THE PHYSICS OF ROTATING CYLINDRICAL STRINGS

Chris Doran, Anthony Lasenby and Stephen Gull

MRAO, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, UK.

August 8, 1996

Abstract

A new gauge-theoretic approach to gravity is applied to the study of rotating cylindrically-symmetric strings. The interior and exterior equations are reduced to a simple set of first-order differential equations, and suitable matching conditions are obtained. The gauge theory formulation affords a clear understanding of the physical observables in the theory, and provides simple conditions for the properties of the fields on the string axis. In this context some errors in previously published work are exposed. Three situations are discussed: the vacuum region; pressure-free matter; and a (2+1)-dimensional 'ideal fluid'. In each case a set of analytic solutions is presented. It is shown that if the fluid is rotating rigidly then closed timelike curves are inevitable, despite the fact that the matter satisfies the weak energy condition.

1 Introduction

In recent years there has been considerable interest in gravity in (2+1) dimensions, and in the string-like solutions obtained when these fields are lifted to (3+1)-dimensional spacetime. Of particular significance are the solutions obtained in spacetime where the matter source is provided by a relativistic scalar field. Such solutions model the gravitational field due to a cosmic string — a topological defect formed in the early universe when GUT-scale symmetries are spontaneously broken [1]. If present, these defects could be responsible for seeding galaxy formation. In (2+1) dimensions the Riemann tensor is determined solely by the matter stress-energy tensor and there is no analogue of the Weyl tensor. It follows that the vacuum region outside a string has vanishing Riemann tensor and can only differ from Minkowski spacetime in its global, topological properties.

In this paper we apply a new approach to gravity to the study of extended string solutions. This approach employs gauge fields in a flat Minkowski spacetime to ensure that all relations between physical quantities are independent of the position and orientation of the matter fields [2, 3]. It may seem surprising that our approach can deal successfully with fields of a purely topological nature, so we shall demonstrate here that the topological picture can indeed be replaced by an equally simple description in terms of gauge fields. This has the advantage that the description of the gravitational effects of strings closely mirrors that of the Aharonov-Bohm effect in quantum mechanics. It is further argued that for rotating strings the gauge field picture is clearer than the topological one, since the latter introduces a periodic time coordinate — a somewhat problematical concept. The gauge theory approach has two further advantages. First, the correct matching conditions at the string boundary are clear. Second, the conditions which the fields must satisfy on the axis are simply stated and applied. These replace the less concrete notion of 'elementary flatness', an idea which we show has led to mistakes in this field in the past.

Our paper starts with a brief summary of the gauge theory formulation of gravity. The relevant equations for a cylindrically-symmetric, time-independent system are then found. In this paper we restrict attention to the case where the z-direction drops out of the dynamics entirely, so that we effectively model gravity in (2+1)-dimensions. Imposing this as the only further restriction means that many of the solutions we discuss do not have the property of boost invariance usually demanded of cosmic string solutions. The situations we discuss are therefore closer to those treated by Deser *et al.* [4] and Jensen & Soleng [5]. These are relevant to gravity in (2+1)-dimensions and may yet be useful for spinning strings in (3+1)-dimensions, as it is not clear that boost invariance can be imposed for spinning strings (though see [6] for one possibility).

The field equations we obtain form a set of four coupled first-order differential equations, which are then studied for various matter configurations. We first study the vacuum equations and show that the solutions fall into two distinct gauge sectors. This classification is new, and arises from simple algebraic considerations rather than topological constructions. The first sector includes the solutions found by Deser *et al.* [4], who constructed point-particle solutions in (2+1) dimensional gravity. The second sector includes solutions which do not appear to have been considered before (possibly because their topological construction is far from obvious). These new solutions have the novel property of exerting an attractive force in the vacuum, despite the absence of a Weyl tensor. Only the limiting case of vanishing angular momentum has been given before, although in a somewhat different setting [7].

We next find simple expressions for the vorticity, shear and angular momentum of the fields. These expressions enable us to construct models by imposing physically sensible restrictions on the string. We then examine a series of models in which the matter is treated as a 2-dimensional ideal fluid with a tension in the third direction. We first look at models having no pressure, which turns out to imply that the string has no angular momentum. We next look at models with pressure but no angular momentum, and find a configuration which matches onto our new class of vacuum solutions. We end with a discussion of solutions for rotating strings. The first such solution for a finite width string was given by Jensen & Soleng [5], but we show here that their field configuration produces an unphysical stress-energy tensor, which is ill-defined on the axis of the string. This problem is easily overcome, and we present a set of analytic solutions for rigidly rotating (shear free) cosmic strings. These solutions are well-defined throughout the interior, and match smoothly onto the vacuum. It is shown that these fields necessarily give rise to closed timelike curves, despite the fact that the source matter satisfies the weak energy condition in (2+1) dimensions. Geodesic plots are also presented, which show the dragging effect of the string rotation, as well as some bizarre acausal effects. The physical relevance of these solutions is discussed further in the concluding section.

2 Gauge Theory Gravity

A satisfactory gauge theory formulation of gravity was first given by Kibble [8], who elaborated on an earlier, unsuccessful attempt by Utiyama [9]. The gauge theory approach leads naturally to an extended version of general relativity (GR) known as a spin-torsion theory [10]. In the absence of any sources of spin, however, it has always been assumed that the gauge theory approach and GR are identical. In [2] we reconsidered the gauge theory formulation using the powerful language of 'geometric algebra' [11, 12]. With this we developed a gauge theory built on arbitrary finite transformations, rather than on infinitesimal Poincaré transformations as was previously the case. It was argued that the Poincaré group does not provide the correct framework for the development of a gauge theory of gravity, and that it is only through considering Dirac fields that one can fully understand the need to gauge the Lorentz group.

The result of the approach developed in [2] is a theory containing two separate gauge fields, one for arbitrary displacements in a flat Minkowski spacetime, and one for Lorentz rotations at a point. The field equations are a set of first-order differential relations between these fields. If the matter has no spin, any solution to the field equations can be used to generate a metric which solves the Einstein equations. The converse is not always true, however; metrics exist in GR which cannot be reached from the gauge-theoretical starting point. The result is a theory which is slightly more restrictive than full GR, although entirely consistent with experiment. The construction of the theory from the Dirac equation also ensures that it is consistent with quantum theory at the level of relativistic wave mechanics.

Although the differences between the gauge theory approach to gravity and GR are potentially very significant, of greater relevance here is that, in situations where the theories do agree, the gauge theory approach is far simpler to compute with. In [2] we described a new method of solving the field equations by eliminating the position gauge field and concentrating on the rotation gauge fields. This approach is not available in traditional GR, which only works with quantities which are rotation-gauge invariant. This new method enhances the similarities between gravitation and Yang-Mills gauge theories. It works directly with the physically-relevant quantities in the theory, and always yields a set of first-order differential equations. These equations are non-linear, but in a way that is much simpler to control than the second-order equations for the metric coefficients obtained in standard GR. Furthermore, the gauge-theory formalism eliminates any possible ambiguity from the physical predictions of the theory, since all physical quantities must be defined in a gauge-invariant way.

In this paper we apply the method described in [2] to the case of cylindrically-symmetric systems. Summation convention and natural units ($G = c = \hbar = 1$) are employed throughout. It is helpful to start by introducing the 'spacetime algebra' — the geometric (or Clifford) algebra of spacetime. This will be familiar to physicists in the guise of the algebra of the Dirac γ-matrices. The spacetime algebra (STA) is generated by four vectors $\{\gamma_\mu\}$, $\mu = 0 \ldots 3$, with an associative (Clifford) product denoted by juxtaposition. The symmetrised product is denoted with a dot and satisfies

$$\gamma_\mu \cdot \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \eta_{\mu\nu} = \mathrm{diag}(+ - - -). \tag{2.1}$$

The remaining, antisymmetric part is denoted with a wedge and generates a bivector (a directed plane segment),

$$\gamma_\mu \wedge \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{2.2}$$

A full basis for the STA is provided by the set

$$\begin{array}{ccccc} 1 & \{\gamma_\mu\} & \{\sigma_k, i\sigma_k\} & \{i\gamma_\mu\} & i \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar,} \\ \text{grade 0} & \text{grade 1} & \text{grade 2} & \text{grade 3} & \text{grade 4} \end{array} \tag{2.3}$$

where $\sigma_k \equiv \gamma_k\gamma_0$, $k = 1 \ldots 3$, and $i \equiv \gamma_0\gamma_1\gamma_2\gamma_3$. The pseudoscalar i squares to -1, anticommutes with all odd-grade elements and commutes with even grade elements. An arbitrary real superposition of the basis elements (2.3) is called a 'multivector' and these inherit the associative Clifford product of the $\{\gamma_\mu\}$ generators. For a grade-r multivector A_r and a grade-s multivector B_s we define the products

$$A_r \cdot B_s \equiv \langle A_r B_s \rangle_{|r-s|}, \qquad A_r \wedge B_s \equiv \langle A_r B_s \rangle_{r+s}, \tag{2.4}$$

where $\langle M \rangle_r$ denotes the projection onto the grade-r components of M. We also employ the commutator product,

$$A \times B \equiv \tfrac{1}{2}(AB - BA). \tag{2.5}$$

Vectors are usually denoted in lower case Latin, $a = a^\mu\gamma_\mu$, or Greek for basis frame vectors. Given a frame of vectors $\{e_\mu\}$, say, the reciprocal frame is denoted by $\{e^\mu\}$ and satisfies $e^\mu \cdot e_\nu = \delta^\mu_\nu$. The vector derivative, ∇, is defined by

$$\nabla \equiv \gamma^\mu \frac{\partial}{\partial x^\mu} \tag{2.6}$$

where the $\{x^\mu\}$ are a set of Cartesian coordinates.

The first of the gravitational gauge fields is a position-dependent linear function $\overline{h}(a)$, or $\overline{h}(a,x)$. This is linear in its vector argument a, and is a general non-linear function of the position vector $x \equiv x^\mu \gamma_\mu$. If arbitrary orthogonal and coordinate frames are introduced, the linear function $\overline{h}(a)$ can be used to define a vierbein. However, its gauge-theoretic purpose is quite different and has nothing to do with frames or coordinate systems [2]. The second gauge field is denoted $\omega(a) = \omega(a,x)$, and is a bivector-valued linear function of its argument a. The position-dependence of $\omega(a)$ is also generally non-linear. On defining the derivative operators

$$L_a \equiv a \cdot \overline{h}(\nabla), \tag{2.7}$$

the $\overline{h}(a)$ and $\omega(a)$ gauge fields are related by

$$[L_a, L_b] = L_c, \tag{2.8}$$

where

$$c = L_a b + \omega(a) \cdot b - L_b a - \omega(b) \cdot a. \tag{2.9}$$

Spacetime rotations (*i.e.* boosts and rotations) are generated by a 'rotor' R. This is an even-grade element satisfying $R\tilde{R} = 1$, where the tilde denotes the operation of reversing the order of vectors in any geometric product. In terms of this rotor, a spacetime rotation of a vector a is performed by

$$a \mapsto Ra\tilde{R}. \tag{2.10}$$

Rotors form a group under the geometric product (the Lie group $\mathrm{spin}^+(1,3) \cong \mathrm{sl}(2,C)$), and the associated Lie algebra is generated by the 6 bivectors in (2.3). Under a local rotation, the gauge fields transform as:

$$\overline{h}(a) \mapsto \overline{h}'(a) \equiv R\overline{h}(a)\tilde{R}, \tag{2.11}$$

and

$$\omega(a) \mapsto \omega'(a) \equiv R\omega(\tilde{R}aR)\tilde{R} - 2L_{\tilde{R}aR}R\,\tilde{R}. \tag{2.12}$$

Under the displacement $x \mapsto x' \equiv f(x)$ the $\overline{h}(a)$ function is defined to transform as

$$\overline{h}(a,x) \mapsto \overline{h}'(a,x) \equiv \overline{h}(\overline{f}^{-1}(a), f(x)) \tag{2.13}$$

where $\underline{f}(a) \equiv a \cdot \nabla f(x)$, and the underbar/overbar notation denotes a linear function and its adjoint. This transformation law ensures that a vector field such as $a(x) \equiv \overline{h}(\nabla\phi(x))$, where $\phi(x)$ is a scalar field, transforms covariantly under displacements.

That is, if we replace $\phi(x)$ by $\phi'(x) = \phi(x')$ and $\overline{h}$ by $\overline{h}'$ then $a(x)$ transforms simply to $a'(x) = a(x')$. Similarly, the $\omega(a)$ field transforms by simply changing its position dependence,

$$\omega(a, x) \mapsto \omega'(a, x) \equiv \omega(a, f(x)). \tag{2.14}$$

The final quantity we need is the Riemann tensor, $\mathcal{R}(a \wedge b)$, which is defined by

$$\mathcal{R}(a \wedge b) = L_a \omega(b) - L_b \omega(a) + \omega(a) \times \omega(b) - \omega(c), \tag{2.15}$$

with c determined by equation (2.9). The Ricci and Einstein tensors and the Ricci scalar are defined by

$$\text{Ricci Tensor:} \quad \mathcal{R}(b) = \gamma_\mu \cdot \mathcal{R}(\gamma^\mu \wedge b) \tag{2.16}$$

$$\text{Ricci Scalar:} \quad \mathcal{R} = \gamma_\mu \cdot \mathcal{R}(\gamma^\mu) \tag{2.17}$$

$$\text{Einstein Tensor:} \quad \mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2} a \mathcal{R}. \tag{2.18}$$

This concludes the definitions of the quantities required for this paper. Further details can be found in [2].

3 Cylindrically-Symmetric Systems

We start with the introduction of a cylindrical polar coordinate system:

$$\begin{array}{ll} t \equiv x \cdot \gamma_0 & \tan\phi \equiv (x \cdot \gamma^2)/(x \cdot \gamma^1) \\ r \equiv \sqrt{[-(x \wedge \sigma_3)^2]} & z \equiv x \cdot \gamma^3. \end{array} \tag{3.1}$$

The associated coordinate frame is

$$\begin{array}{ll} e_t \equiv \gamma_0 & e_\phi \equiv r(-\sin\phi\, \gamma_1 + \cos\phi\, \gamma_2) \\ e_r \equiv \cos\phi\, \gamma_1 + \sin\phi\, \gamma_2 & e_z \equiv \gamma_3, \end{array} \tag{3.2}$$

with the reciprocal-frame vectors denoted as $\{e^t, e^r, e^\phi, e^z\}$. It is also convenient to define the unit vector

$$\hat{\phi} \equiv -\sin\phi\, \gamma_1 + \cos\phi\, \gamma_2 \tag{3.3}$$

and the unit spatial bivectors

$$\sigma_r \equiv e_r e_t, \qquad \sigma_\phi \equiv \hat{\phi} e_t. \tag{3.4}$$

In this paper we are interested in stationary systems for which the z-axis drops out of the dynamics, so that we effectively simulate gravity in (2+1) dimensions. We also impose the requirement that our system be symmetric under simultaneous reversal of the t and ϕ directions. This requires that there be no coupling between the e_r direction and the e_t and e_ϕ directions. The most general form of $\overline{h}$-function meeting these criteria is given by

$$\begin{aligned} \overline{h}(e^t) &= f_1 e^t + r f_2 e^\phi & \overline{h}(e^r) &= g_1 e^r \\ \overline{h}(e^\phi) &= r h_1 e^\phi + h_2 e^t & \overline{h}(e^z) &= e^z \end{aligned} \tag{3.5}$$

where the factors of r have been included on f_2 and h_1 to simplify some later equations. The scalar functions appearing in (3.5) are all functions of r only. The possibility of non-zero off-diagonal terms, as well as a non-constant f_1, means that the function (3.5) violates boost invariance. It is, however, the natural extension to three dimensions of a two-dimensional rotating symmetric fluid.

We will occasionally employ the abbreviations

$$g^t \equiv \overline{h}(e^t), \qquad g^\phi \equiv \overline{h}(e^\phi), \tag{3.6}$$

and

$$g_t \equiv \underline{h}^{-1}(e_t), \qquad g_\phi \equiv \underline{h}^{-1}(e_\phi) \tag{3.7}$$

for the reciprocal vectors. In terms of these the equivalent GR line element generated by (3.5) is [2]

$$ds^2 = {g_t}^2\, dt^2 + 2 g_t \cdot g_\phi\, dt\, d\phi + {g_\phi}^2\, d\phi^2 - {g_1}^2\, dr^2 - dz^2. \tag{3.8}$$

A suitable $\omega(a)$ gauge field consistent with (3.5) is given by

$$\begin{aligned} \omega_t &\equiv \omega(e_t) = -T\sigma_r + (K + h_2) i\sigma_3 & \omega_{\hat{\phi}} &\equiv \omega(\hat{\phi}) = K\sigma_r + (h_1 - G) i\sigma_3 \\ \omega_r &\equiv \omega(e_r) = \bar{K}\sigma_\phi & \omega_z &\equiv \omega(e_z) = 0. \end{aligned} \tag{3.9}$$

Again, the new scalar functions appearing here (T, K, $\bar{K}$, G) are functions of r alone. The terms in h_1 and h_2 are included to ensure that the commutation relations (2.8) employ the functions T, K, $\bar{K}$ and G only. Since all expressions involving $L_z \equiv e_z \cdot \overline{h}(\nabla)$ must vanish, there are only three non-vanishing commutation relations implied by (3.9). These are

$$\begin{aligned} [L_r, L_t] &= T L_t + (K + \bar{K}) L_{\hat{\phi}} \\ [L_r, L_{\hat{\phi}}] &= -(K - \bar{K}) L_t - G L_{\hat{\phi}} \\ [L_t, L_{\hat{\phi}}] &= 0, \end{aligned} \tag{3.10}$$

where $L_t \equiv e_t \cdot \overline{h}(\nabla)$ and $L_r \equiv e_r \cdot \overline{h}(\nabla)$. Note that $L_r = g_1(r)\partial_r$, and we can always make the position gauge choice that $g_1 = 1$. This choice will be made later, after a useful alternative gauge choice has been discussed. Since neither L_t nor $L_{\hat{\phi}}$ contain derivatives with respect to r, the bracket relations (3.10) immediately yield

$$\begin{aligned} L_r f_1 &= T f_1 + (K+\bar{K}) f_2 & L_r f_2 &= -G f_2 - (K-\bar{K}) f_1 \\ L_r h_1 &= -G h_1 - (K-\bar{K}) h_2 & L_r h_2 &= T h_2 + (K+\bar{K}) h_1. \end{aligned} \tag{3.11}$$

The definition (2.15) leads to the following Riemann tensor:

$$\begin{aligned} \mathcal{R}(\sigma_r) &= \alpha_1 \sigma_r + \beta i\sigma_3 \\ \mathcal{R}(i\sigma_3) &= \alpha_2 i\sigma_3 - \beta\sigma_r \\ \mathcal{R}(\sigma_\phi) &= \alpha_3 \sigma_\phi, \end{aligned} \tag{3.12}$$

with all other terms being zero. The scalar functions appearing in $\mathcal{R}(B)$ are defined by

$$\alpha_1 = -L_r T + T^2 - K(K+2\bar{K}) \tag{3.13}$$

$$\alpha_2 = L_r G + G^2 - K(K-2\bar{K}) \tag{3.14}$$

$$\alpha_3 = K^2 - GT \tag{3.15}$$

$$\beta = L_r K + G(K+\bar{K}) - T(K-\bar{K}). \tag{3.16}$$

The Bianchi identity reduces to the single scalar equation

$$L_r \alpha_3 + T(\alpha_2 - \alpha_3) + G(\alpha_3 - \alpha_1) - 2K\beta = 0, \tag{3.17}$$

which is satisfied automatically by virtue of equations (3.13)—(3.16).

The Einstein tensor is given by a simple rearrangement of the above terms:

$$\begin{aligned} \mathcal{G}(e_t) &= -\alpha_2 e_t - \beta\hat{\phi} \\ \mathcal{G}(e_r) &= -\alpha_3 e_r \\ \mathcal{G}(\hat{\phi}) &= -\alpha_1 \hat{\phi} + \beta e_t \\ \mathcal{G}(e_z) &= -(\alpha_1 + \alpha_2 + \alpha_3) e_z, \end{aligned} \tag{3.18}$$

and this is to be equated with the matter stress-energy tensor.

The $\overline{h}$-function (3.5) contains a single rotational gauge freedom, which is the freedom to perform a boost in the σ_ϕ plane. If we make the physical assumption that the matter stress-energy tensor has a future-pointing timelike eigenvector, this gauge freedom can be used to set this eigenvector to the e_t direction. Once this is done all the rotational gauge freedom in the problem has been removed, and we are left with a complete set of field equations. These are:

$$\begin{aligned} -L_r G - G^2 + K(K-2\bar{K}) &= 8\pi\rho \\ K^2 - GT &= 8\pi P_r \\ -L_r T + T^2 - K(K+2\bar{K}) &= 8\pi P_\phi \\ L_r K + G(K+\bar{K}) - T(K-\bar{K}) &= 0, \end{aligned} \tag{3.19}$$

where ρ is the matter density, and P_r and P_ϕ are the radial and azimuthal pressures respectively. The coefficient of $\mathcal{G}(e_z)$ is determined algebraically by the other three coefficients, and the same must therefore be true of the matter stress-energy tensor. It follows that the e_z-component of the Einstein equations contains no new information. (Of course, if we were working in a genuine $(2+1)$ system, the e_z equation would not be present.)

The essential advantage of the above formalism is that, once a choice for g_1 is made, the equations (3.19) can be solved directly, without reference to the functions in $\overline{h}(a)$. Once a solution to (3.19) is found, the remaining functions are given by integrating equations (3.11). We have therefore reduced the field equations to a simple set of coupled first-order equations. To complete the problem we need to define the matter stress-energy tensor. We treat four situations, starting with the case of vacuum fields.

4 Vacuum Solutions

In the vacuum region all of the scalars $\{\alpha_1, \alpha_2, \alpha_3, \beta\}$ are zero, so we are free to perform an r-dependent boost in the σ_ϕ direction. This freedom can be employed to set $\bar{K}$ to zero, so that the vacuum region is described by the equations

$$\partial_r G + G^2 - GT = 0 \tag{4.1}$$

$$\partial_r T - T^2 + GT = 0, \tag{4.2}$$

with K determined by $K^2 = GT$. We have employed the position-gauge freedom to set $g_1 = 1$, which means that the coordinate r represents the proper distance from the axis. On subtracting these equations and integrating we see that

$$G - T = 1/(r + r_0) \tag{4.3}$$

where r_0 is an arbitrary constant of integration. Similarly, adding the equations and integrating yields

$$G + T = c/(r + r_0), \tag{4.4}$$

where c is a second constant of integration.

The restriction that $GT = K^2 > 0$ means that $c^2 > 1$, and we can set

$$c = \pm \cosh 2\alpha. \tag{4.5}$$

There are two distinct vacuum configurations, depending on which sign is chosen for c. In either case, the constant α can be gauged to zero with a further constant boost in the σ_ϕ direction (which does not reintroduce a $\bar{K}$ term). The two vacuum sectors are therefore characterised by the solutions

$$\text{Type I:} \quad G = 1/(r + r_0), \quad T = K = \bar{K} = 0 \tag{4.6}$$

$$\text{Type II:} \quad T = -1/(r + r_0), \quad G = K = \bar{K} = 0. \tag{4.7}$$

All other vacuum solutions can be reached from this pair by r-dependent boosts in the σ_ϕ direction. No globally-defined gauge transformation exists between these solution classes. This two-fold degeneracy in the vacuum solutions has not been previously noted. Of course, since the Riemann tensor vanishes, it is possible locally to gauge transform all of these fields to zero, but this is not possible globally.

Given G, T, K and $\bar{K}$ it is a straightforward matter to integrate equations (3.11) and so recover $\overline{h}(a)$. Applied to Solution I above, this process recovers the solution found by Deser *et al.* [4] and discussed by Jensen & Soleng [5]. The second class of solutions have not been given previously, except for the case of vanishing angular momentum. In this case the solution reduces to one found by Jensen and Kucera [7], who reinterpreted the Taub singularity in terms of cylindrical geometry. We show in Section 9 that the second class of solutions is naturally picked out for the vacuum exterior to bodies that are rotating rigidly. The solutions have a non-zero T, giving rise to an attractive force of magnitude $1/(r+r_0)$. This force is confining — no particle can escape from the string, regardless of what initial velocity it is given. Such behaviour is typical of two-dimensional systems. A further feature of this class of solutions is that the circumference of a circle surrounding the string is independent of distance from the string. This explains how an attractive force is possible even in the absence of a Weyl tensor — the proper size of an object is unchanged as all of its constituents follow radial trajectories towards the string axis. The possible existence of an attractive force of this nature does not appear to have been considered for astrophysical or cosmological systems. Of course, the long-range properties of these solutions look unphysical, but that does not rule out the possibility that this class of solutions may be valid locally around an extended string-like object.

5 Physical Properties of Matter Solutions

In order to construct solutions with matter present, it is helpful to see how the $\{G, T, K, \bar{K}\}$ variables relate to the physical properties of the string. These properties include the acceleration, vorticity, shear and angular momentum of the string. The definitions of acceleration, vorticity and shear in our gauge theory formalism are given in Appendix A. The results in the present setup are:

$$\text{Acceleration} \qquad w = -T e_r, \tag{5.1}$$

$$\text{Vorticity bivector} \qquad \varpi = -(K-\bar{K}) i\sigma_3, \tag{5.2}$$

$$\text{Shear tensor} \qquad \sigma(a) = -\tfrac{1}{2}(K+\bar{K})(a\cdot e_r\, \hat{\phi} + a\cdot\hat{\phi}\, e_r). \tag{5.3}$$

These show that the acceleration is controlled by T, the vorticity by $(K-\bar{K})$ and the shear by $(K+\bar{K})$. In the matter region all of these scalar quantities are physically measurable functions. The same is true of the fourth function, G, which can be determined from the radial pressure.

The remaining physical property of relevance here is the angular momentum contained in the fields. The vector g_ϕ is a Killing vector of the above solution, so the vector $\mathcal{T}(g_\phi)$ is covariantly conserved. It follows that [2]

$$\nabla\cdot[\underline{h}(\mathcal{T}(g_\phi))\det(\underline{h})^{-1}] = 0 \tag{5.4}$$

and the total conserved angular momentum per unit length in the e_t frame is therefore given by the expression

$$J_S = \int_0^{r_s} d^2x\, g^t\cdot\mathcal{T}(g_\phi)\det(\underline{h})^{-1}, \tag{5.5}$$

where r_s is the string radius. In the $g_1 = 1$ gauge this expression evaluates to give

$$J_S = -2\pi\int_0^{r_s} dr\,(\rho + P_\phi) f_1 f_2 (f_1 h_1 - f_2 h_2)^{-2}, \tag{5.6}$$

which shows that a non-zero f_2 is required for angular momentum to be present.

6 Pressure-Free Strings

The simplest matter configuration is one in which both the radial and azimuthal pressures are zero. The coefficient of $\mathcal{T}(e_z)$ is then determined automatically by the density, and for this case $\mathcal{T}(a)$ can be summarised neatly as

$$\mathcal{T}(a) = \tfrac{1}{2}\rho(r)(a - i\sigma_3\, a\, i\sigma_3). \tag{6.1}$$

An equivalent stress-energy tensor was first discussed by Linet [13] (see also the recent book by Vilenkin and Shellard [1]), and describes a simple non-rotating cosmic string. The string stress-energy tensor has a *negative* pressure along the string, $P_z = -\rho$, which is characteristic of relativistic scalar fields and shows that the string is under a tension. The stress-energy tensor (6.1) is also invariant under boosts along the z-axis, which is a condition usually demanded of all cosmic string solutions [1].

From the Einstein equations we see that $\alpha_1 = \alpha_3 = \beta = 0$, and the Riemann tensor therefore has the compact form

$$\mathcal{R}(B) = 8\pi\rho\, B\cdot i\sigma_3\, i\sigma_3. \tag{6.2}$$

It follows from equation (3.17) that $T = 0$, and since $\alpha_3 = \beta = 0$ we must also have $K = \bar{K} = 0$. The Einstein equations therefore reduce to the single equation

$$L_r G + G^2 = -8\pi\rho, \tag{6.3}$$

together with equations (3.11). From equations (3.11) we see that both f_1 and h_2 are constant. These therefore encode a global rotation which can be removed by the displacement

$$x' = \tilde{R}xR, \qquad R = \exp\left(\tfrac{1}{2}i\sigma_3\, h_2 t/f_1\right), \tag{6.4}$$

followed by the rotation defined by R. We can therefore work in a gauge where $h_2 = 0$, and we can also scale the time direction so that $f_1 = 1$.

The remaining equations from (3.11) are

$$L_r h_1 = -G h_1, \qquad L_r f_2 = -G f_2, \tag{6.5}$$

and it follows that $f_2 = \lambda h_1$, where λ is an arbitrary constant. However, $r h_1$ must tend to 1 as $r \mapsto 0$ so that $\overline{h}(a)$ is well-defined on the axis. It follows that h_1, and hence f_2, must diverge as $1/r$. But if f_2 diverges in this fashion then $\overline{h}(e^t)$ is singular on the axis, which is not permitted. It follows that the constant λ must be zero, so the string has no angular momentum, agreeing with the fact that the shear and vorticity are both zero. We therefore see that pressure is necessary for strings to have any angular momentum.

The restriction that the $\overline{h}(a)$, $\omega(a)$ and $\mathcal{T}(a)$ fields be well-defined on the axis replaces the notion of 'elementary flatness' employed in the GR literature (see, for example, Synge [14]). The gauge theory approach provides a clear and unambiguous statement of the necessary and sufficient conditions which must be satisfied by the fields. The conditions are easily implemented, since all the fields are functions of position in flat spacetime. In contrast, we show in Section 8 that the criterion of elementary flatness alone is not sufficient to determine whether a given line element has physically acceptable properties on the axis.

We have now restricted $\overline{h}(a)$ to the simple form

$$\overline{h}(a) = a + (g_1 - 1) a \cdot e_r e^r + (r h_1 - 1) a \cdot e_\phi e^\phi, \tag{6.6}$$

and the remaining equations are

$$L_r h_1 = -G h_1 \quad \text{and} \quad L_r G = -8\pi\rho - G^2, \tag{6.7}$$

with $L_r = g_1 \partial_r$. From these we find that

$$\partial_r (G/h_1) = -8\pi\rho (g_1 h_1)^{-1} \tag{6.8}$$

and hence that

$$G/h_1 = 1 - 4\mu(r), \tag{6.9}$$

where $\mu(r)$ is the mass per unit length,

$$\mu(r) = \int_0^r \frac{ds}{s h_1 g_1} 2\pi\rho s. \tag{6.10}$$

The denominator $r h_1 g_1$ is the spatial determinant of $\overline{h}(a)$, so $\mu(r)$ is a covariant quantity.

To complete the solution we must make a gauge choice for g_1, which amounts essentially to a choice of radial coordinate. The standard choice is to set g_1 to 1, so that the radial coordinate r agrees with the proper distance. In this gauge h_1 is found by solving the equation

$$\partial_r^2(h_1^{-1}) = -8\pi\rho h_1^{-1}, \tag{6.11}$$

and G is then given by $-\partial_r h_1/h_1$. This solution is the same as that employed for non-rotating cosmic strings [1], and was first found by Linet [13]. The case of constant ρ recovers the solution given by Gott [15].

Whilst setting $g_1 = 1$ has obvious advantages, it has the disadvantage that solving for h_1 for a given density profile ρ is often difficult. An alternative gauge choice, motivated in part by the solution for a spherically-symmetric system [2], is to choose g_1 such that $\overline{h}(e^\phi) = e^\phi$. This requires that

$$h_1 = 1/r \tag{6.12}$$

and it follows that

$$G = g_1/r. \tag{6.13}$$

The remaining equation can now be written as

$$g_1 \frac{\partial g_1}{\partial r} = -8\pi\rho r, \tag{6.14}$$

which integrates to give

$${g_1}^2 = 1 - \int_0^r 16\pi\rho s\, ds, \tag{6.15}$$

where the constant of integration is chosen so that $\overline{h}(a)$ is well-defined on the axis. On defining

$$M(r) = \int_0^r 2\pi\rho s\, ds, \tag{6.16}$$

the solution can be summarised neatly by

$$\overline{h}(a) = a + \Big([1 - 8M(r)]^{1/2} - 1\Big) a\cdot e_r e^r. \tag{6.17}$$

It is a simple matter to show that in this gauge M and μ are related by

$$1 - 8M = (1 - 4\mu)^2. \tag{6.18}$$

The useful feature of this gauge is that the solution is straightforward to calculate given the density profile ρ. Provided that the density falls off with radius in such a way that $M(r) < 1/8$ for all r, then (6.17) is valid over the whole of spacetime. If the string has a finite radius r_s, then in the exterior region g_1 has the constant value $\surd(1 - 8M)$, where $M = M(r_s)$. While we anticipate that this second gauge choice may be useful in some applications, for the remainder of this paper we will work exclusively in the $g_1 = 1$ gauge.

6.1 The Exterior Fields and Topology

In the region outside the string the $\overline{h}$-function reduces to the simple form

$$\overline{h}(a) = a + \frac{4\mu}{1-4\mu} a \cdot e_\phi e^\phi, \tag{6.19}$$

where μ is the total mass per unit length. This solution has $T = K = \bar{K} = 0$ and $G = 1/r$. It therefore lies in the gauge class of type I vacuum solutions (4.7), with the constant r_0 set to zero.

The solution (6.19) can be obtained by acting on the identity function first with the displacement defined by

$$f(x) = \tilde{R}xR, \qquad R = \exp(-2\mu\phi i\sigma_3) \tag{6.20}$$

and following this with the rotation defined by the rotor R. But the displacement (6.20) transforms fields at an angle ϕ' in the initial flat spacetime to the angle $\phi = (1-4\mu)\phi'$. Since this latter angle runs between 0 and 2π in the solution (6.19), there is a region of the initial flat spacetime for which the gauge transformation (6.20) is not defined. This region is a wedge of spacetime of angular size specified by the 'deficit' angle $8\pi\mu$.

This is the gauge theory equivalent of the standard picture of the exterior string metric as representing flat spacetime with a wedge of space removed and the edges identified. From the gauge theory perspective, the solution (6.19) has a vanishing Riemann tensor, but no globally-defined gauge transformation exists which takes (6.19) to the identity. This is precisely analogous to the case of the electromagnetic gauge field A in the region exterior to a solenoid in an Aharonov–Bohm type experiment. There the field strength tensor F vanishes in the external region, but A still has physical effects there since no global gauge choice exists that sets A zero everywhere and is well matched at the boundary of the solenoid. Seeing the effects of a string in terms of gauge fields in flat space (as in electromagnetism), rather than in terms of topological surgery upon spacetime, both focuses attention on the physics and illustrates the importance of global solutions in gauge theory gravity.

7 Non-Rotating Strings with Pressure

From the above considerations it is clear that non-rotating matter distributions are generated by setting $K = \bar{K} = 0$. In this case we find that f_2/h_1 is constant, and the properties of the fields on the axis require that $f_2 = 0$. It also follows that f_1/h_2 is constant, and we can use the transformation described at equation (6.4) above to work in a gauge where $h_2 = 0$. The remaining field equations are

$$\begin{aligned} \partial_r G + G^2 &= -8\pi\rho \\ \partial_r T - T^2 &= -8\pi P_\phi \\ GT &= -8\pi P_r, \end{aligned} \tag{7.1}$$

to which solutions can be generated by specifying ρ and P_ϕ and then solving for G and T. At the boundary we must have $P_r = 0$, forcing us to set either $G = 0$ or $T = 0$, which in turn determines the gauge sector for the vacuum region.

One interesting case is that of a two-dimensional ideal fluid, $P_r = P_\phi$. This stress-energy tensor breaks boost invariance, so does not model a cosmic string, but is still of physical interest. The simplest ideal fluid solution is that of constant density,

$$8\pi\rho = \lambda^2. \tag{7.2}$$

With this matter distribution we find that

$$G = \frac{\lambda \cos\lambda r}{\sin\lambda r} \tag{7.3}$$

which ensures that rG tends to 1 on the axis, as required from the condition that $\omega(a)$ be well defined there. The equation for T now becomes

$$\partial_r T - T^2 = \frac{\lambda \cos\lambda r}{\sin\lambda r} T, \tag{7.4}$$

which integrates simply to give

$$T = \frac{\lambda \sin\lambda r}{\cos\lambda r + A}, \tag{7.5}$$

where A is a second constant of integration. The string boundary is located where the pressure falls to zero, which occurs at $\lambda r_s = \pi/2$.

At the boundary this solution has $G = 0$ and $T \neq 0$, so matches onto the type II vacuum solution of equation (4.7). It follows that $A < -1$, in order to avoid the fields diverging at some finite distance from the string. The remaining functions f_1 and h_1 are found easily by integration, and the resulting line element is

$$ds^2 = \frac{(\cos\lambda r + A)^2}{(1+A)^2} dt^2 - \frac{\sin^2\lambda r}{\lambda^2} d\phi^2 - dr^2 - dz^2 \tag{7.6}$$

in the interior $r \leq \pi/(2\lambda)$, and

$$ds^2 = \frac{\lambda^2 (r + r_0)^2}{(1+A)^2} dt^2 - \frac{1}{\lambda^2} d\phi^2 - dr^2 - dz^2 \tag{7.7}$$

in the exterior $r \geq \pi/(2\lambda)$. The constant r_0 is determined by $r_0 = -(2A+\pi)/(2\lambda)$.

Since the vacuum solution is of type II, the matter exhibits an attractive force in the vacuum region outside the string despite the absence of a Weyl tensor. The fact that $G = 0$ at the boundary of the string means that the total mass energy in the fields lies on the critical value of 1/4. This is characteristic of all non-rotating strings which match onto a type II vacuum. Unlike the type I vacuum reached in the case of vanishing pressure, there is no simple picture of the type II vacuum region in terms of topological surgery on flat spacetime. The coordinate transformation required to obtain the line element (7.7) from the Minkowski line element

$$ds^2 = dt'^2 - dx'^2 - dy'^2 - dz'^2 \tag{7.8}$$

is

$$\begin{aligned} t' &= (r + r_0) \sinh[\lambda t/(1 + A)] \qquad & y' &= \phi/\lambda \\ x' &= (r + r_0) \cosh[\lambda t/(1 + A)] \qquad & z' &= z. \end{aligned} \tag{7.9}$$

This transformation involves a strange mixing of spacelike and timelike coordinates, as well as the replacement of the Cartesian coordinate y' by the periodic coordinate ϕ. The topology of this transformation is not easy to visualise and the identification of this second type of solution only becomes straightforward once the gauge theory formalism is adopted. The external line element (7.7) is the same as one given in [7]. However, the matter distribution given here to which the vacuum is matched is much simpler than that given in [7]. Indeed, it is curious that such a simple matter distribution (a constant density ideal fluid) should give rise to an exterior field with the pathological features already noted for type II vacuum solutions.

8 The solution of Jensen & Soleng

The first published solution describing the interior of a rotating string was that of Jensen & Soleng [5]. Their solution is defined by the metric

$$ds^2 = dt^2 + 2M\, dt\, d\phi - (A^2 - M^2) d\phi^2 - dr^2 - dz^2, \tag{8.1}$$

where

$$A = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda} r) \tag{8.2}$$

and

$$M = 2\alpha\Big((r - r_s)\cos(\sqrt{\lambda} r) - \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda} r) + r_s\Big). \tag{8.3}$$

Here λ is a positive constant, α is a constant with $\alpha \leq 1$, and r_s is the radius of the string. The solution (8.1) is generated by an $\overline{h}(a)$-function of the general form

$$h_1 = \frac{1}{A} \cosh u \qquad f_1 = \cosh u - \frac{M}{A} \sinh u \tag{8.4}$$

$$h_2 = \frac{1}{A} \sinh u \qquad f_2 = \sinh u - \frac{M}{A} \cosh u, \tag{8.5}$$

where the boost factor u is arbitrary up to the constraint that $u = 0$ on the axis of the string. That $u = 0$ on the string axis is enforced by the requirement that $\overline{h}(a)$ is well defined there. If we now analyse the solution in the gauge in which $u = 0$ everywhere we find that the Einstein tensor (3.18) contains a non-zero β term, with

$$\beta = -\alpha\lambda. \tag{8.6}$$

(This is the 'heat flow' term discussed in [5].) But the presence of this β term means that the stress-energy tensor is not properly defined on the string axis. The value of $\mathcal{T}(\gamma_1)$, for example, is dependent on the direction in which the string is approached.

This conclusion is independent of the choice made for u, since we are constrained to set $u = 0$ on the axis. It follows that the metric given by Jensen & Soleng does not produce a well-defined matter distribution and is physically inadmissible. This is surprising, since the metric (8.1) does satisfy the elementary flatness criteria [5]. This observation illustrates one way in which the gauge theory formulation of gravity given in [2] is superior to GR. The gauge theory approach deals solely with linear functions defined over (flat) spacetime, and there is never any doubt as to the conditions these functions should satisfy. Furthermore, the gauge theory approach focuses attention directly on physically relevant quantities, such as the eigenvalues of the stress-energy tensor, so that it quickly becomes apparent if a solution has unphysical properties. Standard GR, in contrast, usually imposes conditions at the metric level. As we have seen, this approach is not necessarily successful in distinguishing between physical and unphysical metrics.

9 Rigidly Rotating Strings

A further problem with the Jensen and Soleng solution is the fact that their stress-energy tensor does not always satisfy the weak energy condition [16]. This is due to their introduction of a heat flow term, which means that the pressure and density terms discussed in [5] are not the true invariant quantities defined by the fluid frame. Both of the problems with the Jensen and Soleng solution are straightforwardly overcome by removing the heat flow term, producung a physically acceptable matter distribution. The simplest model is again that of a two-dimensional ideal fluid, $P_r = P_\phi$. The stress-energy tensor is then automatically well defined on the axis, provided that the density and pressure are finite there. The two natural physical models to consider are where the fluid is vorticity-free ($\bar{K} = K$), or shear free ($\bar{K} = -K$). The latter case corresponds to a rigidly rotating string, and is the situation we analyse here. The equations governing this setup are

$$\partial_r K - 2KT = 0 \tag{9.1}$$

$$\partial_r G + G^2 = -8\pi\rho + 3K^2 \tag{9.2}$$

$$\partial_r T - T^2 = -8\pi P + K^2 \tag{9.3}$$

$$K^2 - GT = 8\pi P, \tag{9.4}$$

and we still need to make a choice of the density. A suitable choice (the same as that made by Jensen & Soleng [5]) is to set

$$8\pi\rho = 3K^2 + \lambda^2, \tag{9.5}$$

where λ is an arbitrary positive constant. This ansatz ensures that the density is always positive, so the matter does satisfy the weak energy condition (in the (2+1)-dimensional sense). One can solve for G and T immediately, yielding

$$G = \frac{\lambda \cos\lambda r}{\sin\lambda r} \quad \text{and} \quad T = \frac{\lambda \sin\lambda r}{\cos\lambda r + A}, \tag{9.6}$$

as was also obtained in Section 7. Again, the constant A satisfies $A < -1$.

We next solve for K and obtain

$$K = \frac{B}{(A + \cos\lambda r)^2}, \tag{9.7}$$

where B is a further constant. The density and pressure are now given by equations (9.5) and (9.4) respectively. The boundary of the string occurs where the pressure $P = 0$, and this must be reached before $r > \pi/\lambda$. Having solved for the covariant quantities, we can return to equations (3.11) to find a suitable form for the $\overline{h}$-function. First we see that f_1/h_2 is a constant, so that a rotation/displacement of the type defined at equation (6.4) can again be employed to set $h_2 = 0$. The remaining functions are easily found by integration:

$$f_1 = \frac{1 + A}{\cos\lambda r + A} \tag{9.8}$$

$$h_1 = \frac{\lambda}{\sin\lambda r} \tag{9.9}$$

$$f_2 = \frac{-B({f_1}^2 - 1)}{\lambda(A + 1)\sin\lambda r}. \tag{9.10}$$

For f_1 the arbitrary time-scale factor has been used to set $f_1 = 1$ on the axis. It is simple to verify that this solution is well-defined on the axis of the string. The corresponding line element is

$$\begin{aligned} ds^2 = {} & \frac{(\cos\lambda r + A)^2}{(1 + A)^2}\,dt^2 + \frac{2B}{\lambda^2(A + 1)^3}(1 - \cos\lambda r)(2A + 1 + \cos\lambda r)\,dt\,d\phi \\ & - \frac{\sin^2\lambda r}{\lambda^2}\left(1 - \frac{B^2(1 - \cos\lambda r)^2(2A + 1 + \cos\lambda r)^2}{\lambda^2 \sin^2\lambda r(1 + A)^4(A + \cos\lambda r)^2}\right)d\phi^2 - dr^2 - dz^2. \end{aligned} \tag{9.11}$$

9.1 Matching

To find a vacuum solution to which the above solution matches, it is simplest to return to the vacuum equations and consider the case where $K + \bar{K} = 0$. This gives the general vacuum setup for matching to a rigidly rotating matter distribution. The general form of the solution to these equations is:

$$G = \frac{-\alpha^2}{(r+r_0)[(r+r_0)^2-\alpha^2]} \tag{9.12}$$

$$T = -\frac{r+r_0}{(r+r_0)^2-\alpha^2} \tag{9.13}$$

$$K = \frac{\alpha}{(r+r_0)^2-\alpha^2} \tag{9.14}$$

where r_0 and α are constants to be determined by the fields at the boundary. It can be shown that this solution falls into the second class of vacuum solutions, as defined by equation (4.7).

The physical requirement that the vacuum fields be non-singular away from the source (*i.e.* that we are not in the supercritical regime) places restrictions on the various constants in the interior solution. In particular, if r_s is the string radius, the restriction that G is negative at the boundary means that $\pi/2 < \lambda r_s < \pi$. It also follows from the fact that GT is positive at the boundary that $A < -1$. A simple means for generating matched solutions is to first choose λ, r_s and r_0 and to then determine α, A and B from the three equations corresponding to the requirement that G, T and K are continuous at the boundary. A further choice of sign must be made to specify B, which amounts to a choice of direction of string rotation. The $\overline{h}(a)$ function in the vacuum is then found by integration, and turns out to be:

$$f_1 = -(1+A)(\alpha/B)^{1/2}[(r+r_0)^2-\alpha^2]^{-1/2} \tag{9.15}$$

$$h_1 = (\alpha/B)^{1/2}\lambda^2\frac{[(r+r_0)^2-\alpha^2]^{1/2}}{(r+r_0)} \tag{9.16}$$

$$f_2 = \frac{\alpha}{f_1(r+r_0)}\Big(f_1^2-1\Big). \tag{9.17}$$

The associated line element is

$$\begin{aligned} ds^2 = {} & \frac{B}{\alpha(1+A)^2}\Big((r+r_0)^2-\alpha^2\Big)\,dt^2 - \frac{B^2(r+r_0)^2}{(1+A)^2\alpha^2\lambda^4}\Big({f_1}^2-{f_2}^2\Big)\,d\phi^2 \\ & + \frac{2B}{\lambda^2(A+1)}\left(1-\frac{B}{\alpha(1+A)^2}\Big((r+r_0)^2-\alpha^2\Big)\right)dt\,d\phi - dr^2 - dz^2. \end{aligned} \tag{9.18}$$

An example of the fields due to a rigidly rotating string is shown in Figure 1. The class of rigidly rotating solutions is of particular interest because these solutions always admit closed timelike curves at some distance from the string. This is clear from the fact that, at large r, f_1 goes as $1/r$ whereas f_2 tends to a constant value. Beyond the point where the magnitude of f_2 overtakes that of f_1, a closed circular path orbiting the string becomes timelike. It further follows that closed timelike curves exist at spatial infinity for this solution. This is the reason why our solution

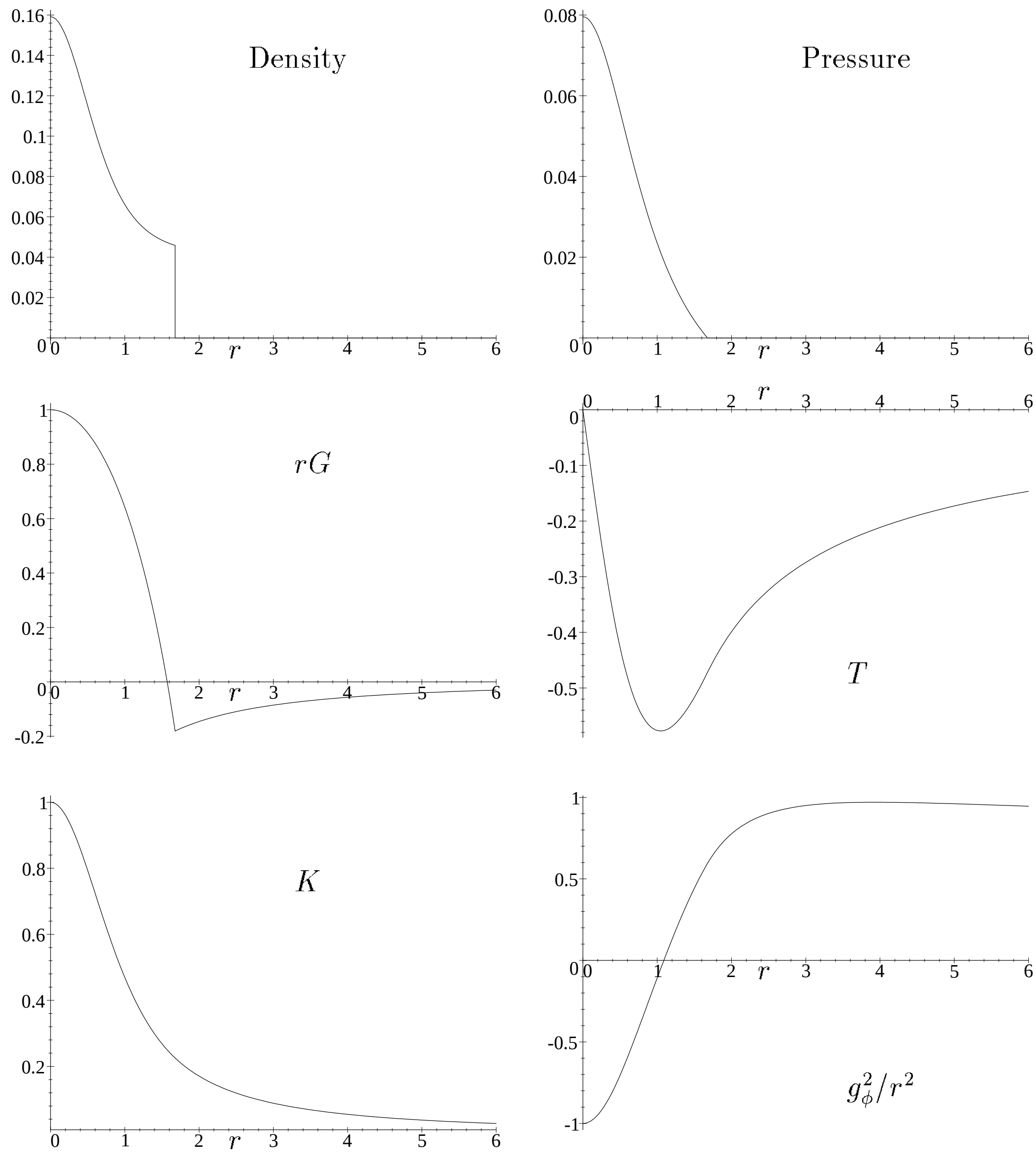

Figure 1: *An example of the fields inside and outside a rigidly rotating string. The equations for the fields are as described in the text. This solution has* $A = -2$, $B = 1$ *and* $\lambda = 1$, *which implies a string radius of* $r_s = 1.678$. *At the string boundary the pressure falls to zero, and there is a jump in the density. The corresponding jump in the Riemann tensor is caused by the kink in the curve of* G. *Both* T *and* K *are continuous and differentiable at the boundary. The final plot shows* g_ϕ^2/r^2. *When this term is positive, closed timelike curves exist. For the above solution this point is reached in the string interior.*

is not in conflict with the result of Menotti and Seminara [17]. These authors showed that in (2+1)-dimensional gravity an axially-symmetric source satisfying the weak energy condition cannot give rise to closed timelike curves. However, their result required the absence of closed timelike curves at spatial infinity, which is not the case for the present solution.

9.2 Physical Properties

Two remaining physical quantities of interest are the mass and angular momentum of the string. The total mass is given by the integral

$$\mu = \int_0^{r_s} dr\, 2\pi\rho/h_1 \tag{9.19}$$

which evaluates to

$$\mu = \frac{1}{4}\Big(1 - \frac{B^2}{\lambda^2(A+1)^3}\Big). \tag{9.20}$$

The total mass-energy for this type of string is therefore always greater than the critical value of 1/4, and can essentially take any value beyond this. The mass reduces to 1/4 when the angular momentum vanishes ($B = 0$) and we recover the solution described in Section 7. The total angular momentum is given by the integral (5.6) and evaluates to

$$J_S = \frac{B}{4\lambda^4(A+1)^4}\Big(B^2 + (A+1)^2\frac{B}{\alpha}\Big). \tag{9.21}$$

where the matching conditions have been employed to simplify the expression.

The fact that the solution both admits closed timelike curves and exerts a confining force in the vacuum region makes it interesting to study geodesic motion in this background. Figure 2 shows one such plot for a particle which starts with an initial outward velocity. The string attraction pulls the particle in, and the angular momentum sweeps the particle round the string. For much of its path the particle moves in the $-t$ direction, so that when it returns to the same point in space it arrives at an earlier time. This is of course pathological, and it is remarkable that such behaviour results from so simple a model.

10 Conclusions

The gauge theory approach to gravity provides a clear and simple route through to the gravitational field equations. For the case of static, symmetric (2+1)-dimensional systems lifted to spacetime the field equations reduce to a set of four coupled first-order differential equations. The variables in these equations are directly related to

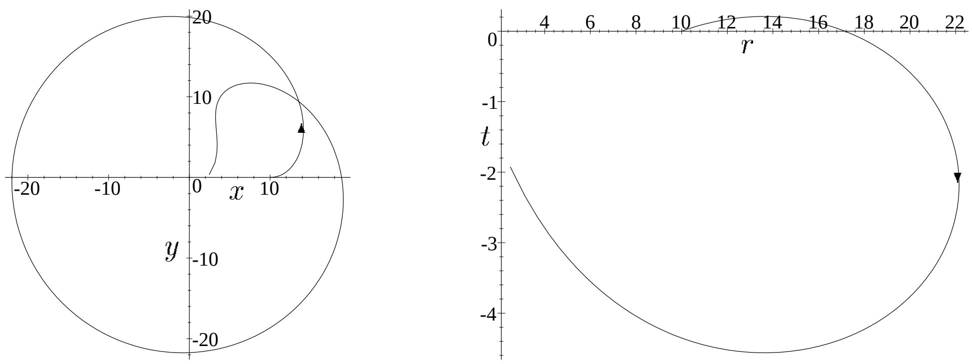

Figure 2: *Geodesic motion in the the fields described in Figure 1. The particle starts with initial coordinates $x = 10$, $y = 0$ and $t = 0$ and an initial velocity of* $\tanh(1.4)$ *radially outwards. The left-hand plot shows motion in the x-y plane, and shows that the particle is attracted towards the string and is swept round as well. The right-hand plot shows the motion in t-r space. In particular, we see that the particle returns to the same point in space at an earlier time than when it left.*

the physical properties of the fluid making up the string — its acceleration, shear and vorticity. The physical content of the equations is therefore manifest, in contrast to the second-order equations one obtains from the traditional metric-based approach.

The equations uncovered here reveal a somewhat richer structure than previously supposed. The vacuum fields fall into two distinct gauge classes, one of which exerts a long range confining force. Matter configurations which match to this new class of vacuum fields are simple to construct, and two analytic solutions were given. The long-range properties of these fields make them ultimately unphysical, but there is no reason to suppose that the solutions will not be relevant near a string of finite extent. This possibility is strengthened by the physical nature of the assumptions that were used in constructing the solutions.

The inclusion of angular momentum leads to the possibility of closed timelike curves and, for rigidly rotating matter, these turn out to be unavoidable. Causality violation would therefore appear to be an inherent feature of gravitational interactions in (2+1) dimensions. Again, the long-range properties of the fields look unphysical, with a rigidly rotating ideal fluid giving rise to closed timelike curves at spatial infinity. This result alone demonstrates how strange gravity is in a (2+1)-dimensional spacetime. Further work is required to determine whether causality violation remains a possibility for strings of finite extent in spacetime. (See [18] for a discussion of some further issues concerning causality violation near strings.)

This paper is the first in a series of papers applying the gauge theory approach of [2, 3] to areas of gravitational physics. In [2] and [3] the approach was applied to spherically-symmetric time-dependent systems, and many advantages over GR were obtained. Elsewhere, integral equation techniques have yielded novel insights into the singularity structure of black holes [19, 20]. In further work we will consider applications to more general cylindrical systems, and to axisymmetric systems. Some preliminary work on this latter topic is contained in [21] where a novel, and very quick, derivation of the Kerr solution is presented.

References

[1] A. Vilenkin and E.P.S. Shellard. *Cosmic Strings and other Topological Defects.* Cambridge University Press, 1994.

[2] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. Gravity, gauge theories and geometric algebra. Submitted to: *Phil. Trans. R. Soc. Lond. A*, 1996.

[3] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. Astrophysical and cosmological consequences of a gauge theory of gravity. In N. Sanchez and A. Zichichi, editors, *Advances in Astrofundamental Physics, Erice 1994*, page 359. World Scientific, Singapore, 1995.

[4] S. Deser, R. Jackiw, and G. 't Hooft. Three-dimensional Einstein gravity: Dynamics of flat space. *Ann. Phys.*, 152:220, 1984.

[5] B. Jensen and H.H. Soleng. General-relativistic model of a spinning cosmic string. *Phys. Rev. D*, 45(10):3528, 1992.

[6] D. V. Gal'tsov and P.S. Letelier. Spinning strings and cosmic dislocations. *Phys. Rev. D*, 47(10):4273, 1993.

[7] B. Jensen and J. Kucera. A reinterpretation of the Taub singularity. *Phys. Lett. A*, 195:111, 1994.

[8] T.W.B. Kibble. Lorentz invariance and the gravitational field. *J. Math. Phys.*, 2(3):212, 1961.

[9] R. Utiyama. Invariant theoretical interpretation of interaction. *Phys. Rev.*, 101(5):1597, 1956.

[10] F.W. Hehl, P. von der Heyde, G.D. Kerlick, and J.M. Nester. General relativity with spin and torsion: Foundations and prospects. *Rev. Mod. Phys.*, 48:393, 1976.

[11] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus.* Reidel, Dordrecht, 1984.

[12] S.F. Gull, A.N. Lasenby, and C.J.L. Doran. Imaginary numbers are not real — the geometric algebra of spacetime. *Found. Phys.*, 23(9):1175, 1993.

[13] B. Linet. The static metrics with cylindrical symmetry describing a model of cosmic strings. *Gen. Rel. Grav.*, 17(11):1109, 1985.

[14] J.L. Synge. *Relativity: The General Theory.* North-Holland Publishing, Amsterdam, 1964.

[15] R. J. Gott III. Gravitational lensing effects of vacuum strings: Exact solutions. *Ap. J.*, 288:422, 1985.

[16] H.H. Soleng. Negative energy densities in extended sources generating closed timelike curves in general relativity with and without torsion. *Phys. Rev. D*, 49(2):1124, 1994.

[17] P. Menotti and D. Seminara. Closed timelike curves and the energy condition in 2+1 gravity. *Phys. Lett. B*, 301:25, 1993.

[18] M. Novello and M.C. Motta da Silva. Cosmic spinning strings and causal protecting capsules. *Phys. Rev. D*, 49(2):825, 1994.

[19] C.J.L. Doran. Integral equations and Kerr-Schild fields I. Spherically-symmetric fields. Submitted to: *Class. Quantum Grav.*, 1996.

[20] C.J.L Doran, A.N. Lasenby, and S.F Gull. Integral equations and Kerr-Schild fields II. The Kerr solution. Submitted to: *Class. Quantum Grav.*, 1996.

[21] C.J.L. Doran, A.N. Lasenby, S.F. Gull, and J. Lasenby. Lectures in geometric algebra. In W.E. Baylis, editor, *Geometric (Clifford) Algebras in Physics.* Birkhauser, Boston, 1996.

[22] S.W. Hawking and G.F.R. Ellis. *The Large Scale Structure of Space-Time.* Cambridge University Press, 1973.

A Definitions of Acceleration, Shear and Vorticity

For the definitions of the physical properties associated with a fluid it is useful to introduce the covariant derivative

$$a\cdot\mathcal{D} \equiv L_a + \omega(a)\times . \tag{A.1}$$

We suppose that the fluid stress-energy tensor has a timelike eigenvector u ($u^2 = 1$) which we identify as the fluid 4-velocity. The acceleration vector w is then defined by

$$w \equiv u\cdot\mathcal{D}u \tag{A.2}$$

and measures the extent to which u departs from geodesic motion.

The vorticity bivector ϖ is defined by

$$\varpi \equiv \gamma^\mu \wedge (\gamma_\mu \cdot \mathcal{D} u) + w \wedge u \tag{A.3}$$

and satisfies $u \cdot \varpi = 0$. To define the shear tensor we require the linear function which projects vectors into the 3-space orthogonal to u:

$$H(a) \equiv a - a \cdot u u. \tag{A.4}$$

In terms of this function the shear tensor $\sigma(a)$ is defined by

$$\sigma(a) \equiv \tfrac{1}{2}\Big(H(a) \cdot \mathcal{D} u + H(\gamma^\mu)(\gamma_\mu \cdot \mathcal{D} u) \cdot a\Big) - \tfrac{1}{3}\gamma^\mu \cdot (\gamma_\mu \cdot \mathcal{D} u) H(a), \tag{A.5}$$

and is a symmetric, traceless linear function [22]. These are the definitions employed in the main text. Frame-free equivalents of these expressions can be given using the conventions developed in [2].