CHARGED PARTICLES AT POTENTIAL STEPS

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Abstract. The behaviour of charged particles at electromagnetic steps is analysed using the powerful mathematical tools provided by the SpaceTime Algebra. Currents predicted in the evanescent region of a Dirac wavefunction strongly suggest that the electron "zitterbewegung" (ZBW) represents a real circulation. At higher potentials, the Klein paradox reveals a crucial difficulty of interpretation of "positronic" wavefunctions that must be overcome before Hestenes' ZBW model can be taken seriously. The problem of radiation reaction is still not solved. Solutions of the Lorentz-Dirac equation for a potential step show crazy teleological features: certain input velocities have no possible future output states. The prospects for realistic electron models are briefly discussed.

1. Introduction

I can safely say to this audience that, as a radio astronomer, I have observed more electrons than anyone else present. A medium-sized radio galaxy displays radio synchrotron emission from about 10^{63} electrons as jets of relativistic electrons and positrons shoot out into space from a black hole deep inside a galactic nucleus. Our interest in these, the most violent objects in the Universe, requires a corresponding understanding of the humble lepton and its radiation mechanisms, particularly at the highest energies.

Like some of the other participants here, I have for a long time been deeply unhappy about the accepted theories of this little particle and I have to admit that I have been unable to heed Feynman's (1967) excellent advice not to ask oneself

> " 'But how can it be like that?' because you will get 'down the drain',
> into a blind alley from which no one has yet escaped.".

Two years ago, however, an unexpected event brightened up my view of this particular drain, when I became aware of the work by David Hestenes on Geometric Algebra. He claimed (Hestenes 1986) that our mathematical language is seriously incomplete, because physicists do not know how to multiply vectors together, and that they are thereby missing the geometrical content of the equations of physics, particularly the Dirac equation. I rapidly became convinced that Geometric Algebra, which includes the algebra of our spacetime, the SpaceTime Algebra (STA), as a special case, is an essential ingedient in correcting our misconceptions about the nature of quantum mechanics.

As a beginner in a strange new field, I have modest aims in this paper and focus attention on the behaviour of a charged particle encountering an electromagnetic potential step, all the time using the powerful mathematical tools provided by the STA. This situation forms a convenient "theoretical laboratory" for the electron, allowing us to examine critically the predictions of presently-available models. After a brief review of the STA and the Dirac equation, I consider the riddle of the charge current in the Dirac theory and the extent to which the behaviour of an electron wave at an electromagnetic potential step throws light upon the new "zitterbewegung" (ZBW) interpretation (Hestenes 1990). Klein's paradox is briefly mentioned, but *not* resolved. I believe that the difficulties of the Dirac current are clearly exposed by this paradox and that our present ways of overcoming it (essentially due to Feynman) are unsatisfactory. "The cure is worse than the disease", as Ed Jaynes is fond of saying.

In the second part of this paper we return to the potential step to examine the mystery of the radiation reaction. This is still an important problem, because it is extremely desirable to have a self-consistent equation of motion for an accelerated charge that takes into account its own radiation. The case of a finite-sized charge distribution is tractable if sufficient care is used in any approximations, but the difficulties for a point charge seem insuperable. The most famous attempt, the Lorentz-Dirac equation (Dirac 1938), fails miserably when confronted with the problem of the potential step, showing unacceptable overall scattering states that prohibit certain ranges of input condition. The STA suggests an alternative equation.

Inspired by the ZBW interpretation of quantum mechanics, we have a brief look at realistic electron models (Barut & Zhanghi 1984). Translated into the STA, these models begin to look very interesting, but certain defects become obvious.

2. The SpaceTime Algebra and the Dirac equation

This paper embraces the ideas and notations developed by Hestenes over the last 30 years (Hestenes 1966, 1986). The SpaceTime Algebra (STA) is a real, Geometric (Clifford) algebra developed on a 4-dimensional flat spacetime with a standard Minkowski metric. The basic ingredients of this algebra are an orthonormal frame of vectors $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, where $\gamma_0^2 = -\gamma_k^2 = 1$. The time-like vector γ_0 defines a Lorentz frame, that we can think of as representing the *laboratory frame*. The $\{\gamma_\mu\}$ satisfy the same algebraic relations as the Dirac γ-matrices, but it must be stressed that they here represent 4 unit vectors in spacetime and *not* the 4 components of a single vector. We shall have no use for a matrix representation of the $\{\gamma_\mu\}$ here, but a translation table is given in the Appendix.

From this basic set of vectors we build up the 16 ($= 2^4$) geometric elements of the STA:

1	$\{\gamma_\mu\}$	$\{\sigma_k, i\sigma_k\}$	$\{i\gamma_\mu\}$	i
1 scalar	4 vectors	6 bivectors	4 pseudovectors	1 pseudoscalar .

The time-like bivectors $\sigma_k \equiv \gamma_k\gamma_0$ obey the same algebraic relations as the Pauli spin-matrices, but in the STA they represent an orthonormal frame of vectors in space *relative* to the laboratory time vector γ_0. The unit pseudoscalar of spacetime is defined as

$$i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3.$$

The STA is the 16-dimensional *real* linear space formed from these geometric objects. We will not need to consider the coefficients to be *complex* scalars, because the STA already contains 10 geometrically distinct square roots of -1, which is plenty enough for our purposes.

Indeed, the STA representation of the complex imaginary is different from one application to another, so that the use of complex numbers as *scalars* may be unnecessary in physics.

A particularly important subalgebra of STA is the set of *spinors*, which, for spacetime, is simply the even subalgebra, comprising the scalar, bivectors and pseudoscalar. We can write a general spinor in the form

$$\psi = a_0 + \mathbf{a} + i(b_0 + \mathbf{b}),$$

where $\mathbf{a} \equiv a_k\sigma_k$ and $\mathbf{b} \equiv b_k\sigma_k$ are *relative vectors*. A very important point of interpretation is the fact that spinors of the STA are geometrical objects in their own right, rather than (as in matrix versions) residing in a complex "spin-space". The notion of "spin-space" is not needed here.

The spinors are important because they define the transformation properties of frames in spacetime. A spinor ψ defines a Lorentz rotation through the transformation

$$\rho e_\mu = \psi\gamma_\mu\tilde{\psi},$$

where the $\{e_\mu\}$ are a new frame of orthogonal vectors and $\tilde{\psi}$ is the *reverse* of ψ, formed by reversing the order of all geometric products. To see this explictly we write a Dirac spinor ψ in canonical form:

ψ	$=$	$\left(\rho e^{i\beta}\right)^{\frac{1}{2}}$	$L(\mathbf{u})$	$R(\theta,\phi,\chi)$.
spinor		complex amplitude	Lorentz transformation to velocity $v = e^{\mathbf{u}}\gamma_0$	Rotation by Euler angles (θ,ϕ,χ)

Explicit forms for $L(\mathbf{u})$ and $R(\theta,\phi,\chi)$ are

$$L(\mathbf{u}) = e^{\mathbf{u}/2} \quad \text{and} \quad R(\theta,\phi,\chi) = e^{-i\sigma_3\phi/2}\, e^{-i\sigma_2\theta/2}e^{-i\sigma_3\chi/2}.$$

Writing the Dirac equation in the STA (see Appendix) we have

$$\nabla\psi i\sigma_3 - eA\psi = m\psi\gamma_0.$$

This admits plane-wave solutions for 4-momentum p (with $p^2 = m^2$),

$$\begin{aligned}\psi &= L(\mathbf{p})R(\theta,\phi,\chi)e^{-i\sigma_3 p\cdot x}\\ &\text{or}\quad L(\mathbf{p})R(\theta,\phi,\chi)i\, e^{+i\sigma_3 p\cdot x}.\end{aligned}$$

The Dirac current $\psi\gamma_0\tilde{\psi} \equiv \rho v$ and the spin current $\frac{1}{2}\psi\gamma_3\tilde{\psi} \equiv \rho s$ can then be interpreted using the STA. The Dirac wavefunction represents a change of frame, providing a Lorentz boost of γ_0 to the comoving velocity v and a rotation of γ_3 to align it with the spin axis of the electron. The phase of the wavefunction gives the rotation of the $\gamma_2\gamma_1$ plane about the spin axis. Of the two other factors contained in $\psi\tilde{\psi} \equiv \rho e^{i\beta}$, ρ can be taken to represent the probability of finding the electron at position x, but the rôle of β is uncertain. We can see that it distinguishes between positive ($\beta = 0$) and negative ($\beta = \pi$) frequency (energy) states.

In the ZBW interpretation of quantum mechanics (Hestenes 1990), the Dirac current is redefined as $\rho v \equiv \psi\gamma_-\tilde{\psi}$, where $\gamma_- \equiv \gamma_0 - \gamma_2$. This would make the worldline of a particle into a light-like helix, making manifest a transverse ZBW as the source of the electron's spin and magnetic moment. The positive frequency solutions above are interpreted as electrons, and the negative frequencies as positrons, so that β measures the extent to which we have a pure particle/antiparticle state. It is safe to say, however, that no entirely satisfactory interpretation of β is yet available.

3. The Dirac electron at a potential step

An elementary application of the Dirac equation is to consider an electron wave incident normally upon a simple electromagnetic potential step of magnitude ϕ, so that ($A = 0,\ z < 0$) and ($A = \phi\gamma_0, z > 0$). In the STA we write the incident, reflected and transmitted waves as follows:

Incident	Reflected	Transmitted
$\psi_I = e^{u\sigma_3/2}\Phi e^{-i\sigma_3 p_I\cdot x}$	$\psi_r = r\ e^{-u\sigma_3/2}\Phi e^{-i\sigma_3 p_r\cdot x}$	$\psi_t = t\ e^{u'\sigma_3/2}\Phi e^{-i\sigma_3 p_t\cdot x}$
$p_I = E\gamma_0 + p\gamma_3$	$p_r = E\gamma_0 - p\gamma_3$	$p_t = E\gamma_0 + p'\gamma_3,$

where Φ is a Pauli spinor describing the spin state, $\Phi = 1$ corresponding to longitudinal spin "up" and $\Phi = -i\sigma_2$ to spin "down". Matching the spinors at $z = 0$ we find

$$\cosh(u/2)(1 + r) = \cosh(u'/2)\ t,$$

$$\sinh(u/2)(1 - r) = \sinh(u'/2)\ t,$$

$$r = \frac{\sinh(u - u')/2}{\sinh(u + u')/2}.$$

This is plausible, the reflection coefficient increasing to unity as the step height approaches the classical reflection point at $e\phi = E - m$.

For steps higher than this, the reflection coefficient is unity and the wavefunction inside the step is evanescent. We have to take some care when the solutions are written in terms of the STA, because we cannot just let p' become imaginary. The evanescent wave is

$$\psi_t = \psi_0 e^{-\kappa\cdot x} e^{-i\sigma_3 p'\cdot x},$$

where ψ_0 is a constant Dirac spinor. We can find a matching condition if $\kappa \cdot p' = 0$ and $(p')^2 - \kappa^2 = m^2$. The solution displays very interesting general features that may give us a clue about the nature of the ZBW. For the case of longitudinal spin we find

$$\psi_0 = e^{i\beta/2} e^{-\kappa z}\Phi e^{-i\sigma_3 Et},$$

where $E - e\phi = m\cos\beta$ and $\kappa = \pm m\sin\beta$ for the two polarisation states, spin "up" taking the positive sign. The evanescent wave has a non-zero β, which flips to $\pm\pi$ as $e\phi$ approaches $E + m$.

If the spin of the incident wave is tranverse (polarised in the $x - y$ plane), then there is a non-zero Dirac current in the evanescent region, the Dirac "velocity" having the value κ/m. The direction of this velocity is perpendicular both to the incident spatial momentum and to the spin. Similar effects can be seen in solutions of the Dirac or the non-relativistic Pauli equations whenever the amplitude of ψ varies with position. Other examples are the hydrogen wavefunction or a Gaussian wave-packet (Hestenes 1979). It is tempting the interpret this phenomenon in terms of the non-cancelling of "Amperian" currents due to the circulation of the ZBW. It seems that the evanescent wave is giving us a "tomographic" view of the ZBW, implying again that the spin of the electron represents a real circulation. There is a need, however, for a sensible interpretation of the longitudinal behaviour, which shows a non-zero value of β.

The Klein paradox and Feynman's resolution

When the size of the potential step exceeds $E+m$ the solution becomes propagating. Defining $E' = e\phi - m$, $(p')^2 = (E')^2 - m^2$ and $\tanh u' = p'/E'$, we find

$$r = \frac{\cosh(u-u')/2}{\cosh(u+u')/2}; \quad r < 1: \ \beta = \pi.$$

This case needs some care in order to match on to a solution with a positive group velocity inside the step. Many standard books give a solution with a negative group velocity, a notable example being Bjorken & Drell (1964).

This result for the reflection coefficient is actually very odd indeed, and I suspect that the reason some books give the wrong solution (with $r > 1$) is simply wishful thinking. We could perhaps understand $r > 1$ as representing the creation of electron/positron pairs at the step, with the positrons continuing into the step. In fact we seem to have an entirely different situation, and the transmitted wave looks more like a "positronic hole", having negative mass and negative charge.

The central problem is, as everybody knows, that the Dirac current $e\psi\gamma_0\tilde{\psi}$ is positive definite, and thus cannot represent a positronic current without some modification. The problem does not arise for the (non-STA) Klein-Gordon equation

$$\left(D^2 + m^2\right)\psi = 0, \text{ where } D \equiv \nabla + ieA$$

(the i is now just the uninterpreted $\sqrt{-1}$ of common usage). The Klein-Gordon reflection coefficients are

$$r = \frac{p-p'}{p+p'} < 1 \quad (e\phi < E - m),$$

$$r = \frac{p+p'}{p-p'} > 1 \quad (e\phi > E + m).$$

The Klein-Gordon charge current behaves sensibly, showing a strong resonance at $p = p'$, which one could interpret as stimulated emission of pairs.

The reason for the striking difference between these solutions is easy to find, because the Dirac equation does not lead to the minimally-coupled Klein-Gordon equation:

$$\nabla\psi i\sigma_3 - eA\psi = m\psi\gamma_0 \Longrightarrow$$

$$\nabla^2\psi + m^2\psi - e^2A^2\psi + 2eA\cdot\nabla\psi i\sigma_3 + eF\psi i\sigma_3 = 0,$$

where $F = \nabla A = \mathbf{E} + i\mathbf{B}$ ($\nabla \cdot A = 0$). There is an extra spin term $eF\psi i\sigma_3$, which couples into the very strong $\mathbf{E}$ field in the step, effectively changing the matching conditions. We see that the *spin* is crucial to this longitudinal ZBW phenomenon, which again cries out for a better physical interpretation.

Why should we bother with the Klein paradox of the Dirac equation when we are going to use field theory? The very simple case of an electromagnetic step has revealed a stark contradiction about the nature of the Dirac current. In almost any other branch of science we would conclude at this point that we have *already* done something wrong, and that we should go back and correct it. It seems, however, that in quantum theory we conclude that we have to do something else as well. This is more than a little odd and, in any case, it is unreasonable to suppose that such a contradiction will go away just by making the theory more complicated.

I am not able to understand or willing to repeat the arguments that lead to the standard resolution of the Klein paradox; a representative reference is Nikishov (1970). It appears that somewhere in the quantum fog the ideas of "exclusion principle" and "no stimulated emission of fermions" are invoked. The conclusion, on the other hand, is crystal-clear: $r = 1$ and there is no longer any paradox.

> I know an old lady who swallowed a fly,
>
> I don't know why she swallowed a fly, perhaps she'll die
>
> (Traditional nursery rhyme).

There seems to be a fundamental truth in the Dirac equation; we can interpret it geometrically. I believe that the same is probably true of the Weinberg/Salam electroweak model (Abers & Lee 1973, Hestenes 1982). But the Dirac theory *doesn't quite work* even when, with apologies to David Hestenes, it is translated into the STA. How should we patch it up? QED? Families of leptons? Quarks? Higgs particles? I suspect that we are in the same unfortunate position as the the old lady in the nursery rhyme who swallowed a fly and, finding it not to her liking, followed it with a series of increasingly indigestible remedies. My diagnosis is that we physicists have now reached the stage where we are attempting to swallow a "goat"and should beware the "horse" that must be waiting for us soon. I honestly believe that most "unifications" (except Weinberg/Salam) are headed in the wrong direction and that we (or at least some of us) should try to turn the clock back and start again using the proper mathematical tools. The SpaceTime Algebra is an essential ingredient in this programme.

Pushing the analogy a little further, the place to start is, I suppose, at the point we swallowed the fly. My guess is that we went wrong when the first infinity appeared in physical theory: the self-energy of a charge. To this end I am sympathetic to the view expressed by Jaynes (1990) about the reality or otherwise of the electromagnetic field, following the lead given by Wheeler & Feynman (1949). In this view the electromagnetic field does not actually exist in spacetime, but represents instead an *information storage device*. The electromagnetic field at any point might be a *summary* of what we need to know about distant charges in order to predict the behaviour of a charge *if there should be one* at that point. But I have already been sufficiently radical in this paper and I shall defer any further heresy. There is, however, a closely related problem concerning *radiation* *reaction.* Is it possible to have a self-consistent equation of motion for a particle that takes account of its electromagnetic radiation?

4. Radiation reaction at a step

The radiation reaction on an accelerated charge is a century-old problem that is still not resolved, despite the valiant efforts of generations of theoretical physicists. The Lorentz-Dirac equation (Dirac 1938) is "self-consistent" in the sense that it correctly accounts for the energy budget of a radiating particle, but it suffers from strange pre-acceleration effects and admits runaway solutions. Dirac derived the equation by expanding the field near a point charge as a power-series in retarded time and collecting the parts that did not diverge at the origin. The standard form of this equation (in SI units) is

$$m\dot{v} - \frac{e^2}{6\pi\epsilon_0 c^3}\left(\ddot{v} + \dot{v}^2 v\right) = eF\cdot v.$$

The derivation of this equation is badly flawed: the power-series in retarded time has only a finite radius of convergence. Burke (1970) shows that, if the same mathematical techniques are applied to the case of spherical oscillations of a rubber ball in air, pre-acceleration and runaway solutions again appear. Careful analysis of *finite-sized* charged distributions do not show such peculiar effects, and *do not* yield the Lorentz-Dirac equation (Jaynes 1980 (unpublished notes), Grandy & Aghazadeh 1983). Although these analyses are fine, I do not believe that the electron has a finite size (though see Jaynes' paper in this volume for a different point of view), and it is still very attractive to have an equation of motion that self-consistently accounts for the radiation of an accelerated point charge. Consequently, the Lorentz-Dirac equation is still used (see, for example, Barut 1988, Barut 1990), usually with the added boundary condition (due to Dirac) that the velocity remains finite as $t \to \infty$. Unfortunately, even with Dirac's condition, the solutions of this equation are particularly strange and physically unacceptable for the apparently innocuous electromagnetic potential step, which we now study.

The usual form of the Lorentz-Dirac equation does not fully reveal its geometrical meaning. Expressing the equation in the STA, we note that $v^2 = 1$, $\dot{v}\cdot v = 0$ implies

$$\ddot{v} + \dot{v}^2 v = \ddot{v} - (v\cdot\ddot{v})v = \frac{1}{2}(\ddot{v} - v\ddot{v}v) \equiv \ddot{v}_\perp,$$

which is the component of $\ddot{v}$ projected perpendicular to v. Because the acceleration $\dot{v}$ and the Lorentz force $eF\cdot v$ are both perpendicular to v, the reactive term must also be perpendicular. Multiplying by v we find

$$\frac{1}{2}(\ddot{v}v - v\ddot{v}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(\dot{v}v - v\dot{v}) = \frac{\mathrm{d}}{\mathrm{d}\tau}(\dot{v}v),$$

where we have again used $v\cdot\dot{v} = 0$. Defining $e^2/(6\pi\epsilon_0 mc^3) \equiv \tau_0$, we can then rewrite the equation in terms of the rest-frame acceleration bivector $\Omega_v \equiv \dot{v}v = \dot{v}\wedge v$:

$$\frac{\mathrm{d}\Omega_v}{\mathrm{d}\tau} - \frac{\Omega_v}{\tau_0} = -\frac{e}{2m\tau_0}(F - vFv) = -\frac{eE_v}{m\tau_0},$$

where $E_v \equiv \frac{1}{2}(F - vFv)$ is the electric field in the rest frame.

Figure 1. Input and output velocities for the Lorentz-Dirac equation at a potential step of $e\phi = 0.1m$. (a) $v_{\rm in}$ is a single-valued function of $v_{\rm out}$. (b) $v_{\rm out}$ is a multi-valued function of $v_{\rm in}$. The dashed curves show the behaviour of a non-radiating particle.

The field-free solutions can be expressed using the STA as $\Omega_v = B\exp(\tau/\tau_0)$ for any simple time-like bivector B, so that $v(\tau) = \exp[B\exp(\tau/\tau_0)]v(0)$. The solutions clearly have very undesirable properties as $\tau \to \infty$ unless we employ Dirac's boundary condition. To apply the Lorentz-Dirac equation to the potential step problem we search all possible finite output velocities $v_{\rm out}$ and integrate the equation *backwards in time* to see what the appropriate input velocities $v_{\rm in}$ must have been.

For a particle incident on a step having higher potential for $x > 0$ there are three distinct cases to consider.

1. Transmission left to right ($v_{\rm in}$ and $v_{\rm out}$ both positive). Note that, if $v_{\rm out}$ is large, then the particle, which has pre-accelerated up to $v_{\rm out}$ before it reached the step, goes through the step quickly, thereby receiving a small influence from it, so that $v_{\rm in}$ must have been large. However, if $v_{\rm out}$ is small the particle spends a longer time in the step and the change in velocity due to the step (not *at* it, though) is also large, so that $v_{\rm in}$ is again large.

2. Transmission right to left ($v_{\rm in}$ and $v_{\rm out}$ both negative). The particle, which has $v_{\rm out} < v_{\rm in}$, has pre-accelerated by an amount that increases as $v_{\rm out}$ is reduced. When $v_{\rm out}$ is sufficiently small, the long-term value of $v_{\rm in}$ is positive, so that the step is re-encountered.

3. Anomalous reflection ($v_{\rm in} > 0$; $v_{\rm out} < 0$). The particle approaches the step from the left, pre-decelerates towards the step and crosses it. On the other side the pre-deceleration of its next step-crossing changes the sign of its velocity and drags it back into the step, from which it emerges, having been reflected.

In Figure 1 the values of $v_{\rm in}$ and $v_{\rm out}$ are plotted for the particular case $e\phi = 0.1\ m$. The dashed curve shows, for comparison, the behaviour of a non-radiating particle. Even if the pre-acceleration properties of the equation are forgiven, the solutions are totally crazy. Figure 1(a) plots $v_{\rm in}$ as a singled-valued function of $v_{\rm out}$, but a time-dweller's view of Figure 1(b) shows that for $v_{\rm in} > 0$ there are ranges of $v_{\rm in}$ with either 2 or 4 values of $v_{\rm out}$. In case that is not bad enough, for low values of $v_{\rm in}$ (corresponding in the non-radiating case to simple reflection) there are *NO* allowed values of $v_{\rm out}$. We have, apparently, lost control of the input velocity! That is the sort of thing which is almost bound to happen when complicated equations are integrated backwards in time.

The firm conclusion of this little investigation is that Dirac's suggested boundary condition is not correct; merely making v finite at $\tau \to \infty$ is not sufficient to rescue the Lorentz-Dirac equation. It should also be noted that there are no mathematical difficulties associated with the idealised case of a sharp step, which can be approached as a well-behaved limit of a finite-width step of any shape. An extra condition on the Lorentz-Dirac equation (Sawada, Kawabata & Uchiyama 1983), which apparently prohibits such steps, cannot remove the difficulty, therefore.

A recent suggestion by Barut (1990) of "renormalising" the Lorentz-Dirac equation, taking solutions without pre-acceleration and removing the runaway part by decree is, I believe, even worse, because it does not conserve energy. Applying it to the finite-width step problem, and letting the step width go to zero whilst keeping the height constant, we find $v_{\rm out} \to v_{\rm in}$. That is extremely unfortunate if the step is a decelerating one as we have supposed, because the particle is now in a region of higher potential, which we could use to accelerate the particle in another, wider, downward step. This leads to an interesting new design of linear accelerator!

Modified Lorentz-Dirac equations

We can throw some light on the problems of the Lorentz-Dirac equation by writing the equation of motion as $m\,\dot{v} = e(F + F_s)\cdot v$, where F_s is a "self-field" representing the inner workings of the particle. It would be natural to suppose that these internal mechanisms would introduce a time lag, so that the effective force might be due to some retarded average of the applied field, for example $F_{\rm ret} \equiv \frac{1}{\tau_0}\int_{-\infty}^{\tau} e^{(\tau'-\tau)/\tau_0} F(\tau')\mathrm{d}\tau' \approx F(\tau - \tau_0)$. A simple example shows why this suggestion is unsatisfactory.

Imagine a classical electron orbiting a nucleus. The instantaneous force is always towards the nucleus, so that the particle is kept in a circular orbit. The retarded force is directed *ahead* of the nucleus so that it would tend to increase the radius of the orbit, and hence the energy. Dirac solved this problem by using the *advanced* force $F_{\rm adv} \equiv \frac{1}{\tau_0}\int_{\tau}^{\infty} e^{(\tau-\tau')/\tau_0} F(\tau')\mathrm{d}\tau'$, so that the response to any impulse precedes its cause. This can be easily seen by rewriting the Lorentz-Dirac equation in terms of F_s:

$$\dot{F}_s - \frac{1}{\tau_0}F_s = -\frac{\mathrm{d}E_v}{\mathrm{d}\tau}.$$

In this form the equation looks extremely dangerous, as we have already shown it to be in practice. An obvious causal modification is

$$\dot{F}_s + \frac{1}{\tau_0}F_s = \frac{\mathrm{d}E_v}{\mathrm{d}\tau},$$

which uses $(2F - F_{\rm ret})$ instead of $F_{\rm adv}$. It responds to an impulse like a mass of $m/2$ and then remembers the rest of its mass over a time $\approx \tau_0$. Applied to the step problem, the modified equation has properties very similar to the sensible branches of the Lorentz-Dirac solutions in Figure 1(b). The disallowed values of $v_{\rm in}$ now have $v_{\rm out} = 0$, and we take the upper branch for large $v_{\rm in}$. However, although the modification produces results indistinguishable to the accuracy of plotting from those of Figure 1(b), they are *not* identical, and the suggestion is *ad hoc*. To make further progress we must have a more detailed model of electron behaviour, but I conclude from this study that we probably have to revise our notions about the rate of radiation from accelerated charges.

5. Realistic classical models

Making classical models of particles with spin is an interesting game, all the more so when the STA is available. An example of such a model has been given by Barut & Zhanghi (1984); in addition to its position x, the particle has an internal mechanism or "clock" determined by a Dirac spinor ψ. Using the STA, and returning to natural units ($\hbar = c = 1$), we translate their Lagrangian as

$$\mathcal{L} = \frac{1}{2}\langle\, \dot{\psi} i\sigma_3 \tilde{\psi}\,\rangle + \langle\, p(\dot{x} - \psi\gamma_0\tilde{\psi})\,\rangle + e\,\langle\, A(x)\psi\gamma_0\tilde{\psi}\,\rangle\,,$$

where $\langle\;\rangle$ means "scalar part of ". The equations of motion are

$$\dot{\psi} i\sigma_3 = \pi\psi\gamma_0,$$

$$v \equiv \dot{x} = \psi\gamma_0\tilde{\psi},$$

$$\dot{\pi} = eF\cdot v,$$

where $\pi \equiv p - eA$. The Hamiltonian is $\pi\cdot v = m$ and the angular momentum bivector is $\frac{1}{2}\psi i\sigma_3\tilde{\psi} + x\wedge p$.

The solution of these equations for zero field can be written in terms of

$$\psi_- \equiv e^{\mathbf{u}/2}\Phi e^{-i\sigma_3 m\tau} \;\;;\;\; \psi_+ \equiv e^{\mathbf{u}/2}\Phi i e^{i\sigma_3 m\tau}.$$

The model can display longitudinal ZBW through interference of positive and negative frequencies but, unlike solutions of the Dirac equation, the negative frequency solution has the same charge/mass ratio as the positive energy one. To reverse the charge/mass ratio, it is necessary for the momentum π to point into the backward light-cone, thereby making the Hamiltonian negative. When the particle encounters a potential step there is a sudden change in its momentum π, but the spinor ψ is continuous. Energy is conserved; the time component of π changing by $e\phi$. The subsequent motion inside the step shows longitudinal ZBW and analysis of the mean velocity shows that the apparent rest mass has decreased.

The behaviour of the model in a magnetic field $F = Bi\sigma_3$ shows that it has no magnetic moment ($g = 0$), because a stationary solution ($v = \gamma_0$, $\pi = m\gamma_0$, $\psi = e^{-i\sigma_3 m\tau}$) can always be found. Following the suggestion by Hestenes (1990) for the Dirac current, we can make the ZBW of this model manifest by redefining the velocity $v = \psi\gamma_-\tilde{\psi}$, where $\gamma_- \equiv \gamma_0 - \gamma_2$ again. Examination of the stationary solutions shows that the negative frequencies have been eliminated and the positive frequencies moved to $2m$. There is now a magnetic moment; computer studies show that $g = 1$ for this modification, with the ZBW orbit plane precessing at one half the gyro frequency eB/m.

I conclude that, whilst Barut's little model is not yet satisfactory to describe the electron, there is still great promise for realistic models, particularly when the STA is employed.

6. Conclusions

We now have a natural language for spacetime physics that simplifies manipulations and gives equations which are fully Lorentz invariant and coordinate-free. The SpaceTime Algebra is able to provide important insights into the geometrical content of Dirac theory. Hestenes' ZBW interpretation of quantum mechanics seems promising, and there are important clues suggesting that the electron has a helical motion associated with its spin, but the longitudinal ZBW remains a mystery.

The problem of radiation reaction will not go away, but we can at least hope that the Lorentz-Dirac equation will finally be laid to rest. The crazy behaviour of this equation is well illustrated by its disastrous failure for the potential step problem.

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Appendix. A translation table for Dirac spinors

In this Appendix the conventional and STA versions of the Dirac spinor are written down explicitly, so that one can translate freely between them. For the standard column spinor we employ the form of the Dirac matrices given by Bjorken & Drell (1964), the components of which are themselves 2-component Pauli spin matrices:

$$\hat{\gamma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \hat{\gamma}_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}.$$

We identify a 4-component complex column vector Ψ in the standard treatment with an even multivector ψ of the STA. To make it quite clear that Ψ is a column vector in "spin-space", we will use Dirac notation. Thus

$$\underset{\text{conventional}}{\Psi \equiv |\psi\rangle} \quad \leftrightarrow \quad \underset{\text{STA}}{\psi.}$$

The STA form of the Dirac spinor can be written in terms of its components:

$$\psi = a_0 + a_k\sigma_k + b_0 i + b_k i\sigma_k.$$

This translates into a matrix as above, and can be converted into the conventional spinor by taking its first column:

$$|\psi\rangle = \begin{bmatrix} a_0 + ib_3 \\ -b_2 + ib_1 \\ a_3 + ib_0 \\ a_1 + ia_2 \end{bmatrix}.$$

We translate the operators of matrix theory as follows:

$$\hat{\gamma}_\mu|\psi\rangle \leftrightarrow \gamma_\mu\psi\gamma_0,$$

$$i|\psi\rangle \leftrightarrow \psi i\sigma_3.$$

We are now in a position to translate the Dirac equation from the conventional form

$$i\hat{\gamma}^\mu\partial_\mu|\psi\rangle - e\hat{\gamma}^\mu A_\mu|\psi\rangle = m|\psi\rangle \leftrightarrow \gamma^\mu\partial_\mu\psi\gamma_0 i\sigma_3 - e\gamma^\mu A_\mu\psi\gamma_0 = m\psi.$$

Multiplying by γ_0 and re-assembling the vectors $A = \gamma^\mu A_\mu$ and $\nabla = \gamma^\mu\partial_\mu$ we obtain the STA version of the Dirac equation

$$\nabla\psi i\sigma_3 - eA\psi = m\psi\gamma_0.$$