MASSIVE, NON-GHOST SOLUTIONS FOR THE DIRAC FIELD COUPLED SELF-CONSISTENTLY TO GRAVITY

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Abstract

We present new, massive, non-ghost solutions for the Dirac field coupled self-consistently to gravity. We employ a gauge-theoretic formulation of gravity which automatically identifies the spin of the Dirac field with the torsion of the gauge fields. Homogeneity of the field observables requires that the spatial sections be flat. Expanding and collapsing singular solutions are given, as well as a solution which expands from a singularity before recollapsing. Torsion effects are only important while the Compton wavelength of the Dirac field is larger than the Hubble radius. We study the motion of spinning point-particles in the background of the expanding solution. The anisotropy due to the torsion is manifest in the particle trajectories.

1 Introduction

The problem of finding cosmological solutions for a Dirac field coupled to gravity in a self-consistent manner has been considered by several authors, for example [1]–[10]. The importance of such solutions derives from the fact that, with the inclusion of a suitable cosmological constant or vacuum polarisation terms, the model provides an alternative to standard (scalar field) inflationary models. The solutions presented in [1]–[5] are derived from the Einstein-Dirac equations, which do not consider the effect of torsion induced by the spin of the Dirac field. The solutions in [7]–[10] do include the spin induced torsion (they solve the Einstein-Cartan-Dirac equations) but they appear to be either massless [5, 6, 7] or ghost solutions [8, 9, 10] (a ghost solution has a vanishing stress-energy tensor for the Dirac field). In this paper we present massive, non-ghost solutions of the Einstein-Cartan-Dirac equations, which we believe are new.

The importance of torsion in cosmology is well known (see Kerlick [11] for an early review, and Wolf [12] for more recent ideas). In particular, the singularity theorems in the presence of torsion suggest that singularity formation may be suppressed in a wide class of models [13]. However, Kerlick [14] has shown that if a Dirac field

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provides the source of matter, then the energy condition in the singularity theorems is weakened by the presence of torsion, leading to an enhanced singularity formation rate.

The identification of torsion with the spin of the matter field [15, 16, 17] follows naturally from the gauge-theoretic approach to gravity [18]. A new approach to gauge theory gravity (GTG) was developed in [19, 20]. In this approach gravitational effects are described by a pair of gauge fields defined over a flat Minkowski background spacetime. The gauge fields ensure the invariance of the theory under arbitrary local displacements and rotations in the background spacetime. All physical predictions are extracted in a gauge-invariant manner, thus ensuring that the background spacetime plays no dynamic role in the physics. Applications of GTG may be found in [21, 22] and the torsion sector of the theory is considered in some detail in [23]. The equations describing the Dirac field coupled self-consistently to gravity were given in [23]. These are derived from a minimally-coupled, gauge-invariant action and are the analogues of the Einstein-Cartan-Dirac equations. The Einstein-Dirac equations, which describe a Dirac field coupled to gravity through the (symmetrised) stress-energy tensor only, cannot be derived from a minimally-coupled action. This is a compelling reason to regard the Einstein-Cartan-Dirac equations as being more fundamental.

The solutions presented here are homogeneous at the level of the gravitational gauge fields. Homogeneity of the observables formed from the the Dirac field requires that the spatial sections be flat. The solutions given in this paper are all singular, and those of cosmological importance have particle horizons present. Torsion effects are only important while the Compton wavelength of the field is greater than the Hubble radius. During this epoch, the effect of torsion is to reduce the rate of expansion of the universe (as measured by the Hubble parameter) for a given value of cosmic time, as expected from the discussion in [14]. One of the solutions given here describes a universe which continues to expand from a singularity, with density parameter Ω varying as $\Omega = 1 + \mathrm{O}(1/m^2t^2)$, where m is the mass of the Dirac field and t is cosmic time since the singularity. The inclusion of torsion leads to anisotropy in the solutions, which we demonstrate by considering the motion of a spinning point-particle in one of the cosmological solutions.

We have found that the geometric algebra of spacetime — the Spacetime Algebra (STA) [24] — is the optimal language in which to express GTG. Employing the STA not only simplifies much of the mathematics, but it often brings the underlying physics to the fore. We begin this paper with a brief introduction to the STA and to gauge theory gravity. We employ natural units ($G = c = \hbar = 1$) throughout.

2 Spacetime Algebra and Gauge Theory Gravity

The geometric (or Clifford) algebra of spacetime is familiar to physicists in the guise of the algebra generated from the Dirac γ-matrices. The spacetime algebra (STA) is generated by four vectors $\{\gamma_\mu\}, \mu = 0 \ldots 3$, equipped with an associative (Clifford) product, denoted by juxtaposition. The symmetrised and antisymmetrised products define the inner and outer products between vectors, denoted by a dot and a wedge respectively:

$$\begin{aligned} \gamma_\mu \cdot \gamma_\nu &\equiv \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \eta_{\mu\nu} = \text{diag}(+ - - -) \\ \gamma_\mu \wedge \gamma_\nu &\equiv \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \end{aligned} \tag{2.1}$$

The outer product of two vectors defines a bivector — a directed plane segment, representing the plane including the two vectors.

A full basis for the STA is provided by the set

$$\begin{array}{ccccc} 1 & \{\gamma_\mu\} & \{\sigma_k, i\sigma_k\} & \{i\gamma_\mu\} & i \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar} \\ \text{grade 0} & \text{grade 1} & \text{grade 2} & \text{grade 3} & \text{grade 4} \end{array} \tag{2.2}$$

where $\sigma_k \equiv \gamma_k\gamma_0, k = 1 \ldots 3$, and $i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3$. The pseudoscalar i squares to -1 and anticommutes with all odd-grade elements. The $\{\sigma_k\}$ generate the geometric algebra of Euclidean 3-space, and are isomorphic to the Pauli matrices. An arbitrary real superposition of the basis elements (2.2) is called a 'multivector', and these inherit the associative Clifford product of the $\{\gamma_\mu\}$ generators. For a grade-r multivector A_r and a grade-s multivector B_s we define the inner and outer products via

$$A_r \cdot B_s \equiv \langle A_r B_s \rangle_{|r-s|}, \qquad A_r \wedge B_s \equiv \langle A_r B_s \rangle_{r+s}, \tag{2.3}$$

where $\langle M \rangle_r$ denotes the grade-r part of M. The subscript 0 will be left implicit when taking the scalar part of a multivector. We shall also make use of the commutator product,

$$A \times B \equiv \frac{1}{2}(AB - BA). \tag{2.4}$$

The operation of reversion, denoted by a tilde, is defined by

$$(AB)\tilde{} \equiv \tilde{B}\tilde{A}, \tag{2.5}$$

and the rule that vectors are unchanged under reversion. We adopt the convention that in the absence of brackets, inner, outer and commutator products take precedence over Clifford products.

We denote vectors in lower case Latin, a, or Greek for a set of basis vectors. A (coordinate) frame of vectors $\{e_\mu\}$ is generated from a set of coordinates $\{x^\mu(x)\}$

via $e_\mu \equiv \partial_\mu x$, where $\partial_\mu \equiv \partial/\partial x^\mu$. The reciprocal frame, denoted by $\{e^\mu\}$, satisfies $e_\mu \cdot e^\nu = \delta^\nu_\mu$. The vector derivative $\nabla (\equiv \partial_x)$ is then defined by

$$\nabla \equiv e^\mu \partial_\mu. \tag{2.6}$$

More generally, the vector derivative with respect to the vector a is denoted ∂_a. Further details concerning geometric algebra and the STA may be found in [24, 25].

The STA offers a unique geometric perspective of Dirac theory [26]. Dirac spinors $|\psi\rangle$ are conveniently represented by even multivectors ψ (see [27, 28] for an explicit map from the Dirac-Pauli representation). The $\{\hat{\gamma}_\mu\}$ operators, and the conventional unit scalar imaginary j have actions which are represented by:

$$\begin{aligned} \hat{\gamma}_\mu |\psi\rangle &\leftrightarrow \gamma_\mu \psi \gamma_0 \qquad (\mu = 0 \ldots 3) \\ j|\psi\rangle &\leftrightarrow \psi i \sigma_3. \end{aligned} \tag{2.7}$$

In this manner, all matrix manipulations are eliminated, revealing the true geometric content of the Dirac theory.

In gauge theory gravity, gravitational effects are described by the action of two gauge fields, $\overline{h}(a)$ and $\Omega(a)$. The first of these, $\overline{h}(a)$, is a position dependent linear function mapping the vector a to vectors. The overbar serves to distinguish the linear function from its adjoint $\underline{h}(a)$, where

$$\underline{h}(a) \equiv \partial_b \overline{h}(b) \cdot a. \tag{2.8}$$

The second gauge field, $\Omega(a)$, is a position dependent linear function mapping the vector a to bivectors. The gauge-theoretic purpose of these gauge fields is described in [19].

The covariant derivative $\mathcal{D}$ is assembled from the (flat space) vector derivative ∇ and the gravitational gauge fields. The action of $\mathcal{D}$ on a general multivector M is given by

$$\begin{aligned} \mathcal{D}M &\equiv \partial_a a \cdot \mathcal{D} M \\ &\equiv \partial_a (a \cdot \overline{h}(\nabla) M + \omega(a) \times M), \end{aligned} \tag{2.9}$$

where we have introduced the gauge field $\omega(a)$ (which also maps from vectors to bivectors) defined by

$$\omega(a) \equiv \Omega \underline{h}(a). \tag{2.10}$$

The covariant derivative contains a grade-raising and lowering component, so that we may write

$$\mathcal{D}M = \mathcal{D} \cdot M + \mathcal{D} \wedge M, \tag{2.11}$$

where

$$\mathcal{D}\cdot M \equiv \partial_a\cdot(a\cdot\mathcal{D}M), \qquad \mathcal{D}\wedge M \equiv \partial_a\wedge(a\cdot\mathcal{D}M). \tag{2.12}$$

The single-sided transformation law of spinors under rotations requires us to introduce a separate spinor covariant derivative $D\psi \equiv \partial_a a\cdot D\psi$ where

$$a\cdot D\psi \equiv a\cdot\overline{h}(\nabla) + \tfrac{1}{2}\omega(a)\psi. \tag{2.13}$$

The field strength corresponding to the $\Omega(a)$ gauge field is defined by

$$R(a\wedge b) \equiv a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega(a)\times\Omega(b). \tag{2.14}$$

From this we define the covariant Riemann tensor

$$\mathcal{R}(a\wedge b) \equiv R\underline{h}(a\wedge b). \tag{2.15}$$

The Ricci tensor, Ricci scalar and Einstein tensor are then given by:

$$\text{Ricci Tensor:} \quad \mathcal{R}(a) = \partial_b\cdot\mathcal{R}(b\wedge a) \tag{2.16}$$

$$\text{Ricci Scalar:} \quad \mathcal{R} = \partial_a\cdot\mathcal{R}(a) \tag{2.17}$$

$$\text{Einstein Tensor:} \quad \mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R}. \tag{2.18}$$

2.1 The Field Equations

The following equations describe a Dirac field of mass m coupled self-consistently to gravity [23]:

$$\text{`wedge':} \quad \mathcal{D}\wedge\overline{h}(a) = \kappa\mathcal{S}\cdot\overline{h}(a) \tag{2.19}$$

$$\text{Einstein:} \quad \mathcal{G}(a) = \kappa\mathcal{T}(a) \tag{2.20}$$

$$\text{Dirac:} \quad D\psi i\sigma_3 = m\psi\gamma_0, \tag{2.21}$$

where $\mathcal{S} \equiv \frac{1}{2}\psi i\gamma_3\tilde{\psi}$ is the spin trivector, and $\kappa \equiv 8\pi$. The matter stress-energy tensor $\mathcal{T}(a)$ is given by

$$\mathcal{T}(a) = \langle a\cdot D\psi i\gamma_3\tilde{\psi}\rangle_1. \tag{2.22}$$

These equations are the GTG analogues of the Einstein-Cartan-Dirac equations. The 'wedge' equation (2.19) is algebraic in $\omega(a)$ and may be solved to give [19]

$$\omega(a) = -\overline{h}(\nabla\wedge\overline{h}^{-1}(a)) + \tfrac{1}{2}a\cdot[\partial_b\wedge\overline{h}(\nabla\wedge\overline{h}^{-1}(b))] + \tfrac{1}{2}\kappa a\cdot\mathcal{S}. \tag{2.23}$$

This completes the definitions required for this paper.

3 A Massive Solution

We aim to find a self-consistent solution for a massive Dirac field ψ that is both homogeneous and isotropic at the level of classical fields. Since classical fields couple to gravity via the $\overline{h}$-function only, they do not feel the anisotropy of the ω-function which arises because of the spin of the Dirac field.

We start by introducing a set of polar coordinates:

$$\begin{aligned} t &\equiv x\cdot\gamma_0 & \cos\theta &\equiv x\cdot\gamma^3/r \\ r &\equiv \sqrt{(x\wedge\gamma_0)^2} & \tan\phi &\equiv (x\cdot\gamma^2)/(x\cdot\gamma^1). \end{aligned} \tag{3.1}$$

We shall make use of the vectors

$$e_t \equiv \gamma_0 \qquad e_r \equiv x\wedge\gamma_0\gamma_0/r, \tag{3.2}$$

which are members of the polar coordinate frame. Given the assumed symmetry at the level of the $\overline{h}$-function, we may choose a gauge in which the $\overline{h}$-function takes the form [19]

$$\overline{h}(a) = a\cdot e_t e_t + \alpha(t)(1 + kr^2/4)a\wedge e_t e_t, \tag{3.3}$$

where α^{-1} is the scale factor of the universe, and $k = -1, 0, 1$ for open, flat and closed universes respectively. This $\overline{h}$-function generates the 'isotropic' line element [19]

$$ds^2 = dt^2 - \alpha^{-2}(1 + kr^2/4)^{-2}(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)). \tag{3.4}$$

In this gauge, the surfaces of homogeneity have $t =$ constant, the fundamental observers have covariant velocity e_t, and t is cosmic time.

Using equation (2.23), we obtain $\omega(a)$ in the following form:

$$\omega(a) = H(t)a\wedge e_t - \tfrac{1}{2}kr\alpha(t)e_r\wedge(a\wedge e_t e_t) + \tfrac{1}{2}\kappa a\cdot\mathcal{S}, \tag{3.5}$$

where $H(t) \equiv -\dot{\alpha}/\alpha$ is the Hubble parameter, with overdots denoting ∂_t. We can now write the Dirac equation (2.21) in the form

$$(e_t\partial_t + \alpha(1 + kr^2/4)e_t e_t\wedge\nabla + \tfrac{3}{2}He_t + \tfrac{1}{2}kr\alpha e_r + \tfrac{3}{4}\kappa\mathcal{S})\psi i\sigma_3 = m\psi\gamma_0. \tag{3.6}$$

Following Isham and Nelson [6] we demand that the (gauge-invariant) observables formed from ψ, such as the projection of the Dirac current onto the velocity of the fundamental observers, should themselves be homogeneous. For the present gauge choice this requires that we take $\psi = \psi(t)$ only. If we substitute this into equation (3.6), then the only term with any dependence on the coordinates (r, θ, ϕ) is $\frac{1}{2}kr\alpha e_r\psi i\sigma_3$. It follows that if we require the observables associated with the Dirac field to be homogeneous, then the universe must be spatially flat ($k = 0$). This conclusion was first given in [6]. Note that this conclusion would still be reached if we attempted to solve the Dirac equation non self-consistently on a homogeneous gravitational background. This is because the only change to (3.6) would be that $\mathcal{S}$ would now be the torsion trivector of the background (so would not be equated to the spin of the Dirac field). As noted in [19], this argument is more restrictive than the self-consistency argument since it does not assume any particular model for the source of the homogeneous background. In the remainder of this paper we restrict attention to solutions with $k = 0$.

The gauge choice made above was motivated by the requirement that the $\overline{h}$-function should be globally defined, for all choices of k. Now that we have restricted k to be zero, it is convenient to perform the position-gauge transformation defined by the displacement [19]

$$x' = f(x) \equiv x \cdot e_t e_t + \alpha x \wedge e_t e_t. \tag{3.7}$$

This brings the $\overline{h}$-function to the simple form

$$\overline{h}(a) = a + rH(t) a \cdot e_r e_t, \tag{3.8}$$

which generates the line element

$$ds^2 = (1 - r^2 H^2) dt^2 + 2rH dt dr - (dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)). \tag{3.9}$$

This gauge choice, which we refer to as the 'Newtonian gauge', was applied to several spherically symmetric problems in [19]. The ω-function transforms to give

$$\omega(a) = H(t) a \wedge e_t + \tfrac{1}{2}\kappa a \cdot \mathcal{S}, \tag{3.10}$$

and the homogeneous field ψ is unchanged. The surfaces of homogeneity still have $t =$ constant, and the covariant velocity of the fundamental observers is still e_t, but their radial coordinates are now proportional to the scale factor α^{-1}.

The Riemann tensor is easily evaluated using the results of Section 3 in [23]. We find that

$$\mathcal{R}(B) = -\dot{H} B \cdot e_t e_t - H^2 B + \tfrac{1}{4}\kappa^2 B \cdot \mathcal{S}\mathcal{S} - \tfrac{1}{2}\kappa (B \cdot \mathcal{D}) \cdot \mathcal{S}, \tag{3.11}$$

for an arbitrary bivector B. It follows that the Einstein tensor is given by

$$\mathcal{G}(a) = 2\dot{H} a \wedge e_t e_t + 3H^2 a - \tfrac{1}{2}\kappa a \cdot (\mathcal{D} \cdot \mathcal{S}) + \tfrac{1}{2}\kappa^2 a \cdot \mathcal{S}\mathcal{S} - \tfrac{3}{4}\kappa^2 \mathcal{S}^2 a. \tag{3.12}$$

The stress-energy tensor $\mathcal{T}(a)$ evaluates to

$$\mathcal{T}(a) = \langle a \cdot e_t \dot{\psi} i\gamma_3 \tilde{\psi} + H a \wedge e_t \mathcal{S} + \tfrac{1}{2}\kappa a \cdot \mathcal{S}\mathcal{S} \rangle_1, \tag{3.13}$$

where we have used the fact that ψ is a function of t alone. Finally, the Dirac equation becomes

$$(e_t\partial_t + \tfrac{3}{2}He_t + \tfrac{3}{4}\kappa\mathcal{S})\psi i\sigma_3 = m\psi\gamma_0, \tag{3.14}$$

from which we deduce

$$\begin{aligned} \dot{\mathcal{S}} &= -3H\mathcal{S} - m\langle\psi\tilde{\psi}\rangle_4\gamma_0 \\ \implies \quad \mathcal{D}\cdot\mathcal{S} &= -2He_t\cdot\mathcal{S}. \end{aligned} \tag{3.15}$$

On substituting equation (3.15) into the expression for $\mathcal{G}(a)$, and $\dot{\psi}$, from the Dirac equation, into $\mathcal{T}(a)$, we may write the Einstein equation (2.20) as

$$2\dot{H}a\wedge e_te_t + 3aH^2 + \tfrac{3}{4}\kappa\mathcal{S}^2e_tae_t - m\kappa a\cdot e_t\langle\psi\tilde{\psi}\rangle e_t = 0, \tag{3.16}$$

where $\langle M\rangle$ denotes the scalar part of M. From (3.16), we deduce the pair of scalar equations

$$3H^2 + \tfrac{3}{4}\kappa^2\mathcal{S}^2 - m\kappa\langle\psi\tilde{\psi}\rangle = 0 \tag{3.17}$$

$$3H^2 - \tfrac{3}{4}\kappa^2\mathcal{S}^2 + 2\dot{H} = 0. \tag{3.18}$$

It is straightforward to show that the update equation (3.18) is implied by the Dirac equation (3.14) and the constraint (3.17). It follows that the system of equations is consistent, and we need only solve equation (3.14) subject to the constraint (3.17).

If torsion is not included in the model, then (3.18) is replaced by

$$3H^2 + 2\dot{H} = 0. \tag{3.19}$$

Comparing with (3.18), and noting that $\mathcal{S}^2 \leq 0$, we see that the effect of torsion is to make $\dot{H}$ more negative for a given value of the Hubble parameter H. It follows that torsion enhances singularity formation in the sense that for a given value of the current Hubble parmeter, the universe will be younger if torsion is included in the model. This conclusion was reached in [14] on the basis of the singularity theorems in the presence of torsion.

Note that if $m = 0$, then equations (3.17) and (3.18) may be solved immediately to give

$$H(t) = \frac{1}{3t}, \tag{3.20}$$

where we have employed the gauge freedom to perform a constant displacement along e_t, to put the singularity at $t = 0$. This solution was found by Isham and Nelson in [6].

To make further progress, we consider the case $\psi\tilde{\psi} \neq 0$ (solutions with $\psi\tilde{\psi} = 0$ can be obtained from solutions of the Einstein-Dirac equations using the results in Seitz [7]). We may then parameterise ψ as [26]

$$\psi \equiv \sqrt{\rho} e^{i\beta/2} R, \tag{3.21}$$

where ρ (> 0) and β are scalar functions of t. The 'rotor' R is an even-grade (t-dependent) element, which satisfies $R\tilde{R} = 1$. Substituting (3.21) into the Dirac equation (3.14) and equating grades on either side, we find:

$$\dot{\rho} = 4m \sin\beta \, i\gamma_0 \wedge \mathcal{S} - 3\rho H \tag{3.22}$$

$$\rho\dot{\beta} = 4(m \cos\beta + 3\pi\rho) i\gamma_0 \wedge \mathcal{S} \tag{3.23}$$

$$\rho \dot{R}\tilde{R} = -2(me^{-i\beta} + 3\pi\rho)\gamma_0 \cdot \mathcal{S}, \tag{3.24}$$

where we have used the fact that $\dot{R}\tilde{R}$ is a bivector. The spin trivector $\mathcal{S}$ is given by $\frac{1}{2}\rho R i\gamma_3 \tilde{R}$. We shall not attempt the general solution of these equations. However, an important simplification occurs if we take $\sin\beta = 0$. Since β is then constant, we see from (3.23) that $\gamma_0 \wedge \mathcal{S} = 0$ since a non-zero, constant ρ is forbidden by equation (3.22). It follows that we must solve

$$\dot{\rho} = -3\rho H \tag{3.25}$$

$$\dot{R} = -(m \cos\beta + 3\pi\rho)\gamma_0 R i\gamma_3, \tag{3.26}$$

subject to the constraint

$$3H^2 - 12\pi^2\rho^2 - 8\pi m\rho \cos\beta = 0. \tag{3.27}$$

In these equations $\cos\beta = \pm 1$. Solutions with $\cos\beta = 1$ are usually regarded as 'particle' (positive energy) solutions, and those with $\cos\beta = -1$ as 'antiparticle' (negative energy) solutions in the absence of gravity [28].

Equations (3.25) and (3.27) have the solution

$$\rho(t) = \frac{1}{6\pi t(1 \pm mt)}, \qquad H(t) = \frac{1 \pm 2mt}{3t(1 \pm mt)}, \tag{3.28}$$

where the $\pm$ signs are for $\cos\beta = \pm 1$ respectively. Again, we have exploited the gauge freedom to put one of the singularities at $t = 0$. These solutions are only valid for $\rho > 0$. This requirement produces a curious asymmetry between the $\cos\beta = +1$ and the $\cos\beta = -1$ solutions. To see this we first consider the case where $\cos\beta = +1$. This solution is valid for $mt > 0$ or $mt < -1$. In the former case, the Universe expands from an initial singularity at $t = 0$. For $mt \gg 1$, $H(t)$ tends to the value for a universe filled with dust (the Einstein-deSitter model). During this epoch, the scale factor of the universe varies as $t^{2/3}$. This $t^{2/3}$ behaviour is found for all

t (> 0) if torsion is not included in the model [6]. We see that torsion effects are only important while the Compton wavelength of the field is larger than the Hubble radius, and that the effect of torsion is to enhance singularity formation. During this epoch ($mt \ll 1$), the scale factor varies as $t^{1/3}$. The model clearly has a particle horizon. The solution valid for $mt < -1$ describes an (unphysical) collapsing universe, which is singular at $mt = -1$. If instead we consider $\cos\beta = -1$, then the solution is valid for $0 < mt < 1$. It describes a universe that expands from an initial singularity at $t = 0$, turns around at $mt = \frac{1}{2}$, and then contracts to a singularity at $mt = 1$. A particle horizon is also present in this model, which continues to exist right up to the singularity at the endpoint of the collapse.

It remains to solve the rotor equation (3.26). We begin by noting that

$$\begin{aligned} & \gamma_0 \wedge \mathcal{S} = 0 \\ \Longrightarrow \quad & \gamma_3 \cdot (\tilde{R}\gamma_0 R) = 0. \end{aligned} \tag{3.29}$$

The rotor R may be decomposed into a product of rotors $R \equiv \Phi L$, where Φ and L are uniquely determined by the requirements that $\Phi\gamma_0 = \gamma_0\Phi$ and $L\gamma_0 = \gamma_0\tilde{L}$. Decompositions of this type are useful when discussing tetrad transport. In particular, a tetrad with covariant velocity γ_0 is spatially rotated by Φ and boosted by L. The condition (3.29) restricts L to the form

$$L = e^{\xi(\sin\eta\, \sigma_1 + \cos\eta\, \sigma_2)/2}, \tag{3.30}$$

where ξ and η are scalars. However, from the rotor equation (3.26), we find that the vector $\tilde{R}\gamma_0 R$ is time independent. Since L is determined uniquely by this vector and γ_0, we find that L must be constant. If we now substitute $R = \Phi L$ into equation (3.26) and note that L commutes with $i\gamma_3$, we find that

$$\dot{\Phi} = -\dot{\chi}\Phi i\sigma_3, \tag{3.31}$$

where the scalar $\chi(t)$ is defined by

$$\dot{\chi} \equiv m\cos\beta + 3\pi\rho. \tag{3.32}$$

Equation (3.31) has the solution

$$\Phi(t) = \Phi_0 e^{-i\sigma_3\chi(t)}, \tag{3.33}$$

where Φ_0 is a constant rotor that commutes with γ_0. This rotor may be eliminated by combining the global position-gauge transformation $x' = \Phi_0 x\tilde{\Phi}_0$, with the global rotation-gauge transformation defined by the rotor $\tilde{\Phi}_0$. Any constant of integration that results from integrating equation (3.32) may also be gauged away in a similar fashion.

The solution to equation (3.32) is

$$\chi(t) = \pm mt - \frac{1}{2}\ln\left(\frac{1 \pm mt}{mt}\right), \tag{3.34}$$

with the $\pm$ signs for $\cos\beta = \pm 1$. The integration constant is chosen so that χ is defined for $\rho > 0$. The general solution for ψ with $\sin\beta = 0$ may then be written as

$$\psi(t) = \sqrt{\rho(t)} e^{i\beta/2} e^{-i\sigma_3 \chi(t)} e^{\xi(\sin\eta\, \sigma_1 + \cos\eta\, \sigma_2)/2}, \tag{3.35}$$

with $\beta = 0$ or π. The spin trivector is given by $\mathcal{S} = \frac{1}{2}\rho i\gamma_3$. The presence of this preferred direction induces anisotropy in the 3-spaces of homogeneity. In the next section we show how this anisotropy can be detected by a spinning point-particle. The constants ξ and η encode gauge-invariant information. For example, the density $\mathcal{J}\cdot\gamma_0$ (where $\mathcal{J} \equiv \psi\gamma_0\tilde{\psi}$ is the covariant Dirac current) 'measured' by the fundamental observers, is proportional to $\cosh\xi$. Note also that the current $\mathcal{J}$ is not along the velocity of the fundamental observers unless $\xi = 0$. It follows that the current defines another preferred direction in the 3-spaces of homogeneity, but this anisotropy is not present in the gravitational gauge fields. It seems unlikely that this anisotropy would be observable for the electrically neutral field considered here. The symmetry implied by the freedom in the choice of ξ and η is a property of the free Dirac equation (it is a continuous charge conjugation), and this is not removed by coupling to gravity. The symmetry is broken if electromagnetic effects are included, suggesting that the inclusion of electromagnetic (self) interaction would remove the anisotropy in the current.

The stress-energy tensor $\mathcal{T}(a)$ is given by

$$\begin{aligned} \mathcal{T}(e_t) &= \rho(2\pi\rho \pm m)e_t \\ \mathcal{T}(\gamma_3) &= 0 \\ \mathcal{T}(\gamma_1) &= +\tfrac{1}{2}\rho H\gamma_2 - \pi\rho^2\gamma_1 \\ \mathcal{T}(\gamma_2) &= -\tfrac{1}{2}\rho H\gamma_1 - \pi\rho^2\gamma_2. \end{aligned} \tag{3.36}$$

As expected, e_t is a timelike eigenvector of $\mathcal{T}(a)$. The only other real eigenvector is γ_3 which is dual to the spin trivector $\mathcal{S}$ (in the STA, the duality operation is performed by the pseudoscalar i). The stress-energy tensor singles out two directions in spacetime as being algebraically special. This reflects the anisotropy of the solution at the level of the gravitational gauge fields. The energy density $\mathcal{T}(e_t)\cdot e_t$ measured by the fundamental observers evaluates to

$$\mathcal{T}(e_t)\cdot e_t = \rho(2\pi\rho \pm m). \tag{3.37}$$

This density is positive, for both $\beta = 0$ and $\beta = \pi$, over the range ($\rho > 0$) for which the solutions are valid. The density parameter (with respect to an Einstein-deSitter universe) is defined by

$$\Omega \equiv \frac{\mathcal{T}(e_t)\cdot e_t}{(3H^2/8\pi)}, \tag{3.38}$$

which evaluates to

$$\Omega = \frac{4 + mt(mt \pm 1)}{3 + mt(mt \pm 1)} \\ = 1 + \mathrm{O}(1/m^2t^2). \tag{3.39}$$

Of course, although $\Omega \neq 1$ for early times ($mt \ll 1$) the universe remains spatially flat.

4 Spinning Point-Particle Trajectories

The presence of torsion introduces anisotropy at the level of the $\omega(a)$ gauge field in the cosmological solutions presented in the previous section. Since classical point-particles follow geodesics (which are dependent only on the $\overline{h}$-function) their trajectories will be insensitive to the presence of torsion. However, in [23] we presented a model for a spinning particle which coupled directly to the ω-field, and hence was sensitive to the presence of torsion. In this section, we present some numerical results for an ideal experiment performed with an ensemble of spinning point-particles in the expanding $\beta = 0$ solution. The anisotropy due to torsion is clear in these results. For the remainder of this section, overdots denote differentiation with respect to λ, the parameter along the path of the particle.

The particle, of mass m_p, is described by its position $x(\lambda)$, a spinor $\Psi(\lambda)$, and an independent momentum $p(\lambda)$. We include an einbein $e(\lambda)$ to ensure reparameterisation invariance. The einbein may be chosen arbitrarily, but we shall find it convenient to make the choice so that λ measures proper time for the particle. The equations of motion, which are derivable from a gauge-invariant action, are given by [23]:

$$v\cdot\mathcal{D}p = \tfrac{1}{2}v\cdot\overline{\mathcal{R}}(S) + \kappa v\cdot\mathcal{S}\cdot p \tag{4.1}$$

$$v\cdot D\Psi i\sigma_3 = m_p e p \Psi \gamma_0 \tag{4.2}$$

$$v = m_p e \Psi \gamma_0 \tilde{\Psi} \tag{4.3}$$

$$p\cdot v = e m_p^2, \tag{4.4}$$

where $v \equiv \underline{h}^{-1}(\dot{x})$ is the covariant tangent vector to the path, and $S \equiv \Psi i\sigma_3\tilde{\Psi}$ is the spin bivector for the particle. The tensor $\overline{\mathcal{R}}(B)$ appearing in (4.1) is the adjoint to the Riemann tensor $\mathcal{R}(B)$, and is defined by

$$\overline{\mathcal{R}}(a\wedge b) \equiv \tfrac{1}{2}\partial_d\wedge\partial_c\langle a\wedge b\mathcal{R}(c\wedge d)\rangle. \tag{4.5}$$

We consider an experiment where an ensemble of particles are prepared, with each particle in a helicity eigenstate as viewed by the fundamental observers. They then move freely for a given length of proper time, after which each particle determines the cosmic time at its current spacetime position. The initial direction of motion varies across the ensemble, but the speed relative to the fundamental observers is the same for each particle.

For a typical particle, take the initial spinor Ψ_0 to be the rotor

$$\Psi_0 = e^{\zeta i\sigma_2/2} e^{u\sigma_3/2}, \tag{4.6}$$

and take the initial value of the einbein to be $1/m_p$. The initial velocity v_0 evaluates to

$$v_0 \equiv m_p e \Psi_0 \gamma_0 \tilde{\Psi}_0 = \cosh u \, \gamma_0 + \sinh u \, (\cos\zeta \, \gamma_3 - \sin\zeta \, \gamma_1), \tag{4.7}$$

whilst the initial spin vector $s_0 \equiv \Psi_0 \gamma_3 \tilde{\Psi}_0$ is

$$s_0 = \sinh u \, \gamma_0 + \cosh u \, (\cos\zeta \, \gamma_3 - \sin\zeta \, \gamma_1). \tag{4.8}$$

We see that $s_0 \wedge e_t \propto v_0 \wedge e_t$, which ensures that the particle is initially in a helicity eigenstate for the fundamental observers. The initial speed relative to the fundamental observers is $\tanh u$, and u is the same for each particle. The parameter ζ controls the angle that the initial 3-velocity makes with the torsion direction (the γ_3 direction in the current gauge choice), and so this parameter is varied across the ensemble. We take the initial momentum to be $p_0 \equiv m_p v_0$.

An analytic treatment is possible for $\zeta = 0$ (initial 3-velocity along the torsion direction). The solution, for the given initial conditions, is

$$\Psi(\lambda) = e^{u(\lambda)\sigma_3/2} e^{\tau(\lambda) i\sigma_3}, \tag{4.9}$$

with momentum $p = m_p v$ and $e = 1/m_p$. Substituting (4.9) into the update equation for Ψ (equation (4.2)) we find the scalar equations

$$\dot{u} = -H(t) \sinh u \tag{4.10}$$

$$\dot{\tau} = -m_p - \pi\rho(t) \cosh u \, . \tag{4.11}$$

Equation (4.1) reduces to the geodesic equation $v \cdot \mathcal{D} v = 0$, on account of the velocity and spin being along the torsion direction. However, the update equation for Ψ implies that the geodesic deviation, $v \cdot \mathcal{D} v$, is proportional to $p \cdot S$, where S is the spin bivector,

$$S \equiv \Psi i\sigma_3 \tilde{\Psi} = i\sigma_3. \tag{4.12}$$

Since Ψ is a rotor, we have that $v \cdot S = 0$, and so the implied geodesic deviation vanishes. It follows that Ψ and p satisfy their respective equations of motion if conditions (4.10) and (4.11) are satisfied. The resulting motion is entirely classical; the particle moves on a geodesic, the spin undergoes parallel transport (it actually remains constant), and $p = m_p v$.

The remaining equation that we require is

$$v \equiv \underline{h}^{-1}(\dot{x}) \quad \Longrightarrow \quad \dot{t} = \cosh u \, . \tag{4.13}$$

It is a consequence of the homogeneity of the universe that the equations for $\dot{x}\wedge e_t$ may be integrated independently, once $u(\lambda)$ and $t(\lambda)$ are known. Equations (4.10) and (4.13) imply that

$$\partial_t u = -H(t) \tanh u, \tag{4.14}$$

from which we find $\sinh u \propto \alpha(t)$ (recall that α is the reciprocal scale factor defined by $H \equiv -\partial_t \alpha / \alpha$). It follows that the particle's proper time λ approaches cosmic time (up to a constant) for $mt \gg 1$. Returning to equation (4.11), we see that

$$\partial_t \tau = -m_p \text{sech} u \; - \pi \rho(t). \tag{4.15}$$

Integrating this equation from $t = 0$ (the initial singularity), we find that the phase of Ψ has made an infinite number of oscillations since the singularity. This should be contrasted to the finite proper time that the particle takes to reach the singularity. This effect is due solely to interaction with the torsion — in the absence of torsion, the second term on the right-hand side of equation (4.11) is absent, and the phase varies linearly with the particle's proper time.

We may now integrate (4.13) numerically to find the elapse of cosmic time during the experiment. We take $m = 1, m_p = 0.1$, and the initial value of u to be 0.1. We launch the particle from $t = 0.2$ for a proper time of 10 units, after which the cosmic time is $t_f = 10.204$.

For $\zeta \neq 0$, we find that Ψ does not remain a rotor, and p does not remain collinear with v. In this case we must integrate the update equations for Ψ, p and t numerically (the equations for $\dot{x}\wedge e_t$ may be integrated independently afterwards), to find the final value of cosmic time t_f. Figure 1 shows the variation of t_f with the parameter ζ. It is clear that such an experiment with an ensemble of spinning point-particles would reveal the anisotropy associated with the torsion component of the ω-function.

5 Conclusions

We have derived new massive, cosmological solutions for the Dirac field coupled self-consistently to gravity. The solutions are derived from a gauge-theoretic approach to gravity, which naturally identifies the spin of the Dirac field with the torsion of the gravitational gauge fields. The solutions are homogeneous, and isotropic at the level of classical fields.

The solution with $\beta = \pi$ describes a universe which expands from an initial singularity, before turning around and recollapsing in a time inversely proportional

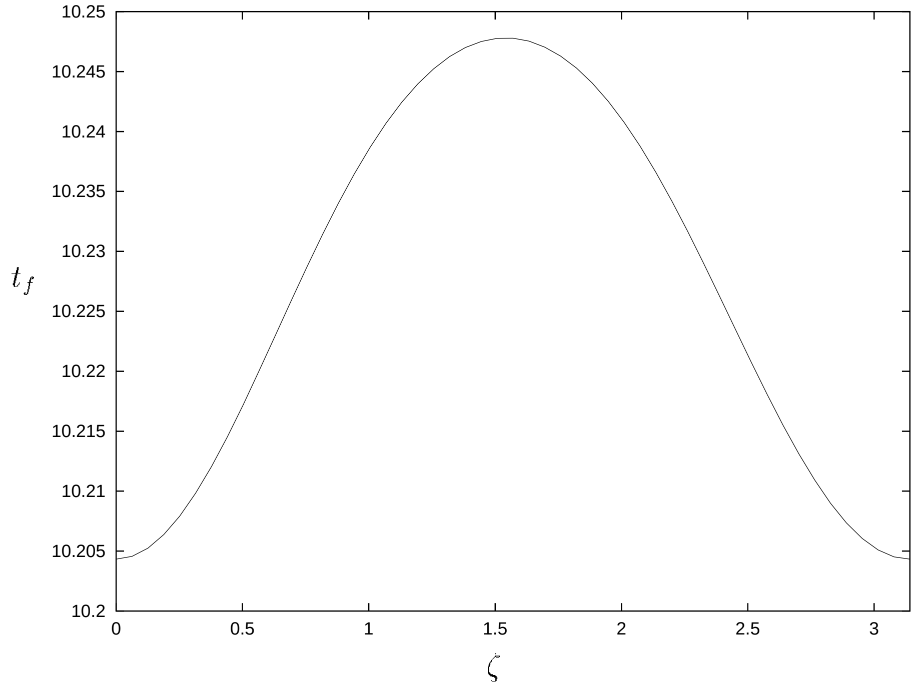

Figure 1: *The final value* t_f *of cosmic time against the angle parameter* ζ. *The value* $\zeta = 0$ *corresponds to an initial relative velocity along the torsion direction. The particle is released at* $t = 0.2$ *and travels for a proper time of* 10 *units. All times are in units of the Planck time.*

to the mass of the Dirac field. For $\beta = 0$, we found a solution that describes an expanding universe. The evolution for late times approaches a dust filled universe ($H(t) = 2/3t$). Torsion is negligible once the Compton wavelength of the Dirac field comes inside the Hubble radius. For early times ($mt \ll 1$) the effect of torsion is to reduce the Hubble parameter for a given cosmic time, thus reducing the age of the universe for a given value of the Hubble parameter. Particle horizons are present in both of these solutions. The spin of the Dirac field leads to anisotropy in the solution, which although present in the eigen-structure of the Einstein tensor, would not be detectable by classical non-spinning point particles. The Dirac current itself picks out a further preferred spatial direction, but this does not induce further anisotropy in the gravitational gauge field. We expect this further anisotropy would not be present in a self-consistent calculation for an electrically charged Dirac field, where anisotropy at the level of the current would have observational consequences.

We exploited the gravitational-coupling of a spinning point-particle to probe the anisotropy due to the spin of the Dirac field. We have demonstrated the anisotropy of the expanding $\beta = 0$ universe, by comparing the elapse of cosmic time for point-particle trajectories of fixed proper time. As an interesting aside, we showed that the particle's phase makes an infinite number of oscillations as it emerges from the singularity.

The presence of a particle horizon, but no inflationary phase is problematic for the solutions presented here. It would be of interest to reconsider the model with a cosmological constant, or vacuum polarisation terms included. It has been shown that, in the framework of the Einstein-Dirac equations, such inclusions can provide an alternative to standard inflationary models [1, 2].

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