Case Study 4
Chaos and Self-Organised Criticality
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We have dealt with non-linear problems in physics by adopting various strategies. In this case study, we deal with extremely non-linear systems.

- In the case of *chaotic systems*, the equations are well defined, but are non-linear, giving rise to *deterministic* chaotic behaviour.

- In the case of *self-organised criticality*, many complex non-linear effects contribute to the observed behaviour, and yet regularities appear. These appear, despite the fact that there is no way we can write down the appropriate equations for the detailed behaviour of the system.
Introduction to Chaos

Chaotic dynamical behaviour is found in systems which follow deterministic dynamical equations, such as Newton’s Laws of Motion. Consider a dynamical system to be started twice, but from very slightly different initial conditions, say, due to a small error.

- In the case of a non-chaotic system, this uncertainty leads to an error in prediction which grows linearly with time.

- In a chaotic system, the error grows exponentially, so that the state of the system is essentially unknown after a short time, despite the fact that the equations of motion are entirely deterministic.

Excellent references: Chaotic Dynamics – an Introduction by G.L. Baker and J.P Gollub; Chaos J. Gleick.
The Discovery of Chaotic Behaviour

The discovery of chaotic behaviour only became possible with the invention of the electronic computer (despite what happens in Tom Stoppard’s excellent play Arcadia).

The first example of chaotic behaviour was discovered by Edward Lorenz of MIT who set up a simple set of non-linear equations to describe the evolution of weather systems.

\[
\begin{align*}
\frac{dx}{dt} &= -\sigma x + \sigma y \\
\frac{dy}{dt} &= -xz + rx - y \\
\frac{dz}{dt} &= -xy - bz
\end{align*}
\]

He describes what happened in 1961 when he put his set of equations onto his new Royal McBee electronic computer in the video. (Video of Edward Lorenz)
The Necessary Conditions for Chaos

The necessary conditions for chaotic dynamical behaviour of any system are now known.

- The system has at least three dynamical variables.
- The equations of motion contain a non-linear term coupling several of the variables.

Only three variables are needed to obtain:

- the divergence of trajectories;
- confinement of the motion to a finite region of phase space of the dynamical variables;
- uniqueness of the trajectories.
The Damped Driven Pendulum

One of the simplest system which can exhibit chaotic motion is a damped sinusoidally-driven pendulum of mass $m$ and damping constant $\gamma$. The system is driven at angular frequency $\omega_D$. Then, the equation of motion is

$$ml\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + mg \sin \theta = A \cos \phi$$

Notice that we use the equation for oscillations of any amplitude. By changing variables, Baker and Gollub rewrite this equation in simplified form:

$$\frac{d^2\theta}{dt^2} + \frac{1}{q} \frac{d\theta}{dt} + \sin \theta = g \cos \phi$$

where $q$ is the damping or quality factor and $g$ is the amplitude of the driving force, which is not the acceleration due to gravity. The equation has been normalised so that the natural frequency of oscillation of the pendulum $\omega$ is 1.
The Damped Driven Pendulum

This equation can be written in terms of three independent first order equations

\[
\begin{align*}
\frac{d\omega}{dt} &= -\frac{1}{q}\omega - \sin \theta + g \cos \phi \\
\frac{d\theta}{dt} &= \omega \\
\frac{d\phi}{dt} &= \omega_D
\end{align*}
\]

These equations describe:

- \(\phi\) is the phase of the oscillatory driving force with constant angular frequency \(\omega_D\);
- \(\theta\) is the amplitude of the oscillation;
- \(\omega\) is the instantaneous angular velocity.

Thus there are three independent free parameters and non-linearities coupling \(\omega\) to \(\theta\) and \(\phi\). Whether the motion is chaotic or not depends on the values of \(g\), \(\omega_D\) and \(q\).
Visualising Chaotic Motions

There are several ways of representing the phenomena involved in reaching the chaotic state. Examples include:

- Phase space diagrams leading to Lorenz attractors
- Poincaré sections
- Power spectra
- Bifurcation diagrams

All these can be used to represent the different types of motion found in the damped driven pendulum. Many surprising phenomena are found.

Notice that part of the problem of representation is that we need three independent dynamical variables and so two-dimensional diagrams miss one of the variables.
Phase Space Diagrams

We begin with some simple examples of the evolution of simple dynamical systems on simple phase diagrams in which the instantaneous angular velocity is plotted against the phase of the oscillator.

Undamped small oscillations

Damped oscillations

Notice that the trajectory homes in on the origin, what is known as the attractor.
Phase Space Diagrams

Undamped large oscillations

Notice that the pendulum can start ‘upside-down’ at $\theta = \pm \pi$ or can swing round in circles with variable angular velocity.

Undamped large oscillations

In this plot, the angular velocity is plotted as a function of time. Notice that the ‘frequency’ changes as the oscillations become large.
Phase Diagrams for Steady State Oscillations

Forced moderate amplitude oscillations with $q_0 = 45^\circ$.

Phase Diagram

$g = 0.5 \quad q = 2$
Phase Diagrams for Steady State Oscillations

Forced large amplitude oscillations.

$g = 1.07 \quad q = 2$
Phase Diagrams for Steady State Oscillations

Forced large amplitude oscillations.

\[ g = 1.35 \quad q = 2 \]
Phase Diagrams for Steady State Oscillations

Forced large amplitude oscillations.

\[ g = 1.45 \quad q = 2 \]
Poincaré Sections

A useful way of characterising the behaviour of the pendulum is to plot on the phase space diagrams the points at which the pendulum crosses the $\omega - \theta$ plane at multiples of the period of the forcing motion, that is, we make stroboscopic pictures of the motion at the forcing frequency $\omega_D$. This is called a Poincaré Section. For a simple pendulum, there is only one point on the Poincaré section.

These diagrams enable much of the complication of the dynamical motion to be eliminated. The Poincaré sections for the four examples discussed above are shown in the following diagrams. The form of the Poincaré section depends upon the phase of the forcing cycle at which the stroboscopic pictures are taken.
Poincaré Sections

Poincaré sections for the four examples of large steady state oscillations
For certain values of the parameters, however, the motion becomes completely unpredictable - the trajectory of the motion never passes through the same point on the Poincaré section.

\[ g = 1.15, \ q = 2 \]
Poincaré Sections and Chaotic Behaviour

When the motion becomes chaotic, the structure of the Poincaré sections becomes self-similar on different scales. The geometrical structure of the diagrams becomes fractal and a fractal dimension can be defined for the structure.

Chaotic attractors have fractional fractal dimension and are called strange attractors. Baker and Gollub show how the fractal dimension of the phase space trajectories can be calculated.
Bifurcation Diagrams

In bifurcation diagrams, the angular frequency is plotted against the forcing amplitude $g$ of the motion. The various period doublings are clearly seen as well as the regions in which the motion becomes chaotic. Note the extreme complexity of these diagrams, with bifurcations and the appearance of ordered orbits in the midst of chaos.
If power-spectra of the motion of the pendulum are taken, the periodic motion results in spikes in the power-spectra. In this case, $g = 0.95$. The peak at frequency $1/3\pi$ corresponds to the drive frequency $\omega_D$. The other sharp peaks are harmonics of the drive frequency.

In the case of chaotic systems, the power-spectra are broad-band. In this case, $g = 1.5$. 
Lorenz Attractor for Chaotic Motion

These diagrams are all sections through the three dimensional behaviour of the system. It is instructive to observe the dynamical motion of the original version of the Lorenz Attractor. Recall there are three independent variables.

(Excerpt from Channel 4 video Chaos by Chris Hawes.)
Population Dynamics

Chaotic behaviour was discovered by an entirely separate line of development by Robert May. He asked the question, “How does a biological population reach a stable state?”

Suppose the $n$th generation with $x_n$ members results in the $(n + 1)$th generation with $x_{n+1}$ members. Then, we can write

$$x_{n+1} = f(x_n)$$

If we write $x_{n+1} = \mu x_n$, we would obtain unlimited, unsustainable growth of the population. Hence, introduce a cut-off if the population becomes too large.

$$x_{n+1} = \mu x_n (1 - x_n)$$

This is a difference equation similar to that introduced by Verhulst to model the development of populations, the Verhulst equation

$$\frac{dx}{dt} = \mu x (1 - x).$$
Logistic Maps

The evolution of the population can be evaluated for different values of $\mu$ on what is called a logistic map. The value of $x_{n+1}$ is plotted on the ordinate against $x_n$ on the abscissa and in the next step the value $x_{n+1}$ is plotted on the abscissa as the input for the next iteration. For different values of $\mu$, we can find:

- Stable population
- Limit cycle between two states
- Oscillation between four states with period doubling
- Chaotic behaviour
Logistic Map

\[ x_{n+1} = 2x_n(1 - x_n) \]

Thus, starting at \( x_1 = 0.2 \), we find the sequence,

- \( x_2 = 0.32 \)
- \( x_3 = 0.4352 \)
- \( x_4 = 0.49160192 \)
- \( x_5 = 0.4998589445 \)
- \( x_6 = 0.4999999999602 \)

The population becomes stable at \( x_n = 0.5 \).
Logistic Map

\[ x_{n+1} = 3.3 x_n (1 - x_n) \]

Thus, starting at \( x_1 = 0.2 \), we find the sequence,

- \( x_2 = 0.528 \)
- \( x_3 = 0.8224128 \)
- \( x_4 = 0.4819649551 \)
- \( x_5 = 0.8239266326 \)
- \( x_6 = 0.4787369711 \)

The population oscillates between \( x_n = 0.48 \) and \( x_n = 0.83 \). There is a limit-cycle in the behaviour of the population.
Logistic Map

\[ x_{n+1} = 3.53 \times x_n(1 - x_n) \]

Thus, starting at \( x_1 = 0.2 \), we find the sequence,

- \( x_2 = 0.5648 \)
- \( x_3 = 0.8676773888 \)
- \( x_4 = 0.4052910823 \)
- \( x_5 = 0.8508366798 \)
- \( x_6 = 0.4480050931 \)

The population becomes stable at the values

\[ x_n = 0.37, 0.52, 0.83, 0.88. \]
Logistic Map

\[ x_{n+1} = 3.9 \cdot x_n(1 - x_n) \]

The system becomes chaotic.
Period Doubling

The passage to chaos occurs through a sequence of period doublings and is a common feature of dynamical systems. The ratio of spacings between consecutive values of $\mu$ at the bifurcations tends to a universal constant, the Feigenbaum number.

$$\lim_{k \to \infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} = 4.6692016091029909\ldots$$

This number is a universal constant for the period-doubling route to chaos for maps with a quadratic maximum. It is an example of *Universality* in chaotic dynamical systems.
Bifurcation Diagram for the Logistic Maps

The bifurcation diagram for the logistic maps $x_{n+1} = \mu x_n (1 - x_n)$ shows all the features of bifurcations, period-doubling, the onset of chaotic behaviour previously as a function of the parameter $\mu$. 
Physical Applications of these Ideas

There are huge numbers of applications of these ideas to complex physical systems.

- Lasers
- Chemical systems
- Electronic circuits with non-linear elements
- Turbulence (see video clip from Chaos programme.)
- Jupiter’s great red spot (see video clip from Chaos programme.)
- Tumbling motion of satellites in Orbit about Planets – the case of Hyperion.
The Tumbling of Hyperion

Hyperion has an orbital period of 21 days and its dimensions are 190 x 145 x 114 km. Its orbit is eccentric about Saturn and so it experiences a varying gravitational force. Although the orbital motion is stable, Hyperion tumbles chaotically.

Poincare section for the position angle of the major axis of Hyperion and its rate of change.
Self-Organised Criticality

Let us now tackle the even more extreme cases in which many non-linear effects contribute to the behaviour of the system.

In what is known as *self-organised criticality*, many complex non-linear effects contribute to the observed behaviour, despite which regularities appear. These occur despite the fact that there is no way we can write down the appropriate equations for the detailed behaviour of the system.

The ideas originate from the appearance of some remarkable *scaling laws* in complex systems. The common feature of all these examples is that they are so complex that there is no way in which we can write down the relevant equations which might describe all aspects of the system.

Let us consider a number of examples.
The Gutenberg-Richter Law for Earthquakes

The frequency of occurrence of earthquakes of different magnitude on the Richter scale. The Richter scale is logarithmic in the energy release $E$ of the earthquake. The frequency distribution is a power-law in $E$.

Consider the huge number of effects responsible for the formation of an earthquake:
- Plate tectonics
- Mountains, valleys, lakes
- Geological structures

The linearity of the law suggests that, despite the complexity, there is some underlying systematic behaviour. Large earthquakes are just the extreme form of smaller earthquakes.
In 1963, Mandelbrot analysed the variation of cotton prices over a period of 30 months. The distribution was ‘spikey’. He determined the distribution of the change in the cotton price from month to month and then plotted the probability distribution of the relative variation on a logarithmic scale. The variations follow a power-law distribution. Other commodities, and hence the stock exchange itself, also follow this pattern.

Consider the many non-linear effects which go into the determination of the cotton price. Supply, demand, rushes on the stock market, speculation, psychology, insider dealing, etc.
John Sepkopski spent 10 years in libraries searching the fossil records in order to determine the fractions of biological families present on the Earth which became extinct over a period of 600 million years. The data were grouped into periods of 4 million years. Notice the huge variation in the extinction rate. The plot of the frequency of events follows a power-law distribution, with a few large extinction events and larger numbers of smaller events.
Notice that the epoch of the extinction of the dinosaurs, at about 65 million years before present, was by no means the largest extinction event.

It was probably due to the collision of an asteroid with the Earth. The location of the collision has been found to be on the coast at the Gulf of Mexico.
The term fractal was introduced by Benoit Mandelbrot to describe geometrical structures with features on all length scales. In his great book *The Fractal Geometry of Nature*, he made the profound observation that the forms found in nature are generally of a fractal character.

A classic example is coastline of Norway. It contains fjords within fjords, within fjords, . . . .

The simplest way of quantifying its fractal nature is to ask how many boxes of length $\delta$ are needed to cover the complete coastline. The first few boxes on large scales are shown. The length of the coast follows a power-law distribution $L(\delta) \propto \delta^{1-D}$, where $D = 1.52 \pm 0.01$ is the fractal or Hausdorff dimension.
Galaxies in the Universe

The distribution of galaxies on the sky can be described by a two point correlation function

\[ N(\theta) \, d\Omega = N_0[1 + w(\theta)] \, d\Omega \]

where \( N_0 \) is a suitably defined average surface density and \( w(\theta) \) describes the excess probability of finding a galaxy at angular distance \( \theta \) within the solid angle \( d\Omega \).

Observations of large samples of galaxies have shown that the function \( w(\theta) \) has the form \( w(\theta) \propto \theta^{-0.8} \) corresponding to a spatial correlation function

\[ \xi(r) \propto r^{-1.8} \quad \text{for} \quad r \leq 8 \text{ Mpc} \]

This corresponds to a fractal distribution. The formation of these structures unquestionably involves non-linear gravitational interactions.
$1/f$ Noise

$1/f$ noise is found in many different circumstances. It has been found in electronic circuits where it manifests itself as spikes far exceeding the random Johnson noise.

Two typical examples of $1/f$ noise are (a) the time variability of quasars and (b) global temperature variations since 1865. They are compared with (c) the spectrum of *white noise*. Generally, the $1/f$ noise spectrum has the form

$$I(f) \propto f^{-\alpha} \quad \text{where} \quad 0 < \alpha < 2$$

In both cases, the observed behaviour is due to the effect of a large number of non-linear processes. In the case of electronic circuits, precautions must be taken so that the large spikes do not upset the operation of the device.
Zipf first showed that there is systematic behaviour in the ranking of city sizes, just as there is in the case of earthquakes. A large number of non-linear effects determine the size of any city, and yet they all conspire to create a power-law distribution of size.

The frequency with which words are used in English also has a power-law scaling law. What are the myriad of social and perceptual processes which result in a power-law rather than some other distribution?

Classical music is $1/f$ noise. Boulez and Stockhausen are not fractal. (Mandelbrot, private communication)
Sand and Rice Piles

Remarkably, the paradigm for all these different types of complex non-linear behaviour may be the behaviour of sand or rice piles. As sand is poured steadily onto the top of the pile, the sand pile takes its shape by undergoing a series of avalanches. The avalanches have many different sizes, but they result overall in a fixed angle for the sand or rice pile. This has become the archetype of self-organised criticality.
Sand Piles and Rice Piles

The evolution of model sand and rice piles can be followed by computer simulation. In a simple example given by Per Bak, 1, 2 or three grains are placed randomly on a $5 \times 5$ grid to begin with. Then, grains are added to the central square. When any element of the grid has 4 grains, they are redistributed to the four neighbouring sites. If any of these then have four grains, these are also redistributed, and so on, until that avalanche is over. Then, the next grain is added to the central square.

A Model Sand-pile

\[
\begin{array}{ccccc}
1 & 2 & 0 & 2 & 3 \\
2 & 3 & 2 & 3 & 0 \\
1 & 2 & 3 & 3 & 2 \\
3 & 1 & 3 & 2 & 1 \\
0 & 2 & 2 & 1 & 2 \\
\end{array}
\]
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\[
\begin{array}{ccccc}
1 & 2 & 0 & 2 & 3 \\
2 & 3 & 2 & 3 & 0 \\
1 & 2 & 4 & 3 & 2 \\
3 & 1 & 3 & 2 & 1 \\
0 & 2 & 2 & 1 & 2 \\
\end{array}
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**A Model Sand-pile**

```
1 2 0 2 3
2 3 3 3 0
1 3 0 4 2
3 1 4 2 1
0 2 2 1 2
```
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A Model Sand-pile

```
1 2 0 2 3
2 3 3 4 0
1 3 2 0 3
3 2 0 4 1
0 2 3 1 2
```
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A Model Sand-pile

\[
\begin{array}{cccccc}
1 & 2 & 0 & 3 & 3 \\
2 & 3 & 4 & 0 & 1 \\
1 & 3 & 2 & 2 & 3 \\
3 & 2 & 1 & 0 & 2 \\
0 & 2 & 3 & 2 & 2 \\
\end{array}
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A Model Sand-pile

```
1 2 1 3 3
2 4 0 1 1
1 3 3 2 3
3 2 1 0 2
0 2 3 2 2
```
Sand Piles and Rice Piles

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*A Model Sand-pile*

```
1 3 1 3 3
3 0 1 1 1
1 4 3 2 3
3 2 1 0 2
0 2 3 2 2
```
Sand Piles and Rice Piles

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A Model Sand-pile

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\begin{center}
A Model Sand-pile
\end{center}

\begin{center}
\begin{tabular}{cccccc}
1 & 3 & 1 & 3 & 3 \\
3 & 1 & 2 & 1 & 1 \\
2 & 1 & 0 & 3 & 3 \\
3 & 3 & 2 & 0 & 2 \\
0 & 2 & 3 & 2 & 2 \\
\end{tabular}
\end{center}
Example of a Model of a Sandpile on a $5 \times 5$ grid

In this example, a single grain added to the centre results in nine toppling events in seven iterations. We can also plot out the cells on which the avalanches occurred.

A Model Sand-pile

```
0 0 0 0 0
0 1 1 1 0
0 1 2 1 0
0 0 1 1 0
0 0 0 0 0
```

Per Bak and his colleagues have repeated these simulations with much larger grids, for example, on a $50 \times 50$ grid.
Evolution of a Sandpile on a $50 \times 50$ Grid

These snapshots show a propagating avalanche in the sandpile model. The light blue area shows regions which have toppled at least once.
The Size Distribution According to the Sandpile Model

By carrying out the experiment many times, Bak and his colleagues determined the size distribution of the avalanches. The results of the numerical experiments have the form of a power-law with slope 1.1, almost exactly the value found in the Gutenberg-Richter law.
Self-Organised Criticality

The key results of the model sand-pile are:

- The simulation gave a power-law distribution of avalanches, of exactly the same form as the Gutenberg-Richter law.
- The sand-pile became ‘critical’ without any fine-tuning.
- The profile of the sand-pile is a fractal.
- The same behaviour occurs for many different types of models.
- It has not been possible an analytic model for any self-organised self-critical system.

Is this the paradigm for all the other examples of scaling phenomena?
Other Examples

There are many other examples:

- Acoustic emission from volcanoes.
- Crystal, cluster or colloid growth.
- Forest fires
- The game of life
- Cellular automata

All these self-organised critical phenomena involve many highly non-linear phenomena.
The inspiration comes from a somewhat unexpected source and involves the problems of tiling an infinite plane by small numbers of tiles of fixed geometric shapes. It is obvious that an infinite plane can be tiled by simple tiles of regular geometry, for example, by squares, triangles, and so on. What is much less obvious is that there exist shapes which can tile an infinite plane without leaving any holes, but which have the extraordinary property that they are non-periodic. The pattern of tiles never repeats itself anywhere on an infinite plane. I find this a quite staggeringly non-intuitive result.
Is All Physics Computable?

Roger Penrose has been able to reduce the numbers of shapes required to tile an infinite plane non-periodically to only two. One example of a pair of tiles which have this property is shown above. Penrose refers to the two shapes as ‘kites’ and ‘darts’. The configuration of the tilings cannot be predicted at any point in the plane since the patterns never recur in exactly the same way. Furthermore, how would a computer be able to establish whether or not the tiling was periodic or not? It might be that analogues of the tiling problem turn up in different aspects of physical problems.