu\sigma\rho} = (g^\mu \wedge g_\nu) \cdot \mathcal{R}(g_\sigma \wedge g_\rho). \tag{A.3}$$

Many of the usual elementary results of General Relativity carry over into convenient and compact forms in gauge theory gravity expressed in geometric calculus. For example, the Bianchi identity

$$R_{\rho\sigma\alpha\beta;\gamma} + R_{\rho\sigma\beta\gamma;\alpha} + R_{\rho\sigma\gamma\alpha;\beta} = 0 \tag{A.4}$$

may be written as

$$\dot{\mathcal{D}} \wedge \dot{\mathcal{R}}(B) \equiv \partial_a \wedge [a \cdot \mathcal{D}\mathcal{R}(B) - \mathcal{R}(a \cdot \mathcal{D}B)] = 0 \tag{A.5}$$

and the contracted Bianchi identity

$$G^{\alpha}{}_{\beta;\alpha} = 0 \tag{A.6}$$

becomes

$$\dot{\mathcal{D}} \cdot \dot{\mathcal{G}}(a) \equiv \partial_b \cdot [b \cdot \mathcal{D}\mathcal{G}(a) - \mathcal{G}(b \cdot \mathcal{D}a)] = 0 \tag{A.7}$$

The formalism of Killing vectors is also easily accomodated in geometric calculus. Killing's equation for a Killing vector K,

$$K_{\alpha;\beta} + K_{\beta;\alpha} = 0 \tag{A.8}$$

may be writen in covariant form as

$$a \cdot (b \cdot \mathcal{D}K) + b \cdot (a \cdot \mathcal{D}K) = 0. \tag{A.9}$$

Killing's equation may be contracted with $\partial_a \cdot \partial_b$ to demonstrate the result that K is covariantly conservered, *i.e.*, $\mathcal{D} \cdot K = 0$. This, in turn, may be combined with the contracted Bianchi identity (A.7) and the symmetry of the stress-energy tensor to obtain the result that $\mathcal{T}(K)$ is also covariantly conserved. Killing vectors also generate conserved quantities along particle free-fall worldlines. If the worldline is parameterised by proper time τ then

$$\partial_\tau(v \cdot K) = v \cdot \mathcal{D}(v \cdot K) = K \cdot (v \cdot \mathcal{D}v) + v \cdot (v \cdot \mathcal{D}K) = 0 \tag{A.10}$$

where the first term on the right vanishes due to the geodesic equation (2.117) and the second term vanishes due to Killing's equation (A.9) for $a = b$. Thus we see that the quantity $v \cdot K$ is constant along particle trajectories.

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