Polarisation

- Some emission processes are intrinsically polarised e.g. synchrotron radiation.

- Scattering processes can either increase or decrease the amount of polarisation.

- Magnetic fields can rotate the plane of polarisation through the Faraday effect.

- We can characterise a source’s polarisation through its Stoke’s Parameters: I, Q, U and V.
Stoke’s Parameters: I, Q, U and V

- Consider 4 different filters, each of which under natural unpolarised illumination will transmit half the incident light.

![Diagram of filters and beam](image)

- **I** = 2$S_0$ – the total flux
- **Q** = 2($S_1 - S_0$) – “resemblance” to horizontal linear polarisation.
- **U** = 2($S_2 - S_0$) – “resemblance” to linear polarisation orientated in direction of $+45^\circ$.
- **V** = 2($S_3 - S_0$) – “resemblance” to right-handed circular polarisation.
- **I^2 = Q^2 + U^2 + V^2** for completely polarised radiation.
- **Q^2 + U^2 + V^2 = 0** for completely unpolarised radiation.
- **\( \Pi = \sqrt{Q^2 + U^2 + V^2}/I \)** is the degree of polarisation.
Summary of Key Points

- Flux density, $F_\nu$, is power per unit area per unit frequency, measured in $\text{Jy} \equiv 10^{-26}\text{Wm}^{-2}\text{Hz}^{-1}$

- Specific intensity, $I_\nu$ is flux density per unit solid angle. It is independent of distance.

- For blackbody sources $I_\nu = B_\nu$ (Planck function).

- For uniform sources $F_\nu = I_\nu \Omega$ where $\Omega$ is often either $\Omega_{(\text{source})}$ or $\Omega_{(\text{beam})}$ for compact or extended sources respectively.

- A source’s polarisation can be characterised by its Stoke’s parameters. $I$ gives the total flux. $Q$ and $U$ give the degree of linear polarisation, $V$ the degree of circular polarisation.
Blackbody Radiation

Thermodynamics

We can characterise the radiation from a blackbody — one in which matter and radiation are in thermodynamic equilibrium — from thermodynamic arguments alone. Adkins (Equilibrium Thermodynamics, CUP) gives a thorough treatment.

Treating the radiation as a gas of photons, we see that the pressure exerted by radiation is $p = \rho c^2/3 = u/3$; the photon flux incident on a small area is $nc/4$, so the energy flux is just $uc/4$.

Consider two perfectly reflecting cavities containing radiation, each separately in equilibrium at temperature $T$. We make small holes in each cavity and connect them with a tube and a filter which transmits only a narrow range of radiation at frequency $\nu$. The flux of radiation leaking out of A into B via the tube is then $u^A_\nu c dA d\nu /4$, where $u^A_\nu$ is the energy density per unit wavelength in the radiation field in A in the direction of the tube.

The second law of thermodynamics prohibits any net energy flow between two bodies at equal temperatures, so we conclude that $u^A_\nu = u^B_\nu$.

This quantity $u_\nu$ must also be isotropic and independent of volume.
If we now consider squeezing an isothermal piston containing radiation, of total energy $U = uV$, we can apply the first law $dU = T \, ds - p \, dV$ to obtain $d(uV) = u \, dV$ so that

$$u = T \left( \frac{\partial S}{\partial V} \right)_T - p = T \left( \frac{\partial p}{\partial T} \right)_V - \frac{u}{3}$$

which integrates to give $u = a \, T^4$. The radiative flux is then

$$\text{flux} = \frac{ac}{4} \, T^4 = \sigma \, T^4$$
Full expressions for the radiant intensity

A perfectly reflecting cubic cavity at temperature $T$ and of side $a$ contains equilibrium radiation. The wave modes (using travelling wave boundary conditions) must be of the form

$$\Psi_{lmn} \propto \exp \left[ 2\pi i \left( \frac{lx}{a} + \frac{my}{a} + \frac{nz}{a} \right) \right] \exp(2\pi i \nu t)$$

with integer values of $l, m, n$.

The density of states $g$ per volume $d^3k$ of $k$-space is thus $g(k) = (a/2\pi)^3 = V/(8\pi^3)$ for a total cavity volume $V$. This can written in terms of the density of states in a given direction of a given wavenumber $k$:

$$g(k) \, d^3k = g(k, \Omega) \, k^2 \, dk \, d\Omega = \frac{k^2 \, V}{8\pi^3}$$

Assuming two independent polarisation states for the photons (either orthogonal plane polarisations or opposite circular polarisations) we obtain the spectra density of available photon states:

$$g(\nu, \Omega) = g(k, \Omega) \frac{dk}{d\nu} \times 2 = \frac{2V \nu^2}{c^3}$$

We multiply this by the mean energy per state and divide by the volume $V$ to obtain the spectral energy density per unit
solid angle introduced earlier. Remember that the occupancy number for photons is \((\exp[h\nu/kT] - 1)^{-1}\), so that

\[
u_\nu(\Omega) = \frac{2\nu^2}{c^3} \frac{h\nu}{\exp(h\nu/k_BT) - 1}
\]

Thus the brightness \(B_\nu\) of blackbody radiation is given by

\[
B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/k_BT) - 1}
\]

The total flux from the surface of a blackbody is then

\[
\Phi(T) = \pi \int_0^\infty B_\nu \, d\nu = \left(\frac{2\pi^5 k_B^4}{15c^2h^3}\right) T^4
\]

The prefactor is the Stefan-Boltzman constant, \(\sigma = 5.67 \times 10^{-8} \text{W m}^{-2} \text{K}^{-4}\). We can also now find the energy density per unit frequency \(u_\nu\) and hence the total energy density by using the relationship \(a = 4\sigma/c\):

\[
u = \left(\frac{8\pi^5 k_B^4}{15c^3h^3}\right) T^4
\]
Describing Polarisation

Monochromatic waves

- The most general form for a pure monochromatic wave is one of elliptical polarisation, formed by taking two plane waves at right angles with a phase difference. Thus three parameters specify the wave — two electric field amplitudes and the phase difference.

\[
E_x = E_1 \cos(\omega t - \phi_1) \\
E_y = E_2 \cos(\omega t - \phi_2)
\]

- Thus a pure monochromatic wave is always polarised.
Stokes Parameters

- We conventionally use the four *Stokes Parameters* to describe the polarisation:

\[
I = \langle E_1^2 + E_2^2 \rangle \\
Q = \langle E_1^2 - E_2^2 \rangle \\
U = \langle 2E_1 E_2 \cos(\phi_1 - \phi_2) \rangle \\
V = \langle 2E_1 E_2 \sin(\phi_1 - \phi_2) \rangle
\]

- It follows from these definitions that \( I^2 = Q^2 + U^2 + V^2 \). Thus there are still only *three* free parameters here.

- Stokes \( I \) is proportional to the *total flux* in the wave.

- Stokes \( V \) measures *circularity*, setting the axis ratio of the ellipse. \( V = 0 \) for linear polarisation, \( V > 0 \) for right handed elliptical polarisation.

- The remaining parameters Stokes \( Q \) or \( U \) measure the orientation of the ellipse. \( Q = U = 0 \) for circular polarisation.
Quasi-monochromatic waves

- In general the amplitudes $E_1$ and $E_2$, and phases $\phi_1$ and $\phi_2$, are functions of time. The wave is no longer monochromatic, but if the observing bandwidth is small it makes sense to talk about quasi-monochromatic waves. There is now the possibility that there is an unpolarised component in the wave.

- A completely unpolarised wave will have $Q = U = V = 0$ since the time averages in the definitions of $U$ and $V$ go to zero, and the field amplitudes in each direction must be on average equal.

- One can usefully decompose a wave $(I, Q, U, V)$ into two components, one completely polarised component

$$\left( \sqrt{Q^2 + U^2 + V^2}, Q, U, V \right)$$

and one completely unpolarised:

$$\left( I - \sqrt{Q^2 + U^2 + V^2}, 0, 0, 0 \right)$$

- The degree of polarisation $\Pi$ is the ratio of the polarised to total intensity:

$$\Pi = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}$$
Faraday Rotation

Faraday Rotation: Linearly polarised light which passes through a plasma which contains a uniform magnetic field has its position angle rotated.

- The modes of propagation of radio waves in a magnetised plasma are right- and left-handed elliptically polarised.
- The two modes experience different refractive indices, $n$ given by

$$n^2 = 1 - \frac{(\nu_p/\nu)^2}{1 \pm (\nu_g/\nu)\cos\theta}$$

$\nu_p = (\frac{e^2 N_e}{4\pi^2 \epsilon_0 m_e})^{1/2}$ is the plasma frequency. $\nu_g = \frac{eB}{m_e}$ is the gyro-frequency.

$\Rightarrow$ The phase velocities of the two modes are different and one sense of polarisation runs ahead of the other.

- Linearly polarised light can be considered as a superposition of equal components of right- and left-handed elliptically polarised light.
- On propagating through the plasma the different phase velocities result in a phase difference, $\Delta\phi$ between the two modes.
- This is equivalent to linearly polarised light with plane of polarisation rotated by $\theta = \Delta\phi/2$. 
\[ \frac{\theta}{\lambda^2} = 8.12 \times 10^3 \int_0^l N_e B_\parallel dl \]

\( \theta \) in radians, \( \lambda \) in metres, \( N_e \) in particles per cubic metre, \( B_\parallel \) in tesla, \( l \) in parsecs.

- \( \theta/\lambda^2 \) is known as the rotation measure.
The flux density observed by a telescope is often expressed in terms of an Antenna Temperature, $T^*_A$. The * indicates that a correction has been applied for atmospheric opacity.

- $T^*_A$ is the equivalent temperature of the power received at an antenna from a source.

- Hence, for a uniform source which fills the telescope beam (i.e. an extended source):
  
  - the antenna temperature is equal to the source’s brightness temperature.
  
  - for a source with a thermal spectrum, $T^*_A$ is (of course) independent of observing frequency.
  
  - observing a thermal source with the same telescope at different frequencies, the received flux density is the same, because the flux density is proportional to both the intensity ($I_\nu \propto \nu^2$) and the beam solid angle ($\Omega \propto \nu^{-2}$). (This is only strictly true if the the aperture efficiencies are the same at the two frequencies).
For a source that is smaller than the telescope beam (often called a *point* source), antenna temperature is a less useful concept, and the following are true:

- the antenna temperature observed is smaller than the source’s actual brightness temperature by a factor equal to the ratio $\Omega_{beam} : \Omega_{source}$.
- for a source with a thermal spectrum, $T_A^* \propto \nu^2$.  

Effective area of a Gaussian beam

The response, or primary beam, $\beta$ of a telescope falls off as a Gaussian as one moves away from its centre.

$$\beta(\theta) = \exp\left(\frac{-\theta^2}{2}\right)$$

The Full Width Half Max (FWHM), $\theta_{FWHM}$ is the width of the beam when the amplitude is half the maximum value.

$$\beta(\theta_{FWHM}) = 1/2 \Rightarrow \theta_{FWHM} = 2\sqrt{2\ln 2}$$

The effective area of the Gaussian beam is $A_{eff}$.

$$A_{eff} = \int_{-\infty}^{\infty} 2\pi \theta \beta(\theta) d\theta$$

$A_{eff} = 2\pi \int_{0}^{\infty} \exp(x) dx = 2\pi$

$$A_{eff} = (\theta_{FWHM})^2 \frac{2\pi}{\left(2\sqrt{2\ln 2}\right)^2}$$

$$= (\theta_{FWHM})^2 \frac{\pi}{4 \ln 2}$$
“Brightness” Question

A Red Giant lies at 20 light years from the Earth. It has half the Sun’s surface temperature and 40 times its radius. A White Dwarf lies at 10 light years from the Earth. It has double the Sun’s surface temperature and 1/40 times its radius. Assuming that all three radiate as black bodies, which of the Sun, the Red Giant or the White Dwarf have the highest

- luminosity?
- surface brightness at 30 GHz?
- flux density at 30 GHz at the Earth?
Answer

- $L \propto R^2 T^4$

$$\frac{L_{RG}}{L_{\odot}} = 100; \quad \frac{L_{WD}}{L_{\odot}} = \frac{1}{100}$$

$\Rightarrow$ The Red Giant is most luminous

- $I_\nu \propto T$ (we are in the R-J region)

$\Rightarrow$ The White Dwarf has the highest surface brightness.

- $S_\nu = I_\nu \Omega; \quad \Omega = \frac{\pi R^2}{d^2}$

$$\Omega_{\odot} \gg \Omega_{RG} \gg \Omega_{WD}$$

$\Rightarrow$ The Sun has the highest flux density.

$(S_{\odot} \approx 10^7 \text{ Jy}; \quad S_{RG} \approx 5 \times 10^{-3} \text{ Jy}; \quad S_{WD} \approx 3 \times 10^{-8} \text{ Jy at } 30 \text{ GHz})$