SYSTEMS

Part II  Experimental and Theoretical Physics
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Overview:

What is a system? Goals of analysis: stability and return to equilibrium, classification, parameter estimation.

Linear Systems:


- **Sampled systems**: Rationale. Effect of sampling on frequency content. Aliasing and the sampling theorem. Finite length data sequences and leakage. Other windows. Discrete Fourier Transform. Effect of sampling in frequency domain, and periodic properties (“circular convolution”).

Non-linear systems:


*Items in italics are not for examination.*

BOOKS

Linear system theory is covered in a variety of books and approaches, and there is no clear preference. There are many texts on “Signals and Systems”, many of which make an easy subject look much more difficult than it is. Useful texts include:


The treatment of non-linear behaviour follows closely that in

The basics of non-linear dynamics are also covered quite well in

There is also an admixture of material from
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1. Introduction

1.1. What is a system?

For the purposes of this lecture course, we can define a system as being anything that produces an output in response to one or more inputs. One way of looking at systems, therefore, is as physical implementations of mathematical operators. For true physical systems these operators are normally differential, but we may also have to consider difference operators as well (for example, when analysing sampled data, or performing computer simulations).

Clearly, almost everything we deal with in physics can be regarded as a system, but in this course we are primarily interested in changes in behaviour in response to a variation of one or more inputs. It is traditional to separate inputs into two sorts — those that are intended to produce a change in the output variable, and those that characterize a particular setting of the system. The latter are often referred to as parameters, or control variables, but there is no essential difference between these and more conventional inputs.

1.2. Goals of analysis.

In general, we are interested in the dynamical behaviour of systems, and their response to excitations. You have already explored this topic in some depth in IB dynamics, and indirectly in most of your other courses. Those of you who attended the TPI lectures have carried this even further. One thing that you saw there was the notion of stochastic dynamics — that is, the idea that non-kinematic variables describing a system could also evolve with time. In this course we use dynamics in the broadest sense, to describe the evolution of any such physical variable.

We may be interested simply in whether or not a particular system is in equilibrium for some combination of control variables, and if so, whether the equilibrium is stable or unstable. More particularly, we may wish to know how a system behaves when displaced from equilibrium — by what path does it return, and how long does it take to settle? It turns out that systems have a very limited repertoire of behaviours — that is, many physically diverse systems share relatively few types of response — and one of the goals of this course is to see how these can be classified.

Another goal follows on directly from your Experimental Methods course in Part IB. Parameter estimation is a very common problem in physics: given a particular model for a system, and some data showing how the outputs depend on the inputs, can one determine the unknown model parameters, and if so, should one believe the model or not? This sort of analysis also casts light on the design of
experiments: what sort of data are required in order to be able to deduce the unknowns to the desired degree of accuracy?

1.3. Outline of the course

The course falls naturally into two sections.

In the first half, we revisit linear time-independent (LTIV) systems (that is, those where the output can be written as a superposition of the responses to each input taken separately). The Green function approach follows naturally from this definition, and leads directly to the formulation of the response in terms of a convolution. By use of the convolution theorem, we can then transform a system governed by a differential operator into a purely algebraic one. This leads us to the notion of a filter. There is a very direct relationship between the filter response and the eigenvalues (or natural frequencies) of the system, and this is explored in some depth.

Sampled-data systems are often treated from the point of view of difference operators, but in general we can treat these as special cases of continuous LTIV systems in which the inputs and outputs are taken to be zero except at the sampling instants. Sampling can have quite profound effects on the system response, and because of the practical importance, we spend a fair amount of time on this.

By this point in your career, you have probably concluded that all physics boils down to the solution of a few (generally linear) differential equations. The more perceptive among you will have noticed that whenever our underlying mathematical model is fundamentally non-linear, we tend to shy away, and perform a perturbation analysis by linearising around a solution gained by some other method. This does not reflect any underlying belief that world is in general well-described by linear models, but more our lack of understanding of how to deal with most non-linear ones. Unfortunately, linearisation necessarily suppresses much of the most interesting behaviour, shown by a variety of physical, economic and biological systems. This behaviour — and its classification — is the main topic of the second part of the course.
2. Analysis of Linear Systems

2.1. Superposition

A linear system can be loosely defined as any system in which the change in the output for any input is always proportional to the change in that input. More precisely, if for any two or more input functions \( x(t) \) and \( y(t) \), the system has response \( f(x) \) and \( f(y) \), then, if \( \alpha \) and \( \beta \) are scalars, it is also true that

\[
2.1 \quad f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).
\]

This is the principle of superposition.

2.2. The Convolution Theorem and the Transfer Function

Superposition leads directly to the Green Function. If can represent the input to the system as a sum of elementary inputs, then we can immediately write the output as the sum of the outputs for each of those inputs taken separately. One convenient elementary input is the delta function\(^1\), \( \delta(t - t') \), for which the Green Function is normally referred to as the impulse response. Any arbitrary input, \( x(t) \), can be represented as a sum of delta functions:

\[
2.2 \quad u(t) = \lim_{\tau \to 0} \sum_{n=0}^{\infty} u(n\tau)\delta(t - n\tau).
\]

Then if the response to a single delta function input, \( \delta(t - t') \), is \( h(t - t') \), the response \( x(t) \) can be written as:

\[
2.3 \quad x(t) = \lim_{\tau \to 0} \sum_{n=0}^{\infty} u(n\tau) h(t - n\tau).
\]

Taking the limit, we obtain:

\[
2.4 \quad x(t) = \int_{-\infty}^{\infty} u(t') h(t - t')dt = u(t) * h(t).
\]

This is the convolution theorem. If the Green Function is causal – that is, the output is zero for all times before the input is applied – then the upper limit of integration may be replaced by \( t \), while the lower limit may be replaced by the time when the input is first applied (usually at time \( t = 0 \)).

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\(^1\) The delta function is only one of many possibilities as a basis function. You may wish to consider the unit step function, \( U(t) \), which is defined as being 0 for all times prior to \( t=0 \), and 1 for all later times. It is easy to see that the appropriate expansion for \( u(t) \) would then be \( u(t) = \lim_{\tau \to 0} \sum_{n=0}^{\infty} u'(n\tau) \tau U(t - n\tau) \), where \( u'(t) \) is the derivative of \( u \) with respect to \( t \).
Consider the Fourier Transform representation of a periodic signal, \( U(\omega) = \int_{-\infty}^{\infty} u(t) e^{i\omega t} \, dt \). It is simple to show that the Fourier Transform of the convolution in the previous section can then be written as \( X(\omega) = U(\omega) H(\omega) \) (note the use of capitals to denote functions in the frequency domain – but they are sometimes denoted by an over-tilde, for example \( \tilde{u}(\omega) \)). That is, the integral relationship in the time domain reduces to a simple product in the frequency domain. We refer to \( U(\omega) \) as the spectrum of the input, and \( H(\omega) \) as the Transfer Function.

\[
H(\omega) \triangleq \frac{X(\omega)}{U(\omega)}. 
\]

The spectrum of the output is produced simply by multiplying the value of the input spectrum at each frequency by the (generally complex) value of the Transfer Function at the same value of \( \omega \). \( H(\omega) \) is therefore clearly just a complex filter. Since our derivation is entirely general, it follows that the response of any linear system can be represented by a simple filter.

Real signals have conjugate symmetric Fourier Transforms, so input \( U(\omega) \), output \( X(\omega) \) and transfer function \( H(\omega) \) are all guaranteed to be conjugate symmetric.

For a passive filter, the output power must be less than or equal to the input power, and the filter response must therefore tend to zero as \( \omega \to \infty \). The effect of a passive filter is therefore always to attenuate the high frequencies in the input signal.

We have seen that there are two exactly equivalent ways of representing the response of any linear system. In the time domain it is characterized by the impulse response, \( h(t) \), and in the frequency domain by the Transfer Function \( H(\omega) \).

**Example 1. Damped RLC harmonic oscillator**

Consider the RLC damped harmonic oscillator shown in the Figure

The differential equation describing this circuit is

\[
L \dot{q} + R \dot{q} + \frac{q}{C} = v(t),
\]

where \( q \) is the charge on the capacitor and \( v(t) \) is the driving voltage.

The free response to initial conditions (i.e. the Complementary Function) can be found by setting the RHS to zero, and solving using the trial solutions \( q(t) = Ae^{\alpha t} + Be^{\beta t} \).
We find that \( s = -R/2L \pm \sqrt{(R/2L)^2 - 1/LC} \), or in canonical form, \( s = -\gamma \pm i\omega \), \( \gamma = R/2L \), \( \omega = \sqrt{\omega_0^2 - \gamma^2} \), and \( \omega_0^2 = 1/LC \).

We can obtain the impulse response most easily by physical reasoning: during the impulse, the applied voltage is infinite, while the initial current and charge are both zero. The effect of the impulse is therefore to produce a step-change in the current, but to leave the charge on the capacitor at zero. Thereafter the circuit undergoes free, damped, oscillations. Working through the algebra we obtain the impulse response,

\[
h(t) = q(t) = \frac{1}{\omega L} e^{-\gamma t} \sin \omega t \times U(t).
\]

The response to an arbitrary input can now be evaluated by convolution, so that, in general,

\[
q(t) = \frac{1}{\omega L} \int_0^t v(t') e^{-\gamma(t-t')} \sin \omega(t - t') \, dt'.
\]

**Example 2. Damped harmonic oscillator frequency response**

For the same circuit as above, we can also derive the frequency response directly, by assuming a sinusoidal input \( e^{i\omega t} \) and finding the particular integral. In this way, we obtain

\[
-\omega^2 LQ(\omega) + i\omega R Q(\omega) + Q(\omega)/C = V(\omega),
\]

which gives the transfer function

\[
H(\omega) = \frac{Q(\omega)}{V(\omega)} = -\frac{1}{\omega^2 L - i\omega R - 1/C}.
\]

The roots of the characteristic equation are \( i\omega = R/2L \pm \sqrt{(R/2L)^2 - 1/LC} \), or in canonical form, \( i\omega = \sqrt{\omega_0^2 - \gamma^2} \pm i\gamma \), with \( \gamma = R/2L \), and \( \omega_0^2 = 1/LC \), as before. The numerator of the transfer function can be written as a polynomial in \( \omega \), giving the form

\[
H(\omega) = \frac{Q(\omega)}{V(\omega)} = \frac{-1/L}{(\omega + \omega' - i\gamma)(\omega - \omega' - i\gamma)}.
\]

That is, a pair of complex roots. If \( R \) is set to zero, then the damping coefficient \( \gamma \) is also zero, and the frequency response goes to infinity at frequencies plus or minus \( \omega \).
The above example illustrates the close relationship between the natural frequencies of the system (as given by the free response) and the infinities, or poles of the transfer function. Clearly, for an undamped system excited resonantly – i.e. at one of its natural frequencies – we expect the amplitude of the response to be infinite. If the system is lightly damped (i.e. $\gamma \ll \omega_0$) then for frequencies close to $\omega_0$, the response is large, but no longer infinite, with a maximum at $\omega' = \left( \omega_0^2 - \gamma^2 \right)^{1/2}$.

2.3. Eigenfunction approach

It is worth reflecting for a moment on how this method works. What we have shown so far is that for any input waveform of the form $e^{\omega t}$, the output of the system can be calculated simply by multiplying the input by a simple complex number, which is a function only of the input frequency $\omega$. Viewed this way, it can be seen the function $e^{\omega t}$ is an eigenfunction of the system, in that output is simply the input multiplied by a value which does not depend on time.

At first sight this doesn’t seem to help much with the case of a general input. However, if we can decompose the input into a sum of terms in $e^{\omega t}$, then linearity allows us to calculate the response to each one individually, and then sum them together again to yield the total output.

But decomposing a function of time into a sum of terms in $e^{\omega t}$ is exactly what the Fourier Transform does! So the general rule for linear systems is:

1. Represent the input function of time as a sum (or integral) over terms in frequency, using the Fourier Transform.
2. Re-weight each frequency by the (complex) value of the transfer function at that frequency.
3. Add the re-weighted terms back together again, to reconstitute the output as a function of time, using the Inverse Fourier Transform.

The only problem arises when we cannot perform the decomposition represented by the Fourier Transform – that is, the Fourier Integral does not converge. An obvious example is the unit step function, $U(t)$ which is zero for all times prior to $t=0$ and one thereafter. It turns out however that the function $e^{\omega t}$ is not the most general eigenfunction. In fact, it is quite easy to show that for any Linear Time-Invariant (LTI) system that the function $e^{st}$, with $s$ any complex number, is also an eigenfunction. It turns out that it is often possible to decompose a function of time into a set of complex exponentials, even when the Fourier Transform is not defined.
2.4. Laplace transforms

By analogy with the Fourier Transform, we are looking for a method where we can:

1. Decompose the input waveform into a sum (or integral) of complex exponentials.
2. Re-weight the coefficient for each value of $s$ by a complex value representing the Transfer Function as a function of $s$.
3. Re-sum or integrate the resulting weighted basis functions to generate the output as a function of time.

The above is a description of the Laplace Transform. It is similar to the Fourier Transform, but has an additional convergence factor in the kernel:

$$\mathcal{L} \{ f(t) \} = F(s) = \int_{0}^{\infty} f(t) e^{-st} \, dt$$

where $s = \sigma + j\omega$ is the complex frequency (see diagram). Note that the Laplace transform may be defined in either the one-sided form, as shown, or in a two-sided form with limits of integration from minus infinity to plus infinity. Almost all applications of the Laplace Transform, however, are to time-domain systems, with the input prior to time $t = 0$ being zero by convention.

For the step function, therefore,

$$\mathcal{L} \{ U(t) \} = F(s) = \int_{0}^{\infty} e^{-st} \, dt = \frac{1}{s}$$

We can represent this in the complex-$s$ plane as a simple pole at the origin:

The Laplace Transform has a number of useful properties, represented in the table below:

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convolution</td>
<td>$u(t) \ast f(t)$ $\overset{L}{\rightarrow}$ $U(s) \ast F(s)$</td>
</tr>
<tr>
<td>Differentiation</td>
<td>$\frac{df(t)}{dt}$ $\overset{L}{\rightarrow}$ $s \cdot F(s) - f(0)$</td>
</tr>
<tr>
<td>Integration</td>
<td>$\int_{0}^{t} f(t) , dt$ $\overset{L}{\rightarrow}$ $\frac{F(s)}{s}$</td>
</tr>
<tr>
<td>Shift (left)</td>
<td>$f(t + T)$ $\overset{L}{\rightarrow}$ $e^{-sT}F(s)$</td>
</tr>
<tr>
<td>Scale change</td>
<td>$f(\alpha t)$ $\overset{L}{\rightarrow}$ $\frac{1}{\alpha}F\left(\frac{s}{\alpha}\right)$</td>
</tr>
</tbody>
</table>

All these follow from the basic definition, and you are invited to prove them, by way of practice. Note that, as with the Fourier Transform, the rule for differentiation allows differential equations in the time domain to be converted to algebraic equations in the frequency domain.
Note on the Region of Convergence

For the general case, it is necessary to define the range of values of $\sigma$ for which the Laplace Transform exists. This gives the Region of Convergence, or ROC. In general it is necessary to know the ROC as well as the algebraic form of the transform in order to choose the correct inverse.

For example, consider the function $e^{-at}$ for $t>0$, which has the transform $(s + a)^{-1}$, with a pole at $s = -a$. This function converges (in the sense that it approaches zero for large values of $|s|$) only for values of $s$ with a real part greater than $a$. The same function, but defined only for $t<0$, has the same transform, but the ROC is now that part of the plane with $\text{Re}(s) < a$.

Since the ROC is determined purely by the real part of $s$, it always takes the form of a strip lying parallel to the imaginary axis in complex frequency space.

2.5. Inverse Laplace Transform – analytic solution (not for examination)

The Inverse Laplace Transform has the form:

$$ \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) e^{st} \, ds. \quad 2.8 $$

The real part of the limits of integration can take any value, as long as the path lies within the ROC.

One way to do this is via a Bromwich contour integral, closing the curve at infinity (as illustrated in TP1 for those who did it). For $t<0$ the result is zero, and we need to close the curve to the right of the right-most pole; for $t>0$, we close the curve to the left, so as to surround the pole(s). For the sorts of problems dealt with in this course, the only singularities are poles (possibly multiple) and the contour integral evaluates to $2\pi i$ times the sum of the residues, where the residue for a pole of order $m$ is given by:

$$ \text{Res}_m(z) = \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z). \quad 2.9 $$

For a single pole, therefore, the residue is just

$$ \text{Res}_1(z) = (z-a) f(z). \quad 2.10 $$

Consider a function $F(s) = (s + a)^{-1}$, which has a single pole located at $s=-a$. The integrand in the inverse transform is $F(s) = e^{st} (s + a)^{-1}$. The residue is then just $e^{at} \left|_{s=-a} \right. = e^{-at}$, and the contour integral evaluates to $2\pi i e^{-at}$, giving the inverse transform $e^{-at}$ as expected.
2.6.  Other methods for the Inverse Laplace Transform

You are not expected to perform the integrals of the previous section in this course, although it is worthwhile noting how they come about. Instead, as with Fourier Transforms, we use a set of very simple transforms. These are tabulated in many places (see, for example, Erdelyi, “TABLES OF FOURIER AND LAPLACE TRANSFORMS”), and a set of useful transform pairs is provided as an appendix to these notes.

For linear systems of finite order (the only ones dealt with in this course), then there is an even simpler technique. Generally for these systems, \( F(s) = \frac{A(s)}{B(s)} \). This can be expanded to partial fraction form.

\[
F(s) = \frac{c_0}{s - p_0} + \frac{c_1}{s - p_1} + \cdots + \frac{c_n}{s - p_n},
\]

where \( c_n = (s - p_n) F(s) \bigg|_{s = p_n} \). We can then integrate each term independently to obtain \( f(t) = c_0 e^{p_0 t} + \cdots c_n e^{p_n t} \).

Note that in the case of a repeated root (e.g. a double pole) we need also to include terms of the form \( t e^{-\alpha t} \), etc (c.f. the treatment of a repeated root in the time domain, where we add terms of the form \( t e^{-\alpha t} \)).

The most common exception to this happy behaviour is a simple time delay. In such a system the phase shift from input to output increases linearly with frequency, and cannot be represented as a rational function of \( s \).

2.7.  A general recipe for Linear Systems problems

Putting all this together, we deduce the following procedure for tackling linear systems problems:

1. Deduce the transfer function (if necessary from the differential equation)
2. The Impulse response of the system follows directly from the transfer function, which you will recall is the Laplace Transform of the output when the input = \( \delta(t) \).
3. The response to any other input follows by multiplying its Laplace transform by the transfer function.
4. Invert the resulting product to obtain the time response.
Example 3. Transfer function of an RC filter.

As an example of finding the transfer function, consider the simple RC circuit shown at the right:

In the time domain we have

\[ x(t) = i(t)R \]
\[ u(t) = \frac{1}{C} \int_0^t i(t')dt' + i(t)R \]

In the frequency domain this becomes:

\[ X(s) = I(s)R \]
\[ U(s) = \frac{I(s)}{sC} + I(s)R, \text{ and hence} \]
\[ U(s) = \frac{X(s)}{sRC} + X(s). \]

So the transfer function is

\[ H(s) = \frac{X(s)}{U(s)} = \frac{s}{s + 1/\tau}, \]

where \( \tau = RC \).

Let us apply our recipe to repeat the analysis of the LCR damped harmonic oscillator, using Laplace Transforms so that we can deal with arbitrary inputs.

Example 4. Damped harmonic oscillator response to unit step

Find the response of the circuit of Example 1. to a unit step input. The charge and current at time \( t = 0 \) immediately before application of the step function, are both zero.

As a reminder, the differential equation is:

\[ L\ddot{q} + R\dot{q} + q/C = v(t). \]

Taking the Laplace Transform of both sides, and using the rules for differentiation given in the table (assuming zero initial conditions for now), we obtain

\[ (Ls^2 + Rs + 1/C)Q(s) = V(s). \]

Hence the transfer function, \( H(s) \), is given by

\[ H(s) = \frac{Q(s)}{V(s)} = \frac{1}{Ls^2 + Rs + 1/C} = \frac{1/L}{(s + \gamma + i\omega)(s + \gamma - i\omega)}, \]

with \( \omega^2 = \omega_0^2 - \gamma^2 \) as before. To find the response to a unit step, write \( V(s) = 1/s \):

\[ Q(s) = \frac{H(s)}{s} = \frac{1/L}{s(s + \gamma + i\omega)(s + \gamma - i\omega)} = \frac{1/L}{s(s + a)(s + a')}. \]
In order to find the time response, we first need to find the partial fraction expansion, which is just

\[ LQ(s) = \frac{1}{aa^*s} + \frac{1}{a(a - a^*)(s + a)} - \frac{1}{a^*(a - a^*)(s + a^*)} \]

\[ = \frac{1}{\omega_0^2s} + \frac{1}{2i\omega'a(s + a)} - \frac{1}{2i\omega'a^*(s + a^*)} \]

and hence taking the inverse Laplace Transform of each term separately (using \(aa^* = \gamma^2 + \omega'^2 = \omega_0^2\)),

\[ Lq(t) = U(t) \left[ \frac{1}{\omega_0^2} + \frac{1}{2i\omega'} \left( \frac{e^{-at}}{a} - \frac{e^{-a^*t}}{a^*} \right) \right] \]

\[ = U(t) \frac{1}{\omega_0^2} \left[ 1 + \frac{1}{2i\omega'} \left( a^* e^{-at} - a e^{-a^*t} \right) \right] \]

Substituting for \(a\) in the last term, and expressing the exponentials in terms of sin and cos, we obtain the final answer,

\[ q(t) = U(t)C \left[ 1 - e^{-\gamma t} \left( \cos \omega't + \frac{\gamma}{\omega} \sin \omega't \right) \right] . \]

An alternative method is to express the Laplace Transform as a sum of non-fundamental terms. Consider the same problem, but, to simplify things, with zero damping (so that \(\gamma = 0; a = i\omega' = i\omega_0\)).

**Example 5. Inverse Laplace Transform by use of tables.**

Using \(Q(s)\) as obtained in the first part of the previous example, express it instead as:

\[ Q(s) = \frac{1}{\omega_0^2} \left[ \frac{1}{s} - \frac{s a}{s^2 + \omega_0^2} \right] \]

The first term can be inverted directly as before: the second term can be evaluated using a table of transforms:

\[ \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + \omega^2} \right\} = \cos \omega t , \]

yielding the same result as obtained the hard way in Example 4.

**2.8. Pole-zero representation of transfer functions**

We have already been introduced to the *poles* of the transfer function, which are the complex values of \(s\) for which the denominator is zero, or equivalently, for which the function approaches infinity. The *zeros* of the transfer function are the values for which the numerator (and hence the transfer function) are zero. The poles and zeros can be plotted in the (complex) \(s\)-plane. Conventionally we use the symbols ‘\(\times\)’ for a zero and ‘\(\bullet\)’ for a pole.

Consider the case of a single pole on the negative real axis. We can infer the frequency response of this system directly from the pole-zero plot. In order to obtain the frequency response, we need to plot the
magnitude and phase of the transfer function (i.e. the filter function) as \( \omega \) is varied from 0 to infinity. That is, we move along a path vertically along the positive \( \omega \) axis on the diagram. The magnitude of the transfer function is then given by the product of the distances of the point on the axis from each of the zeros, divided by the product of the distances from the poles.

As a second example, consider the representation of the function

\[
F(s) = \frac{(s - z)(s - z^*)}{(s - p)(s - p^*)},
\]

depicted below.

Close to the zeros, the magnitude is small; close to the poles the magnitude is correspondingly large. The phase of the transfer function is similarly given by taking the sum of the angles from the point on the axis to the zeros, minus the sum of the angles from the same point to each of the poles.

Examples of magnitude and phase plots for the case of zeros at \( \pm 2 \) and poles at \( -1 \pm 2.5i \) are shown at right.

2.9. System building blocks

Most of the discussion so far has been about the analysis of arbitrary systems, assuming only that the impulse response or frequency response is known. There are times, however, when we wish to realize a particular transfer function. Quite often we simply want a filter which will bring out some feature of our data, but we may also be interested in controlling an existing system, as for example to complete the feedback loop in a Scanning Tunnelling Microscope, or when requiring a telescope to point very accurately at an astronomical source in the presence of windshake and other disturbances.

There are two ways of implementing such transfer functions electronically. Often we simply sample the data and apply the transfer function digitally, either in a general purpose computer or in some sort of Digital Signal Processing (DSP) chip. The sampling causes various
effects, which are the subject of section 4. Here we consider analogue systems, which for the most part are constructed from high-gain operational amplifiers, or op-amps.

You have already seen op-amps on many occasions. As a quick reminder, the basic negative feedback circuit is shown here. If the op-amp gain, \( A(\omega) \), approaches infinity, then, for a finite output, the voltage across the two inputs is approximately zero. The inputs have a very high input impedance, so the current there is essentially zero also, and hence all the current through \( Z_n \) also flows through \( Z_f \). Thus

\[
H(\omega) = -\frac{Z_f}{Z_{in}}. \tag{2.13}
\]

Now if we replace \( Z_f \) with a capacitor, and make \( Z_{in} \) a resistor, then we get the transfer function

\[
H(s) = -\frac{1}{sRC}. \tag{2.14}
\]

By adding a resistor in parallel with the capacitor, we can then realize a general single-pole transfer function,

\[
H(s) = -\frac{1}{R_{in}} \left( \frac{R_f / sC}{R_f + 1/sC} \right) = \frac{1}{R_{in}C} \left( \frac{1}{s + 1/R_fC} \right). \tag{2.15}
\]

Since the negative input terminal is a “virtual earth”, we can sum currents there. As an example, the following simple circuit combines the functions of an adder and a differentiator. The current into the junction is \( i(t) = u_1(t) / R_1 + C \dot{u}_2(t) + u_3(t) / R_3 \), and the output voltage is therefore:

\[
x(t) = R_f \left( \frac{u_1}{R_1} + C \dot{u}_2 + \frac{u_3}{R_3} \right) \tag{2.16}
\]

There is one more building block we haven’t discussed yet, and that is the differentiator. In principle, we can make a differentiator just by interchanging the capacitor and resistor in the integrator, to give

\[
H(s) = -sRC. \tag{2.17}
\]

We very rarely see this device used, however. The main reason is that integrators can be realized much more accurately than differentiators, because the \( 1/\omega \) response of the integrator matches that of the op-amp open-loop gain (see Example 8, in section 3 for more details). The differentiator on the other hand has a response proportional to \( \omega \), and there is a much reduced range of frequencies over which it can yield an accurate response. Since step functions and other discontinuities have large high-frequency components, this is a real disadvantage. Furthermore, by appropriate choice of independent variable, it is always possible to represent an arbitrary linear system by making use of integrators alone, so that’s what we do.
2.10. Representation of Systems

We can use these components to create more complicated circuits. Rather than drawing the entire circuit each time, however, the circuit can be represented in block diagram form, with each unit of the previous section represented by a gain block – a box labelled with the transfer function. The three basic gain blocks are shown in the table:

<table>
<thead>
<tr>
<th>Function</th>
<th>Block diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency independent gain (or attenuation)</td>
<td>![A]</td>
</tr>
<tr>
<td>Frequency dependent gain</td>
<td>![A/s + a]</td>
</tr>
<tr>
<td>Adding and subtracting signals</td>
<td>![V1-V2-V0]</td>
</tr>
</tbody>
</table>

Consider, for example, the system shown below in block diagram form:

![Block Diagram](image)

which has the transfer function:

$$H(s) = \frac{X(s)}{U(s)} = \frac{1 + As + Bs^2}{s^3 + Fs^2 + Gs + H}$$

This can be represented even more succinctly using signal-flow form. Here, the system is reduced to its basic topology, with operations denoted by arrows, and labelled with their transfer functions. Nodes
indicate summation and/or splitting of the signal – any signal leaving a node is equal to the sum of the signals entering that node.

The system above can therefore be represented as shown at right. This level of abstraction simplifies any real physical problem down to its mathematical fundamentals, and saves you having to worry about how the individual blocks are constructed. It is a useful exercise now to try to derive the transfer function from the signal-flow diagram. The “always-works” recipe is to label each node, and then to evaluate the signal there as the sum of all the signals entering it from other nodes.

This must lead to a system of linear equations in the node-labels, which can then be solved straightforwardly for $X(s)$ as a function of $U(s)$. There are many different short-cuts however, and the above diagram can be expanded simply, starting at the right: Hence:

$$X = \frac{1}{s}(BU - FX + \frac{1}{s}(AU - GX + \frac{1}{s}(U - HX)))$$

(Think about it!)

### 2.11. Designing filters with realizable transfer functions.

For real-time applications, filters must be causal – that is, there must be no output until the input arrives. This places severe constraints on the transfer functions that can actually be realized in hardware. Consider for example, the ideal low pass filter, which has a rectangular bandpass which is non-zero only for $\omega < \omega_{\text{max}}$, and which has zero or constant phase across the band. The impulse response is equal to the Fourier transform, and is a sinc function. Clearly this is non-causal.

At the very least then, causal filters must exhibit two properties:

1. There must be an overall time delay between the input and output, which implies roughly a linear phase slope across the passband.
2. The amplitude function must be continuous, and must be non-zero even at infinite frequency if the impulse response is to be bounded in time.

The topic of Network Synthesis deals with the design of filters with particular desirable properties. One well-known approximation to the ideal low-pass filter is the Butterworth, or maximally-flat filter, so-called because for an $n$th-order filter, the first $n-1$ derivatives are zero at the origin. The transfer function is:

$$|H(\omega)|^2 = \frac{1}{1 + \omega^{2n}}$$

or

$$|H(s)|^2 = \frac{1}{1 + (s/j)^{2n}}.$$
The poles of $H(s)$ all have unit magnitude – that is, are of the form $p_n = e^{j\theta}$ where $(e^{-j\pi/2}e^{j\theta})^{2n} = e^{j(2k+1)\pi}$, and $k$ is used to define $n$ separate roots. Thus:

$$\theta = \frac{\pi}{2} \left( 1 + \frac{(2k + 1)}{n} \right)_{k=0 \rightarrow n-1}$$

That is, the poles are equally-spaced on the unit semi-circle in the left half $s$-plane.

Other well-known analytic filter types include:

- Tchebysheff (many spellings!) or minimax filters, which have minimum ripple in the passband for a given $Q$ value.

- Bessel filters, which approximate purely linear phase, and thus a true time-delay for frequencies in the passband.

- Elliptic filters, which aim to achieve minimum ripple in both the passband and stopbands.

There is also an abundance of software for designing filters numerically, according to custom requirements.

### 2.12. The relationship between the Real and Imaginary parts of a causal signal (non-examinable)

One final comment before we leave this topic: it turns out that if the system is causal – that is there is no output before the input arrives – then this places a powerful constraint on the relationship between the real and imaginary parts of the transfer function. In particular, it must be true that multiplying the response function in time by the unit step does not change the response. We can use the convolution theorem to represent this multiplication as a convolution by the spectrum of the unit step in the frequency domain, $[1/j\omega]$. Hence

$$H(j\omega) = \frac{1}{\pi} H(j\omega) \ast \frac{1}{j\omega}$$

If we split this into real and imaginary parts, and write out the convolution explicitly, we find that

$$\text{Re} \{ H(j\omega) \} = \mathcal{H} \{ \text{Im} \{ H(j\omega) \} \}$$

$$\text{Im} \{ H(j\omega) \} = -\mathcal{H} \{ \text{Re} \{ H(j\omega) \} \}$$

where we define the Hilbert Transform,

$$\mathcal{H} \{ X(j\omega) \} = \frac{1}{\pi} X(j\omega) \ast \frac{1}{j\omega} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X(j\omega')}{\omega - \omega'} \, d\omega'.$$

The relationships, equation 2.24, are known as the **Kramers-Kronig relations**.
3. Linear System Stability

For linear systems, there is a wealth of analytical methods for determining stability. Later on we will look at more general techniques which are applicable to all systems, but for now we stick with those that specifically apply to linear systems.

3.1. Location of Poles in the s-plane

The analysis of linear systems follows directly from the transfer function representation, discussed in Section 2. A stable system is one for which the impulse response tends to zero as \( t \) approaches infinity. For the system

\[
H(s) = \frac{A(s)}{B(s)},
\]

we therefore require that all poles of \( H(s) \) lie in the left half plane (and so give decaying exponentials in the time domain). \( B(s) \) must therefore have no roots with a positive real part. If we can factorise \( B(s) \), then we merely need to examine the roots and we will know whether or not \( H(s) \) is stable.

Example 6. Scanning Tunnelling Microscope control system

We assume that the limitation on the system is the speed of response of the piezo-electric height transducer, which has a time constant \( 1/a \). The tunnelling current decreases as the gap increases, so the gain of this part of the system is negative. Then the response of the overall system is given by:

\[
H(s) = \frac{X(s)}{U(s)} = \frac{KK_p a}{s + a(1 - KK_p K_x R)}.
\]

This is a low-pass filter, with cut-off frequency \( a(1 - KK_p K_x R) \). In order to get the best frequency response, we would like the pole to be far from the imaginary axis. We therefore make \( KK_p K_x R \) as large and negative as possible. The system is stable, however, provided that \( KK_p K_x R < 1 \).
3.2. Routh-Hurwitz criteria

For polynomials of high order, it may be impractical to find the factors directly. The Routh-Hurwitz criteria provide a systematic method for determining whether or not a polynomial of any order has any roots with a positive real part, even when the factors are not known. Even so, the calculations for high-order systems can become complicated, and you only need to learn the results for second and third order systems.

Arbitrary order system (Not for Examination)

For the polynomial \( a_ns^n + a_{n-1}s^{n-1} + \ldots + a_0 \), construct the following set of test determinants:

\[
T_1 = a_{n-1}
\]
\[
T_2 = \begin{vmatrix} a_{n-1} & a_n \\ a_{n-3} & a_{n-2} \end{vmatrix}
\]
\[
T_3 = \begin{vmatrix} a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-5} & a_{n-4} & a_{n-3} \end{vmatrix}
\]

The entry at the top left is always \( a_{n-1} \); subscripts always increase by 1 left to right, and decrease by 2, top to bottom. Non-existent coefficients are zero.

If all \( T_n \) have the same sign, then the system is stable.\(^2\)

Second-order systems

Using the general result in the previous sections, we can show that the roots of \( B(s) = as^2 + bs + c \) are guaranteed to be negative if all of \( a, b, c > 0 \).

Third-order systems

Similarly, the roots of \( B(s) = as^3 + bs^2 + cs + d \) are guaranteed to be negative if all of \( a, b, c, d > 0 \) and also \( bc > ad \).

\(^2\) Many references:
- Pippard, *Response and Stability*, pp 27-28
- Pippard, *Physics of vibration*, p142
Example 7. Application of Routh-Hurwitz criterion for stability

Given

\[ H(s) = \frac{s^2 - 2}{s^3 + s^2 + Ks + 2}, \]

for what values of \( K \) is the system stable?

Answer: Using Routh-Hurwitz for a 3rd order system, we require (i) \( K > 0 \) and (ii) \( bc > ad \), i.e. \( K > 2 \).

A classic example of the application of these ideas is to stability of the common operational amplifier. A general-purpose op-amp, like the type 741 is designed to be stable in all common negative feedback configurations. The difficulty arises because the amplifier requires at least two gain stages to provide the high value of d.c. (zero frequency) gain, \( A_0 \), and each of these gain stages has its own time constant and hence associated phase shifts. Stability is normally provided by means of internal frequency compensation, which adds a third time constant, \( \tau_1 \) to the system. The overall transfer function of the bare op-amp is therefore

\[ A(s) = \frac{A_0}{(1 + \tau_1 s)(1 + \tau_2 s)(1 + \tau_3 s)} \]

Typical values of \( \tau_2 \) & \( \tau_3 \) are \( 10^{-7} \) sec, or about 1 MHz.

Example 8. Stability of a general-purpose operational amplifier.

The least stable configuration in common use is the unity gain buffer. The circuit equations are:

\[ x(t) = a \ast \{ u(t) - x(t) \} \]
\[ X(s) = A(s) \{ U(s) - X(s) \} \]

and the overall transfer function can be easily calculated:

\[ H(s) = \frac{X(s)}{U(s)} = \frac{A}{1 + A} = \frac{1}{1 + 1/A} \]

\[ = \frac{A_0}{(1 + \tau_1 s)(1 + \tau_2 s)(1 + \tau_3 s) + A_0} \]

Now, stability requires (i) that the denominator has no roots in the right half-plane, so

\[ \tau_1 \tau_2 \tau_3 s^3 + (\tau_1 \tau_2 + \tau_1 \tau_3 + \tau_2 \tau_3) s^2 + (\tau_1 + \tau_2 + \tau_3) s + 1 + A_0 = 0 \]

while Routh Hurwitz also requires (ii) that the determinant \( bc > ad \).
\[(\tau_1 \tau_2 + \tau_1 \tau_3 + \tau_2 \tau_3)(\tau_1 + \tau_2 + \tau_3) > \tau_1 \tau_2 \tau_3 (1 + A_0).\]

If \(A_0 \gg 1\) and \(\tau_1 \gg \tau_2, \tau_3\), then the stability conditions simplify to
\[
\tau_1^2 (\tau_2 + \tau_3) > \tau_1 \tau_2 \tau_3 A_0,
\]
or,
\[
\tau_1 > \frac{\tau_2 \tau_3}{(\tau_2 + \tau_3)} A_0.
\]

Substituting values of \(A_0 \sim 2 \times 10^5\) and \(\tau_1, \tau_2 \sim 10^{-7}\) we find that \(\tau_1 > 10^{-2}\) sec, or about 10 Hz. Hence the usual form of the op-amp open-loop gain curve, shown at right.

3.3. The Nyquist Stability Criterion

The Nyquist stability criterion is generally applied to systems with a single feedback loop, such as the unity-gain op-amp considered in the previous section.

Its gain with feedback is given by
\[
G(\omega) = \frac{A(\omega)}{1 - A(\omega) \beta(\omega)}
\]

The quantity \(A(\omega) \beta(\omega)\) is known as the open-loop gain, and at the frequency at which the amplifier is designed to operate, must have a phase of \(-180^\circ\) (i.e. negative feedback).

If there is a frequency at which \(A \beta\) has a magnitude of unity and a phase of \(0^\circ\), the negative feedback becomes positive feedback, \(G(\omega)\) becomes infinite, and the system can oscillate (i.e., produce an output with no input).

The Nyquist approach shows that the system will generally (but not always) be unstable for any \(|\beta| > 1\) and phase of zero.

We consider a closed contour in the s-plane, which includes the whole of the right half plane (top diagram), and the corresponding plot of the locus of \(H(s)\) as the contour is traversed (lower diagram – the dotted curve in the lower figure is a reflection in the real axis corresponding to \(\omega = -\infty\) to \(0\)).

The Nyquist theorem, based on Cauchy’s Theorem of complex variables, links the two diagrams.

The locus of \(H(s)\) encircles the origin \(N\) times, where \(N\) is the number of zeros less the number of poles encircled, in the same direction, by the contour in the s-plane. If the system is stable before feedback, then \(A(s)\) has no poles in the right hand half plane. It follows that the closed loop is stable if \(1 + A(s) \beta(s)\) has no zeros in the right half plane.

Therefore the locus of \(1 + A(s) \beta(s)\) must not circle the origin. It is more usual to consider the open-loop gain. The Nyquist criterion for stability then becomes:

\(\triangleright \) “The system is stable if \(A(s) / \beta(s)\) does not encircle the point \((-1,0)\).”
Consider the three plots shown at right. The first system is unconditionally stable, since the plot of the open-loop gain does not encircle the point (-1,0). The second is unstable, while the third is stable in principle, but an unstable regime has to be traversed – for example, while switching on.

The advantage of this approach is that an analytical expression for $A\beta$ is not required, so stability can be inferred from a measurement of the loop gain. The diagram also shows how close to instability the system is.

For the op-amp considered earlier, the phase of $A$ is $-90^\circ$ over most of the frequency range: in particular it is designed so that, for the frequency at which the phase is $-180^\circ$, $|A|$ is less than unity so that even with $\beta=1$, insufficient signal is fed back to make the system potentially unstable.
4. Linear Discrete-time / Sampled-data systems

With the increasing use of digital controllers, discrete-time systems are now probably at least as important as continuous-time systems such as those we have (implicitly) been discussing until now. Fortunately, the tools we have developed for continuous time systems all carry over very simply into the discrete time systems.

Discrete-time systems arise whenever we deal with samples of the signal (be it a voltage, position, water-level, temperature etc). This may happen by design, as when we use a digital computer to control some physical process (a scientific instrument, for example, such as the STM discussed earlier). It may also happen implicitly, as when we model a process using a computer. Whether we simply simulate the process (for example, the evolution of the orbits of the stars in a globular cluster), or whether we attempt to “solve” the equations of motion numerically, either way we have replaced an intrinsically continuous process with one which is sampled.

Of course, we cannot represent analogue signals to arbitrary accuracy in our computations. The quantisation errors introduced in this way were discussed in Part 1B Experimental Methods, and we will not revisit them here. For the purposes of the analysis in this section, we assume that all signal amplitudes can be represented to arbitrary accuracy, so that we concentrate entirely on the effect of sampling in time and frequency. What is the effect on the system response of this sampling?

4.1. Representation of sampling

The simplest sampler consists of a (hypothetical) device which can measure the amplitude of a signal at a given instant of time, and then make it available to some sort of computational process to arbitrary accuracy. The key assumption is that the sampling is instantaneous: that is, the sample really does represent the value at an instant. Any real sampler, of course, will average over some period of time, but we ignore that for now.

Given this assumption, the sampled version of the waveform $x(t)$ can be represented at the (equally spaced) instants $k\tau$ by the sequence:

$$x(t) = \sum_{k=-\infty}^{k=\infty} x_k \delta(t - k\tau)$$

with $\delta(t)$ being the Dirac delta function. The function is thus defined at the sampling instants, and is zero at all other times. We can find the Fourier and Laplace transforms of $x(t)$ in exactly the same way as we would for any other function of time. Since all the operations involved
(summation, multiplication by delta functions) are linear, we have a wide choice of approaches.

In the first instance, let’s work out the Laplace transform of \( x(t) \) directly:

\[
X(s) = \mathcal{L} \{ x(t) \} = \int_0^\infty x(t) e^{-st} \, dt
\]

\[
= \int_0^{+\infty} \sum_{k=-\infty}^{+\infty} x(k\tau) \delta(t-k\tau) e^{-st} \, dt
\]

\[
= \sum_{k=-\infty}^{+\infty} x_k \int_0^\infty \delta(t-k\tau) e^{-st} \, dt
\]

\[
= \sum_{k=0}^{+\infty} x_k e^{-k\tau}
\]

The last term is often written as a power of \( e^{-s\tau} \), giving \( X(s) = \sum_{k=0}^{+\infty} x_k (e^{-s\tau})^k \), or with the substitution \( z = e^{s\tau} \), \( X(s) = \sum_{k=0}^{+\infty} x_k z^k \). Similarly, the Fourier Transform can be shown to be:

\[
X(\omega) = \mathcal{F} \{ x(t) \} = \sum_{k=-\infty}^{+\infty} x_k e^{-ik\omega}.
\]

4.2. A note on normalizing the Fourier transform of the comb function

Consider a function \( f(t) \) consisting of a train of 2N+1 delta-function pulses. The Fourier Transform is:

\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt = \sum_{n=-N}^{N} e^{-i\omega n\tau}
\]

\[
= \sin((2N+1)\omega\tau/2) \over \sin(\omega\tau/2)
\]

In the limit as \( N \to \infty \),

\[
F(\omega) \to A \sum_n \delta(\omega - \frac{2\pi n}{\tau}).
\]

To get \( A \), the area under one peak, integrate over one period:

\[
\int_{-\tau/2}^{\tau/2} F(\omega) \, d\omega = \int_{-\tau/2}^{\tau/2} \sum_n e^{-i\omega n\tau} \, d\omega
\]

\[
= \sum_n \int_{-\tau/2}^{\tau/2} e^{-i\omega n\tau} \, d\omega
\]

\[
= \begin{cases} 1/\tau & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}
\]

Therefore,

\[
\sum \delta(t-n\tau) \mathcal{F} \{ x(t) \} = \frac{1}{\tau} \delta(\omega - \frac{2\pi n}{\tau}).
\]
4.3. **Recovering the original data: Aliasing and Nyquist’s Sampling Theorem.**

The first major problem with sampled data become obvious when we look more closely at the frequency structure of \( x(t) \) – that is, the spectrum \( X(\omega) \). Equation 4.3 shows that \( X(\omega) \) is periodic, with components at all frequencies, even if the original function \( x(t) \) is *band-limited* – that is, limited in spectral extent.

The easiest way to see this is to go back to the equivalent of equation 4.2 for the Fourier Transform case, but to treat the convolution somewhat differently...

\[
X(\omega) = \mathcal{FT}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} \, dt
\]

\[
= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT) e^{-i\omega t} \, dt
\]

\[
= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} x(kT)\delta(t - kT) e^{-i\omega t} \, dt
\]

\[
= \sum_{k=-\infty}^{\infty} \left\{ X(\omega) \star \delta(\omega - \frac{2\pi k}{T}) \right\}
\]

This time, we present the transform as the convolution of the original frequency spectrum and a sequence of evenly-spaced delta functions in the frequency domain.

It is clear that the original spectrum is replicated about each regularly spaced frequency, so that a single frequency in the input spectrum \( X(\omega) \) appears at an infinite number of output frequencies. The output spectrum is necessarily periodic.

As long as the total width of the input spectrum, \( X(\omega) \), is less than the spacing between the elements of the frequency *comb*, then everything is still ok. Consider the problem of recovering the original signal from the sampled data: clearly it is sufficient to pass the sampled data sequence through a band-pass filter which admits the central *image* (the spectrum located around the delta function at \( \omega=0 \)) and rejects the rest.

Note that we can compute the value of \( X(\omega) \) for any arbitrary frequency: the sampling is so far restricted to the original time domain. Functions in the frequency domain are therefore still shown as being continuous.

Now consider what happens if the full width of the original spectrum is greater than the spacing between elements of the delta-function comb in the frequency domain. Clearly at this point the individual convolution images will overlap each other. No amount of filtering will then be able to restore the original signal.
This behaviour is referred to as aliasing. The condition that aliasing not occur is simply that $\Delta \omega \leq 2\pi / T$, and this result is known as Nyquist’s Sampling Theorem, or often just “the Sampling Theorem”.

Note that the effect of aliasing, and the reason for the name, is that it cause frequencies greater than $2\pi / T$ to be reflected about that value, so that they masquerade as being of a lower frequency than they actually are.

As an aside, the filter used to reconstruct the original signal is often known as an interpolation filter. We might for example just use a linear interpolation to “join the dots” as shown. As in this example, most interpolation filters necessarily yield only an approximation to the original signal. The exact reconstruction demonstrated on the previous page was effected by a “top hat” filter function (sometimes called the Shannon window). It is impossible to create such a perfect filter function (consider the convolution).

We return now to the question of whether we can in principle reconstruct the signal exactly.....

### 4.4. Finite length sequences – Windowing and Leakage

Data sequences are rarely of infinite length. The signal they represent is therefore not $x(t)$, but $x(t)$ multiplied by a rectangular window. The original data sequence can thus be written as:

$$x(t) = \sum_{k=-K}^{K} x_k \delta(t - k\tau)$$  \hspace{1cm} 4.9

What is the effect on the frequency structure? Take the Fourier Transform:

$$X'(\omega) = \mathcal{FT} \left\{ \sum_{k=-\infty}^{+\infty} x_k \delta(t - k\tau) \right\} \times (U(t + T/2) - U(t - T/2))$$

$$= \mathcal{FT} \left\{ \sum_{k=-\infty}^{+\infty} x_k \delta(t - k\tau) \right\} * \mathcal{FT} \{ U(t + T/2) - U(t - T/2) \}$$  \hspace{1cm} 4.10

$$= X(\omega) * \text{sinc}(\omega T / 2)$$

where $U(t)$ is the unit step function as before.

That is, the transform of the original infinite sequence has been convolved with a sinc function. Since the sinc function has infinite extent, a single frequency component in the time domain has components at all frequencies except at the zeros of the sinc function.

Another way of looking at this is to say that we have lost resolution. If we are using sampled data to determine the frequency of an oscillatory signal, or to try to distinguish between closely-spaced
spectral components, then the sampling window needs to be sufficiently long to permit this.

**Windowing**

There is a trade-off between the width of the convolving function and the amplitude of the *sidelobes*. By smoothing the ends of the time-domain window, the high-frequency components in the transform can be reduced, but the reduction in the equivalent width of the window leads to a corresponding increase in the width of the convolving function. For example, a triangular “Bartlett” window can be viewed at the convolution of two rectangular windows, each of half the width of the full window, so the Fourier Transform – the new convolving function – is just sinc(ωT/4). There is an enormous literature on windows, but the basic principles are very straightforward, so we’ll leave it at this.

**Leakage**

Although the convolution affects all frequencies equally, the *apparent* effect of the convolution depends on the frequency of the signal and the location of our samples in the frequency domain. Consider a signal of constant angular frequency ω₀, and a window of length T, so that the convolving function in the frequency domain is sinc(ωT/2) as before. The signal is then represented in the frequency domain by a delta function, δ(ω₀), convolved with the sinc function, to give sinc(ω₀T/2). If the transform is also sampled, as is usually the case (see next section), at intervals of 2π/T = 2π/Nτ = ω₀, then the spacing between zeros in the sinc function is the same as the sampling interval. If the sinc function is centered on a sample, then its contribution at all other sample points will be zero.

However, if ω₀ is not commensurate with the sample frequency, ωₛ, then the contribution at nearby frequency samples may be large. This phenomenon is known as “leakage”.
4.5. Effect of sampling the spectrum ("circular convolution").

We can in principle calculate the spectrum of the sampled data sequence for any frequency we choose. In practice, we rarely compute the spectrum at more frequencies than we need to. For a finite length time sequence of length $N$, there are only $N$ independent (real) measurements. As noted earlier, the spectrum of a real signal is conjugate-symmetric, and hence the spectrum can be completely described by $N$ complex values, consisting of $N/2$ conjugate pairs. These values are distributed over the unambiguous spectral range of $[-\pi/T, +\pi/T]$. Actually, this is $N+1$ values at spacing $2\pi/NT$, but the values at the start and end of the sequence are identical (as a result of the periodicity of the Fourier Transform caused by sampling in the time domain).

As noted earlier, these non-redundant samples in the frequency domain fall at the maximum and zeros of the convolving sinc function $\text{sinc}(\omega T/2) = \text{sinc}(\eta T/2)$, i.e., at intervals of $2\pi/T$, where the length of the data sequence, $T$, is equal to the number of samples, $N$, times the sampling interval, $\tau$.

Let us apply a (multiplicative) filter to the sampled data in the frequency domain, and then transform back to the time domain. Since the inverse Fourier Transform is identical to the forward transform except for the sign of the exponent, the effect is very similar to that of sampling the original signal. That is, the inverse transform is now also periodic, with period $T$, and data now extend beyond the limits of the original finite sequence. These additional (redundant, because periodic) data are normally simply discarded, so that the inverse transform is described by the same number of points as the original sequence.
The effect of the filter is to convolve the time domain data with the Inverse Fourier Transform of the filter function. Consider the very simple (delta function) convolving function shown in the diagram below, which has the effect of shifting the time sequence to the right.

Since the time-domain data are now periodic, the convolution brings data from the left into the window at the same time as it pushes it out to the right. It is exactly as if the convolution were wrapping around from one end of the window to the other. The usual solution to this problem is to “pad” the original data sequence with zeros to the left and right, using at least as many zeros as the maximum width of the convolving function.

The complete process of sampling, windowing, transformation, resampling and inverse transformation is illustrated in the next figure.

- The first three rows show the generation of a finite length sequence from an infinite length continuous sample on the left, and on the right the corresponding spectral structure.
- In the fourth and fifth rows, the spectrum is sampled, resulting in the convolution of the time domain signal with a comb of delta functions. It is periodic in time with period $T$.

### Summary – finite data sets.
There are three principal consequences:

- Sampling in the time domain leads to **Aliasing**.
- Windowing in the time domain leads to **Leakage** (i.e., convolution)
- Sampling in the frequency domain leads to **circular convolution**.
Part II Systems

4.6. \textit{z}-transforms (not for examination)

In the previous section we saw the advantages that could come from making the substitution \( z = e^{s\tau} \), where \( \tau \) is the sampling interval. The resulting functions of \( z \) are known as \( z \)-transforms.

To derive the \( z \)-transform relationships, start with the Laplace Transform of \( f(t) \), where \( f(t) \) is defined only at times 0, \( \tau \), 2\( \tau \), … \( n\tau \)…

\[
F(s) = \int_{-\infty}^{\infty} f(t)e^{-st} dt = f(0) + f(\tau)e^{-s\tau} + ...
\]

\[= \sum_{k=0}^{\infty} f(k\tau)e^{-sk\tau} \quad 4.11\]
\[= \sum_{k=0}^{\infty} f_k z^{-k}\]

where \( z = e^{s\tau} \) and \( f_k \) is the \( k \)-th sample.

The \( z \)-transform is a power series in \( z^{-1} \). For causal systems, only negative powers of \( z \) appear – that is, there are no samples prior to \( t=0 \). Multiplication by \( z^{-1} \) is equivalent to delaying the sequence by one sample period, which is in a way obvious, since \( z^{-1} = e^{-s\tau} \) is the time-shift operator for the Fourier and Laplace transforms.

As an example of the manipulation of \( z \)-transforms, consider a sampled impulse response which is an exponential decay. If we think of the sequence \( \{h_k\} \) as the impulse response, then the transfer function follows from the \( z \)-transform:

\[
H(z) = 1 + e^{-T/\tau} z^{-1} + e^{-2T/\tau} z^{-2} + ...
\]

\[= \frac{1}{1 - e^{-T/\tau} z^{-1}} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \quad 4.12 \]

Therefore, the result of applying an input sequence \( u_k \) and convolving with \( h_k \) follows from multiplying the \( z \)-transform of \( u_k \) with \( H(z) \) and then performing the inverse transform.

We can implement the convolution by a series of time delays, multiplications and summations.

\[
x_k = u_k + a u_{k-1} + a^2 u_{k-2} + ... \quad 4.13
\]

In the present case it looks like we will need an infinite series of delays/additions, since there are an infinite number of elements in the impulse sequence. However, a manipulation of the transfer function gives an easier route.

\[
X(z) = H(z)U(z) = U(z)\left(1 - az^{-1}\right) \quad 4.14
\]

The perform the inverse transform:

\[
x_k = u_k + a x_{k-1} \quad 4.15
\]
This can clearly be implemented using the scheme shown at right. It is an example of a recursive implementation, also known as an Infinite Impulse Response (IIR) filter.

4.7. Stability of Digital Systems
Digital systems are as prone to instability as continuous (analogue) systems. However, we have already seen that we can treat any digital system as a continuous system simply by regarding it as the product of some underlying continuous signal with a train of delta functions. We are therefore able to apply all the same technology as we did before: there must be no poles in the right half-plane, and we can demonstrate that this to be the case by applying the Routh Hurwitz criteria.

There are many places where this is likely to be relevant. As well as straightforward digital control systems (for, say, some sort of instrument), the same analysis is used to understand the stability of various computational algorithms. Below I give the example of a digital filter, which is an algorithm used to operate on a data sequence. A similar analysis could be applied to many other algorithms however, such as digital simulations of gravitational dynamics or electromagnetic wave propagation, many of which take place on a finite grid.


Suppose that we want to produce a smoothed version of the data sequence, \(\{x_n\}\), by taking a running average, \(y_n = 0.25 \times (x_{n-1} + 2x_n + x_{n+1})\). (This is known as a Hanning filter). Find the frequency response, and show that the filtering operation is stable.

Recall that the Laplace transform shift operator is \(e^{-st}\), so that \(\mathcal{L}\{x(t + T)\} = e^{-sT}X(s)\). Hence we can write the smoothing operation as

\[
Y(s) = 0.25 \times (e^{sT}X(s) + 2X(s) + e^{-sT}X(s))
\]

giving a transfer function

\[
H(s) = \frac{Y(s)}{X(s)} = 0.25 \times (e^{sT} + 2 + e^{-sT})
\]

\[
= \frac{(e^{sT} + 1)^2}{4e^{sT}}
\]

This has zeros when \(e^{sT} + 1 = 0\), or \(s = i(2m + 1)\pi\) — that is, an infinite set of zeros spaced up the imaginary (frequency) axis at intervals of \(2\pi\). The only pole occurs for \(e^{sT} = 0\), which implies a real part, \(\text{Re}(s) = -\infty\), repeated every \(2\pi\) in the imaginary coordinate. Since the pole is in the left half plane, the filter is unconditionally stable (that is, the output remains bounded for all frequencies \(\omega\)).
5. Random Signals (Noise)

The earlier discussion has been mostly about signals which are deterministic i.e. their values have been known at every point in time. Random signals are non-deterministic - their future values cannot be predicted and they can only be described statistically.

Random signals arise through a variety of mechanisms. At one level, equipartition ensures that all quadratic degrees of freedom in the system have an associated average thermal energy of $kT/2$, and this is the source of the Nyquist or Johnson noise associated with dissipation in a resistor. Equally, since currents and optical energy are carried by discrete quanta, there is also quantization noise associated with any real signal. This is also known as shot noise. Since the statistics of such noise are often Poisson-distributed, it may also be known as Poisson noise.

These mechanism produce true noise – that is, an additional unwanted voltage that produces an error in the desired signal. However, it is worth noting that for most purposes even the signal is in some sense random. If it were not – if we knew what was going to happen – there would be no need for the signal! It is therefore important to understand how a system will react when presented with an unknown input. Typically, although we don’t know what the input will be, nonetheless we have some idea of its general form, and often of its detailed statistics. For example, in modelling communications channels, speech and data will generally have quite different statistics, while the design of the control system for a scanning tunnelling microscope, or the spectrometer for a radio telescope, also requires some knowledge of the structure of the desired “image”.

In this section we first look at what we can say about the response of a linear system, given only the basic statistics of the signal, but not the signal itself. We then briefly look at the characteristics of real noise, before finally looking at ways of reducing the effect of noise on the signal.


We shall be considering random, or noise, signals whose statistical properties do not change with time (‘stationary’ signals) and whose time averages are the same as their ensemble averages (‘ergodic’ signals).

Random signals can be characterised by their mean, variance and probability distribution (which will often be a Normal distribution). In addition it is necessary to know something about the time-scale of
the signal fluctuations. This is described in the time domain by the
Auto-correlation Function (ACF)

\[ \Gamma(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^*(t)x(t+\tau)\,dt. \] 5.1

The ACF is conjugate symmetric, and is real for a real signal. For \( \tau = 0 \), the ACF takes the mean-square value of \( f(t) \). If the mean is zero this is just the variance. If the mean is non-zero, then it may be useful to consider the Auto-Covariance of \( x(t) \), which is the ACF of \( x(t) - \bar{x}(t) \).

A random signal can also be described in the Frequency domain by its Power Spectral Density, or Power Spectrum, which is just the total power per unit bandwidth at any frequency \( \omega \):

\[ P(\omega)\,d\omega = |X(\omega)|^2\,d\omega. \] 5.2

The d.c \((\omega=0)\) value of the power spectrum is the square of the mean of \( f(t) \).

The power spectrum is obviously symmetrical about \( \nu = 0 \). When defining noise power spectra it is usual to consider only positive frequencies. The power spectral density will then be twice that of the two-sided PSD.

5.2. The Wiener-Khinchine Theorem and Parseval’s Theorem

The ACF and Power Spectrum are related by the Wiener-Khinchine theorem. Noting that \( \Gamma(t) \equiv x(t) \ast x(-t) \) and that, for real signals, \( x(-t) \leftrightarrow X^*(\omega) \), we can use the Convolution theorem to show that the ACF and the Power Spectrum are a FT pair (cf. Coherence in Optics) i.e.

\[ \Gamma(t) \leftrightarrow P(\omega). \] 5.3

A useful way of normalizing the noise power spectrum, given the total power in the time domain, is to apply energy conservation:

\[ \int_{-\infty}^{\infty} f^2(t)\,dt = \int_{-\infty}^{\infty} |P^2(\nu)|\,d\nu. \] 5.4

This result is known as Parseval’s Theorem.
Example 10. Parseval’s Theorem

Consider a 1V pulse lasting 1 second.
\[ \int_{-\infty}^{\infty} f^2(t) \, dt = 1 = \int_{-\infty}^{\infty} |F(\nu)|^2 \, d\nu \]
i.e. \[ \int_{-\infty}^{\infty} \left( \frac{\sin \pi \nu}{\pi \nu} \right)^2 \, d\nu = 1 \]

(For more discussion of the mathematical justification for interpreting the FT of the ACF as a Power Density Spectrum see Girod et al. §17.5.2 and §18.2.6)

5.3. White Noise

A useful concept is that of White Noise. This has a constant Power Spectrum (independent of frequency) and a corresponding ACF, \( \Gamma = \delta(t) \). Since it has infinite energy it is not physically realisable. Conceptually, white noise can be thought of

(i) in time, as the superposition of an infinite number of randomly occurring \( \delta \)-functions and

(ii) in frequency, as the addition of an infinite number of sine waves of random frequency and phase and uniform spectral density.

When passed through a linear system, each frequency will be weighted in proportion to the system frequency response, \( H(\omega) \). The output is formed from the superposition of these frequency components but, since they have random phase, their powers must be added.

For a (uniform) input noise power spectrum of \( N \) volt\(^2\) Hz\(^{-1}\) the output noise power will be proportional to

\[ \frac{N}{2\pi} \int_0^\infty |H(\omega)|^2 \, d\omega \text{ volt}^2 \]  

5.4. Johnson Noise

If quantum effects and any reactive components are ignored, any resistance or other loss in a system is a source of white noise. The best known example is that of Johnson noise – noise caused by the random motion of charge-carriers in a resistive medium – but this is just a particular case of the fluctuation-dissipation theorem, which shows that fluctuations in the dynamical variable are inevitable in any dissipative system – see the TPI course, or http://www.utdallas.edu/~hellums/docs/ThermalNoise.pdf for more
details. Another topical example is that of noise in the position of the test mass in a gravitational wave detector.

The simplest way to derive the result for a resistor is to consider the case of a transmission line of length \( l \) and impedance \( R \), terminated at each end by a (resistive) matched load, also of resistance \( R \). Now, each mode of the transmission line must have a mean energy of \( kT/2 \). Each mode propagates at a velocity \( c \) so that the energy absorbed by the matched load is \( kTc/2l \) Watts. The modes are evenly spaced with a frequency separation of \( c/2l \), so the energy absorption per unit time is then \( kT \) W Hz\(^{-1} \). Now, detailed balance must apply, so that the power sourced by each resistor is equal to the power absorbed by it, and hence the power generated by the resistor is also \( kT \) W Hz\(^{-1} \). Since the power is delivered to a matched load, the mean squared voltage and current appearing across the resistor terminals must be

\[
\overline{V^2} = 4kTR \text{ volts}^2 \text{ Hz}^{-1}
\]

\[
\overline{I^2} = 4kT/R \text{ amps}^2 \text{ Hz}^{-1}
\]

Note that the power is the same in all frequency intervals, so that we have a flat power spectrum, or white noise. This does not converge – it implies infinite total power – but in real life of course, there must be some bandwidth limit caused by a stray capacitance, such as the capacitance across the resistor itself, and this limits the noise voltage.

Note that this expression for Johnson noise assumes that only positive frequencies are considered – that is, we are dealing with one-sided spectral densities.

### 5.5. Equivalent Noise Bandwidth

It is convenient to quantify the amount of noise passed by a given system or network in terms of its Equivalent Noise Bandwidth (ENB), defined as the bandwidth of a filter having the same peak response (usually at \( \omega = 0 \)) and passing the same total noise power: e.g. for an R-C filter of time constant \( \tau \),

\[
|F(\omega)|^2 = \frac{1}{1 + (\omega \tau)^2},
\]

and hence

\[
ENB = \int_0^\infty |F(\omega)|^2 d\omega = \frac{\pi}{2\tau} = \frac{\pi}{2RC} \text{ rad s}^{-1}.
\]
5.6. Detection of Signals in Noise

It is often necessary to filter the signal to increase the signal-to-noise ratio (SNR). The form of the filter will in general depend on both the signal we wish to detect, and on the spectrum of the noise.

Consider a signal consisting of 3 δ-functions, each with an rms error or ‘noise’ ±σ. The signal is passed through a filter which adds the δ-functions with weights a, b, c. What are the optimum relative weights for maximum signal-to-noise ratio (SNR)?

<table>
<thead>
<tr>
<th>Δt</th>
<th>Filter output</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3T</td>
<td>0</td>
</tr>
<tr>
<td>-2T</td>
<td>3c</td>
</tr>
<tr>
<td>-T</td>
<td>3b-2c</td>
</tr>
<tr>
<td>0</td>
<td>3a-2b+c</td>
</tr>
<tr>
<td>+T</td>
<td>2a+b</td>
</tr>
<tr>
<td>+2T</td>
<td>a</td>
</tr>
<tr>
<td>+3T</td>
<td>0</td>
</tr>
</tbody>
</table>

The power in the noise is independent of the signal, and hence the total noise amplitude is proportional to \( \sqrt{a^2 + b^2 + c^2} \). So maximising SNR is equivalent to maximising the output under the
constraint that \(a^2 + b^2 + c^2 = \text{const.}\) We can use undetermined multipliers to show that maximum SNR is obtained when \(a, b \& c\) match the amplitudes of the signal. This is equivalent to convolving the signal with a time-reversed version of itself.

The frequency response of this (real) filter is \(F^*(\omega)\) and, for obvious reasons, it is known as a Matched Filter.

Note, however, that without an added time delay, this filter will be non-causal and therefore not physically realisable (although there would be no problem applying it to stored data).

For another slant on this problem, see http://cnx.rice.edu/content/m10101/latest/ and http://cnx.rice.edu/content/m10101/latest/?format=pdf, as well as the first part of http://www.ece.utexas.edu/~bevans/courses/realtime/lectures/12_Matched_Filtering/lecture12.ppt

5.7. **Wiener Filter (non-examinable)**

Usually the noise spectrum is *not* white – for example, it may consist primarily of “1/f” noise. We would like to take account of the signal-to-noise ratio at different frequencies when estimating the signal.

Consider the system of the previous section. What is the filter, \(H(\omega)\), which minimizes the error between the original signal and the reconstructed signal? Write the error power as

\[
|e(t)|^2 = |y(t) - s(t)|^2 .
\]

It can be shown that the optimum filter has the transfer function

\[
H(\omega) = \frac{P_{sx}(\omega)}{P_{xx}(\omega)} = \frac{P_{sx}^*(\omega)}{P_{xx}(\omega)}
\]

where \(P_{sx}(\omega)\) is the cross spectral density.
6. Introduction to non-linear dynamics

The methods described in the previous section tell us whether or not a linear system has one or more unstable modes. However, the range of behaviours exhibited by linear systems is relatively small, and they do not show all the features evinced in more general non-linear systems.

The dynamics of a general non-linear system can be described by a set of coupled differential equations

\[
\dot{x}_1 = f_1(x_1 \ldots x_n) \\
\vdots \\
\dot{x}_n = f_n(x_1 \ldots x_n)
\]

For example, damped harmonic motion with the second order (linear) DE

\[
m\ddot{x} + b\dot{x} + kx = 0
\]

can be written a set of coupled first-order equations as

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2
\]

We examine, in turn, the one-variable system (“flow on the line”), the two-variable system (“flow on the plane”) and the three-variable system (“3-D flow”). We also look briefly at the special case of a one-variable system where the motion takes place on a circle (when the variable is phase angle, for example). In general, an \(n\)-variable system requires \(n\) equations in the form 6.1 to represent it.

6.1. Flows on the line

We start with an examination of the possible trajectories of a system. That is, we plot the path in a \(2n\)-dimensional space, where the dimensions are the \(n\) independent coordinates and their corresponding momenta. Here, we take a fairly loose view of this definition, and we will generally just use the independent coordinates and their time-derivatives. We begin by examining the one-dimensional flow – that is, the dynamics of a single first-order DE,

\[
\dot{x} = f(x).
\]
6.2. Fixed points of a 1-D flow

The function $f$ is single-valued for all $x$. The dynamics therefore take place along a line (the $x$ axis).

In the notation of Strogatz, the phase-plane plot represents a vector field on the line: the velocity vector $\dot{x}$ is shown for every $x$. The trajectory is a plot of $\dot{x}$ as a function of $x$. The time coordinate is thus implicit – we could, for example, mark off time ticks along the curve given any starting value of $x$, and hence $\dot{x}$, but the main properties of the system are apparent directly from the phase-plane plot.

We can immediately identify two types of fixed point. These are values of $x$ for which $\dot{x}$ is zero, so that the system is, momentarily at least, at rest.

- A **stable** fixed point results whenever $\dot{x}$ is zero and the slope of the $\dot{x}$ vs $x$ curve $-\frac{d\dot{x}}{dx}$ is negative. This ensures that for small fluctuations away from the fixed point, as shown in green arrows on the plot, the velocity $\dot{x}$ is in a sense to bring the system back to the fixed point. A stable fixed point is also known as a sink or an attractor.

- An **unstable** fixed point, on the other hand, has $\frac{d\dot{x}}{dx}>0$, so that small fluctuations result in a motion directed away from the fixed point. Other names for an unstable fixed point include source or repeller.

One other type of fixed point is possible, and is known as a **half-stable point**: 

Note the notation: stable fixed points are denoted by filled circles; unstable fixed points by open circles, and half-stable points by half-filled circles, as shown in the examples.

**Example 11. Autocatalytic chemical reaction – a non-linear dynamical system**

Consider the reaction

$$A + X \xrightarrow{\frac{k_1}{k_2}} 2X.$$  

The presence of $X$ stimulates further production of $X$ – hence the term “autocatalytic”. (This is one model for the growth of amyloid plaques in the brain in diseases such as BSE and CJD: the presence of a small amount of plaque, $X$, catalyses the conversion of normal protein, $A$, to plaque.) There are two variables in the process: $a$, the concentration of reactant $A$, and $x$, the concentration of reactant $X$.

If the concentration of $A$ is always large, then it will be effectively constant. The problem then reduces to dynamics in one variable.
Given the rate constants for forward and reverse reactions, \( k_1 \) and \( k_2 \), the equation governing the dynamics is
\[
\dot{x} = k_1 ax - k_2 x^2.
\]
We can sketch the trajectory in the phase-plane, as shown. It is also straightforward to sketch the concentration vs time, as in the right hand figure. Since \( \dot{x} \) is linearly proportional to \( x \) in the vicinity of the fixed points, the approach to equilibrium must be exponential.

### 6.3. Dynamic variables and control variables

In the example above, \( x \) and \( a \) are dynamic variables: that is, they are the variables which change with time. The two other variables – \( k_1 \) and \( k_2 \) – are control variables. In that particular case, varying the control variables did not change the general character of the dynamics, but only the details. Consider now the system described by
\[
\dot{x} = x^2 + a.
\]
As \( a \) is increased from a negative value, the two equilibria – one stable, and one unstable – first approach each other, then merge to form a half-stable fixed point, and finally annihilate. The control parameter, or variable, \( a \), thus determines the stability of the system.

In general, complex dynamical systems have fewer control parameters than dynamical variables. We are interested in situations, such as that shown above, where a change in one or more of the control parameters leads to discontinuities – i.e., qualitatively different dynamics, such as a change from stable to unstable behaviour. This is the basis of Catastrophe Theory. The key result from catastrophe theory is that the number of configurations of discontinuities depends on the number of control variables, and not on the number of dynamical variables.

In particular, if there are four or fewer control variables, there are only seven distinct types of catastrophe, and in none of these is more than two dynamical variables involved. In the next section we consider all cases up to two control parameters. For simplicity we restrict ourselves to a single dynamical variable, \( x \), with little loss of generality.
6.4. Potential methods

The existence of stable, unstable and half-stable fixed points (i.e. equilibria) suggests another way of looking at the dynamics, in terms of an underlying potential, which we shall here denote by $V(x)$. Stable equilibria are local minima in $V(x)$, unstable equilibria are local maxima and half-stable fixed points are points of inflection.

In this course we are dealing with the evolution of arbitrary dynamical systems (as loosely interpreted), and hence there may not actually be a true potential energy (although there often is). In terms of the equation $\dot{x} = f(x)$, we can define the potential to be

$$f(x) = -\frac{dV}{dx}. \quad 6.6$$

For a first-order system (and hence one-dimensional motion) we have to imagine a particle with an inertia which is negligible in comparison with the damping force.

The negative sign implies that the force on a particle is always “downhill”, towards lower potential. This can be shown simply by applying the chain rule to the time-derivative of the potential and applying the definition of the potential:

$$\frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} = -\left(\frac{dV}{dx}\right)^2 \quad 6.7$$

$$\frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0. \quad 6.8$$

Thus $V(t)$ decreases along trajectories, and the particle always moves towards lower potential.

In summary, the potential has the following properties:

1. $-dV/dx$ is force-like (i.e., is in the direction of motion).

2. Equilibrium positions, $x^*$ (fixed points) are given by $-dV/dx = 0$.

3. The stability of the fixed point is determined by $d^2V/dx^2 \mid_{x^*}$.

---

3 See Response and Stability, chapters 4 & 5, and Strogatz, chapter 3.
6.5. Forms of the potential curve.

The potential function can always be approximated by a Taylor series, so that

\[ V(x) = a + bx + cx^2 \ldots \]

We can ignore \( a \), since it is just a constant and does not affect the dynamics. In the vicinity of a single fixed point (i.e. equilibrium) we can also eliminate \( b \) by shifting the coordinate system to put the fixed point at the origin (although \( b \) cannot be ignored for multiple fixed points). This leaves us with

\[ V(x) = cx^2 + dx^3 + ex^4 \ldots \]

We can now enumerate the possibilities.

Harmonic Potential

This is the simplest possible form, and the only one possible for purely linear systems:

\[ V(x) = \alpha x^2. \]

There is a single fixed point, \( x^* = 0 \), for all \( \alpha \). If \( \alpha > 0 \) then the fixed point is stable; if \( \alpha < 0 \), then it is unstable.

Asymmetric cubic potential – The limit point instability

The potential has the form

\[ V(x) = \alpha x + x^3. \]

For \( \alpha > 0 \), no equilibrium position is possible. For \( \alpha < 0 \), then there is always one stable and one unstable equilibrium.

Here we introduce the idea of control space. We can plot the location of the fixed point, \( x^* \), as a function of the control parameter, \( \alpha \), as shown below.

On the control space plot, the solid line denotes the location of the stable equilibrium, while the dashed line indicates the locus of the unstable equilibrium, both as a function of \( \alpha \).

The form of the instability shown here is usually known as a limit point instability (see Pippard). Alternatively, it is sometimes called a
saddle-node bifurcation (e.g., Strogatz), or a fold (in books on catastrophe theory).

The phase-plane trajectories for this system were shown earlier, for the system with \( \dot{x} = x^2 + a \) – see below. This is the origin of the term “saddle-node bifurcation” – as \( \alpha \) is decreased through zero the fixed point is first created, and then bifurcates into two – one stable and one unstable.

**Example 12. Weighted Pulley**

The gravitational potential is given by

\[
V = mR\theta - Mr \sin \theta = \alpha A\theta - B \sin \theta.
\]

For small \( \theta \), we can approximate this as

\[
V \approx (A - B)\theta + \frac{B}{6} \theta^3.
\]

That is, \( V = \alpha \theta + \theta^3 \), with \( \alpha = 6(A - B)/B \).

The system will thus be stable as long as \( \alpha < 0 \), that is, \( B > A \), or \( Mr > mR \). 


Show the radial component of motion of a particle moving in a central field can be described in terms of an effective potential. In a classical model of a multi-electron atom, electrons are assumed to move in a modified electrostatic potential \( V = -(k/r)e^{-r/a} \) where \( k \) and \( a \) are constants. Show that in such a potential, circular orbits are unstable unless \( r/a < (1 + \sqrt{5})/2 \).

We have

\[
\frac{1}{2}mr^2 + \frac{J^2}{2m\omega^2} + V(r) = \text{const}
\]

or

\[
V_{\text{eff}} = \frac{\beta}{r^2} - \frac{k}{r} e^{-r/a}.
\]

The critical point is given by

\[
\frac{dV}{dr} = \frac{2\beta}{r^3} - \left( \frac{k}{r^2} + \frac{k}{ar} \right) e^{-r/a} = 0
\]

\[
\frac{d^2V}{dr^2} = \frac{6\beta}{r^4} - \left( \frac{2k}{r^3} + \frac{2k}{ar^2} + \frac{k}{a^2r} \right) e^{-r/a} = 0
\]

Eliminating \( \beta \) leads to
Part II Systems

\[ \frac{r^2}{a^2} - \frac{r}{a} - 1 = 0, \]

where \( r \) is the radius of the circular orbit, or

\[ \frac{r}{a} = \frac{1 \pm \sqrt{1 + 4}}{2}. \]

The positive root, \( r/a = (1 + \sqrt{5})/2 \) gives the transition between stable and unstable orbits, with \( a \) being the control variable. This is a Limit-Point instability.

Cubic potential with quadratic term – The transcritical bifurcation

The potential this time includes a term in \( x^2 \) rather than a linear term as in the previous section.

\[ V(x) = x^3 + \alpha x^2 \]

The effect of this is to give a double root, and hence a fixed point, at the origin, regardless of the location of the third root.

The bifurcation diagram is shown below. This is generally known as the transcritical bifurcation. One physical example of such a system is the laser, as described in Problem 17 of the Question Sheet.

Symmetric quartic potential – The pitchfork bifurcation.

The potential is

\[ V(x) = x^4 + \alpha x^2 \]

- For \( \alpha \geq 0 \) there is just one stable equilibrium
- For \( \alpha < 0 \) there is one unstable equilibrium and two stable equilibrium points.
First, take the positive sign from 6.14, which generates the sequence of potential curves to the left.

In this case we refer to the Stable Symmetric Transition. It is also known as a Pitchfork Bifurcation (see Strogatz) from the shape of the bifurcation diagram, as shown at right.

One example of this sort of potential is the Euler strut.

When the length of the strut (corresponding to $-\alpha$) is below some critical value, the strut is stable in the vertical direction. When $\alpha$ is increased beyond this value, it can adopt either of the two stable equilibria, both of which are offset from $x = 0$. There is no hysteresis when $\alpha$ is then reduced again – we return smoothly to a state with a single fixed point at $x^* = 0$.

If we take the negative sign, the additional quartic term may also act to destabilize the system, as shown in the diagram to the right.

The notation here is a bit different from that in Strogatz (see section 3.4). The differences arise because he proceeds from the equation for $\dot{x}$, and takes the Taylor series for $f(x)$, whereas we have taken a Taylor series for $V$, which integrates each polynomial term and changes the signs.

**Asymmetric quartic potential with two control parameters – the Cusp catastrophe**

We now consider an asymmetric potential, of the form

$$V(x) = \alpha x^2 + x^4 + \beta x,$$

where the $\beta x$ term introduces asymmetry to the symmetric quartic form of the previous section. We now have two control parameters, $\alpha$ and $\beta$. Depending on the sign of $\alpha$, then, we get two different sorts of behaviour.

If $\alpha > 0$, then the linear term merely shifts the position of the fixed point, but does not qualitatively change the dynamics from that of a simple harmonic potential. If $\alpha < 0$, however, the linear term can eliminate the unstable fixed points and one of the stable fixed points as well.

The control space diagram – the bifurcation set – is now two dimensional. Consider the equilibrium surface, or a plot of the location of $x^*$ against $\alpha$ and $\beta$. The bifurcation set is the set of points in the $\alpha - \beta$ plane dividing the plane into different regions of stability, and has a characteristic cusp shape.
As we move from the shaded to the non-shaded region (i.e. across the bifurcation set), there is a sudden change in behaviour, with marked hysteresis when the path is reversed.

Examples of such systems include the Zeeman catastrophe machine and the isotherms of a van der Waals gas – see Pippard for further details.

Summary – basic instabilities.

<table>
<thead>
<tr>
<th>Name</th>
<th>$f(x)$</th>
<th>$V(x)$</th>
<th>Potential</th>
<th>Bifurcation diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>Harmonic potential</td>
<td>$-2\alpha x$</td>
<td>$\alpha x^2$</td>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>Limit-point instability</td>
<td>$-\alpha - 3x^2$</td>
<td>$\alpha x + x^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>--------------------------</td>
<td>-----------------</td>
<td>------------------</td>
<td></td>
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</tr>
<tr>
<td>Saddle-node bifurcation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fold</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Transcritical Bifurcation

- $-2\alpha x - 3x^2$
- $\alpha x^2 + x^3$

Pitchfork Bifurcation

- $-2\alpha x - 3x^2$
- $\alpha x^2 \pm x^4$

Cusp catastrophe

- $2\alpha x + 4x^3 + \beta$
- $\alpha x^2 + x^4 + \beta x$

<table>
<thead>
<tr>
<th>$\alpha&lt;0$</th>
<th>$\alpha=0$</th>
<th>$\alpha&gt;0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta&lt;0$</td>
<td>$\beta=0$</td>
<td>$\beta&gt;0$</td>
</tr>
</tbody>
</table>

Limit-point instability

- Fold
- Transcritical Bifurcation
- Pitchfork Bifurcation
- Cusp catastrophe

Two stable, one unstable equilibrium
6.6. Lyapunov Stability Theorem (not for examination)

We have seen that the stability properties of the system depend completely on the form of the potential function, but to be fair, this is not necessarily a real potential, but merely the integral of $f(x)$ in the equation $\dot{x} = f(x)$. For the first order cases we have been examining, there is no inertia term, and so the potential defines the direction of the “force”, but does not represent a stored energy. The usual analogy, as made in Strogatz, is that we have to imagine a particle rolling down the potential “hill”, but through a very viscous, dissipative, medium.

Fortunately, we can put the potential onto a more mathematical footing, and this is the basis of the central theorem of stability of non-linear systems, which we now quote:

If there exists a function $V(x)$ such that:

1. $V(x)$ is positive definite, becoming zero only at the origin $x = 0$; and
2. $V(x)$ decreases along a trajectory; then
3. The origin is a stable equilibrium.

For example, take $V(x)$ to be $\|x\|$. Then $V(x)$ is a generalized energy and satisfies the requirements.

In general, $V(x)$ is a scalar function of $x$. Both $V(x)$ and its partial derivatives must therefore be continuous. Hence the gradient vector, $\nabla V(x)$, is defined, and

$$\frac{d}{dt} V(x) = \left(\frac{dx}{dt}\right) \cdot \nabla V(x) = f(x) \cdot \nabla V(x).$$

That is, for the system to be stable, the projection of $\dot{x}$ onto the gradient of $V(x)$ must be less than zero (or possibly equal to zero).

If we can find such a Lyapunov function, $V(x)$, then we have proved stability. Unfortunately, there is no recipe for finding such a function, and much of the art of proving non-linear stability devolves on the search for an appropriate function. On the other hand, failure to find a function $V(x)$ satisfying the conditions does not prove that the system is unstable.
7. Flows on the circle

In the previous section we saw that oscillation was impossible for the flow on a line, because there was no way that a one-dimensional phase-plane trajectory could change direction (which requires \( \dot{x} = 0 \), and hence infinite time). The next step in complexity is flows on the circle – that is, in a periodic system. The classic example is that of the non-linear pendulum, where the amplitude, \( \theta \), is not required to be small. The equation of motion is

\[
ml^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma, \tag{7.1}
\]

where \( \Gamma \) is the driving torque (see diagram).

7.1. Flows in the overdamped limit

In the \textit{overdamped} limit, of very large \( b \), we can approximate this by the first order system

\[
b\dot{\theta} + mgL\sin\theta = \Gamma. \tag{7.2}
\]

As with the flows on a line, we can now ignore the inertial term, but note that this system is highly dissipative – you have to imagine that the action takes place underwater, so that we can achieve a steady-state where the energy lost to damping is exactly replaced by the applied torque.

In order to simplify the problem, reduce it to a non-dimensional form,

\[
\theta' = \gamma - \sin\theta \tag{7.3}
\]

where \( \gamma = \Gamma / mgL \), \( \tau = (mgL / b)t \) and \( \theta' = d\theta / d\tau \). Now, if \( \gamma > 1 \), the applied torque is always greater than the gravitational torque, and the pendulum will continue to turn. It will turn faster when coming down, and slower when going up, as shown:

When the dimensionless torque, \( \gamma \), reduces and approaches unity, then the pendulum will take longer to climb past the point of maximum gravitational torque on the slow side, until finally a half-stable fixed point appears at \( \theta = \pi / 2 \). As \( \gamma \) is reduced still further, this fixed point splits into a stable fixed point on the low side, and an unstable fixed point above. Eventually, when \( \gamma = 0 \) the fixed points reach angles of 0 (at the bottom, stable) and \( \pi \) (at the top, unstable).

The corresponding phase-plane diagrams are as shown below.

See \url{http://www.aw-bc.com/ide/Media/JavaTools/pndulums.html} for some interactive demonstrations.
7.2. Another example of flow on the circle: The Josephson Junction

The Josephson junction consists of two superconductors separated by an insulating barrier thin enough to permit tunnelling of Cooper pairs of electrons. Because the Cooper pairs are bosons, they can condense into a single quantum mechanical state, characterized by a phase \( \phi \). Because there is coupling between the two superconductors, the wave function amplitudes are related by:

\[
\begin{align*}
\frac{iy_1}{dt} &= U_1 \psi_1 + K \psi_2 \\
\frac{iy_2}{dt} &= K \psi_1 + U_2 \psi_2
\end{align*}
\]

where \( K \) is a constant of the junction. Substituting \( \psi_1 = \sqrt{\rho_1} e^{i\theta_1} \) and similarly for \( \psi_2 \) we can solve to show that

\[
\begin{align*}
\dot{\rho}_{1,2} &= \pm \frac{2}{h} K \sqrt{\rho_1 \rho_2} \sin \delta \\
\dot{\theta}_{1,2} &= \frac{K}{\hbar} \frac{\rho_{2,1}}{\rho_{1,2}} \cos \delta + \frac{qV}{2\hbar}
\end{align*}
\]

where \( \delta \) is equal to the phase difference, \( \theta_1 - \theta_2 \), and \( q = 2e \).

In fact we have ignored the external circuit. If the system is connected to an external current source, then a current can flow, without changing the net charge density in either superconductor. The current has magnitude \( J = \frac{2K}{h} \sqrt{\rho_1 \rho_2} \sin \delta = J_c \sin \delta \). so for \( J < J_c \) there are two equilibrium values of \( J \) and \( \delta \), just as for the pendulum, with the current \( J \) taking the place of the external torque, and the quantum-mechanical phase difference \( \delta \) taking the place of the rotational angle.

When \( J \) exceeds \( J_c \), then a voltage develops across the junction and the phases on the two sides start to slip with respect to each other, at a rate given by

\[
\dot{\delta} = \frac{qV}{h}.
\]

Incidentally, this gives us a new voltage standard, since \( q \) and \( h \) are both known very accurately, and we can measure frequency to high precision.

The junction also carries an ordinary current, due to leakage, and a displacement current, due to the junction capacitance. These can be represented by a resistor and capacitor respectively, both in parallel with the junction. The equivalent circuit is therefore like this.

Using the known voltage-phase characteristic for the junction, together with a bit of basic circuit theory, we obtain

\[
\frac{hC}{2e} \dot{\delta} + \frac{h}{2eR} \dot{\delta} + J_c \sin \delta = J,
\]
which again has the same form as the equation for the non-linear pendulum.

As in the case of the pendulum, we consider first the overdamped limit (that is, when \( R \) is small). The equation then takes the general form

\[
\frac{\dot{\phi}}{J_c} = \sin \phi, \tag{7.8}
\]

with \( \phi' = \frac{d\delta}{d\tau} \) and \( \tau = \frac{2eJ_cR}{h}t \).

It is relatively straightforward to calculate the current-voltage curve analytically in the overdamped limit. First note that

\[
\left\langle \frac{d\delta}{dt} \right\rangle = \left\langle \frac{d\delta}{d\tau} \right\rangle = \frac{2eJ_cR}{h} \left\langle \phi' \right\rangle, \tag{7.9}
\]

and hence from 7.6

\[
\langle V \rangle = J_cR \left\langle \phi' \right\rangle. \tag{7.10}
\]

For \( J < J_c \), we have a fixed point, and zero voltage: for \( J > J_c \), the solutions are periodic. The period \( T \) is determined largely by the time it takes for the trajectory to traverse the “ghost” of the saddle-node fixed point and is equal to \( \frac{2\pi}{\sqrt{(J/J_c)^2 - 1}} \) – see section 4.3 of Strogatz for details. Now, we must have

\[
\left\langle \phi' \right\rangle = \frac{2\pi}{T}, \tag{7.11}
\]

and so finally,

\[
\langle V \rangle = \begin{cases} 0 & \text{for } J \leq J_c \\ J_cR\sqrt{(J/J_c)^2 - 1} & \text{for } J > J_c \end{cases}. \tag{7.12}
\]

Because of this very sharp non-linearity, Josephson Junctions were of interest as possible detectors of radiation in the GHz and THz range, but in fact the non-linearity is so sharp that it is very hard to get stable operation.

For those who are interested, more details of the system, including a description of how two junctions can be used to make a very sensitive ‘SQUID’ magnetometer, are given in the Feynman Lectures, vol III, section 21.9, and also in Strogatz, section 4.6.
7.3. **Topological interpretation**

There is an excellent section in Ian Stewart’s book\(^4\) on this topic. This section is included just to give you the general flavour.

Consider the phase-plane trajectories of a general non-linear pendulum, which may either be oscillating to and fro, or (if it has enough energy) rotating steadily in one direction.

If the pendulum is oscillating, the trajectory is closed, so that the total phase excursions are less than \(\pi\) radians. Once it goes “over the top”, then the trajectory wraps around from one end of the plot to the other, with clockwise rotation shown at the bottom of the plot, and anticlockwise rotation at the top.

But this suggests that the phase-plane should be wrapped into a cylinder, as shown at right. All trajectories are now closed loops, but of two different sorts: oscillatory trajectories are closed loops on the surface (technically these are called *librations*), while continuous one-way motions go *around* the cylinder (*rotations*).

There is a rough mapping between height on the cylinder and the energy of the motion, in that zero energy occurs at the midplane, while the energy increases in both the up and down directions. We can distort the cylinder so that the energy maps monotonically onto the vertical direction, by bending it as shown at right.

Now, small oscillations take place in anticlockwise loops at the bottom of the U-tube, while rotations extend up one branch or the other – as shown, clockwise rotation is represented by the left branch.

Now, if we add damping, then energy is dissipated, so that the trajectory must be downwards at all points. This is shown in the last figure, to the right. A clockwise rotation slows down, and eventually turns into a to-and-fro oscillation, before dying away completely at zero energy.

The conclusion is that flows on the circle possess a **cylindrical phase space**, which has quite different properties from that of the flat phase-space belonging to flows on the line.

---

8. Flows in two dimensions

As we saw in section 6.1, oscillations are not possible in one-variable systems, so that all the “fixed points” are static. In two variables, we have the possibility of periodic (“closed”) trajectories, which are known as limit cycles, as we would predict from our knowledge of the harmonic oscillator – a classic two-variable system.

Near any fixed point, the general non-linear differential equations can be approximated by a set of linear equations, so we can examine the behaviour near fixed points of any two-variable system by generalizing the harmonic oscillator equations.

8.1. Linear systems and phase portraits.

We showed earlier that $m\ddot{x} + b\dot{x} + kx = 0$ could be written as

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\frac{k}{m}x_1 - \frac{b}{m}x_2
\end{align*}
\]

where the alignment has now been adjusted to emphasise the lack of the $x_1$ term in the first equation.

The behaviour can be visualised by using phase-space diagrams. For a two-variable system we plot the trajectory on the $x_1$ vs $x_2$ diagram – this is also known as a phase portrait.

Phase portraits for the general damped harmonic oscillator are shown below.

Note:
1. Trajectories never cross.
2. For this definition of $x_1$, trajectories must circle the origin in a clockwise direction.
Part II Systems

As noted before, damped harmonic motion is not the most general case. Write

\[ \begin{align*}
\dot{x}_1 &= \alpha x_1 + \beta x_2, \\
\dot{x}_2 &= \gamma x_1 + \delta x_2,
\end{align*} \]

8.2

or in matrix form,

\[ \dot{x} = Ax; \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}. \]

8.3

There are two methods open to us to examine the stability of this system. One is to diagonalize the equations, so that the equations in the new coordinate system are uncoupled, and must therefore show exponential decay or growth (see e.g. Strogatz p129). Instead, we look at the stability of the characteristic function obtained by converting the two first-order equations to a single second order equation:

\[ \begin{align*}
\dot{x}_1 - (\alpha + \delta)\dot{x}_1 + (\alpha\delta - \beta\gamma)x_1 &= u(t) \\
\text{or,} \\
\ddot{x}_1 - T\dot{x}_1 + \Delta x_1 &= u(t),
\end{align*} \]

8.4

8.5

where \( T \) is the trace of \( A \) and \( \Delta \) is its determinant. If we take the Laplace transform of 8.5 then we obtain

\[ H(s) = \frac{X}{U} = \frac{1}{(s^2 - Ts + \Delta)}. \]

8.6

The roots of \( s^2 - Ts + \Delta = 0 \) are thus the poles of \( H \), and the eigenvalues of \( A \): they are either a complex conjugate pair or they both lie on the real axis. There are five possible pole configurations, which describe all possible behaviours of a two-variable system in the vicinity of a fixed point:

- **Complex poles:**
  - in negative half plane – stable spirals
  - in positive half plane – unstable spirals

- **Real poles**
  - in negative half plane – stable node
  - in positive half plane – unstable node
  - one in each half plane – saddle point.

Note that the straight line trajectories for the last three cases are along the eigenvectors of the system – for motion in these directions, \( x_1 \) is always proportional to \( x_2 \). If the system starts anywhere on an eigenvector it remains on it forever – otherwise, the initial condition
has components along more than one eigenvector, which decay at different rates characterized by the corresponding eigenvalues.

As with the flows in one dimension, we can plot the behaviour (although not the equilibrium surface) as a function of the two control parameters, \( T \) and \( \Delta \). For reference, the poles are located at

\[
p = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2}
\]

so that the line \( T^2 = 4\Delta \) divides the regions corresponding to real and complex roots.

There are special cases on the boundaries between the five cases – for example, closed contours of perfect SHM occur along the line \( T = 0 \) for \( \Delta > 0 \)

### 8.2. Non-linearities and limit cycles

The diagrams we have just drawn are for the linear case, for which all attractors and repellers are points (the closed contour of SHM is not an attractor, since motions are not attracted towards it – the formal definition of an attractor will be presented in section 0). In the non-linear two-variable case there is one other possibility, and that is a periodic orbit known as limit cycle.

We define a limit cycle as an isolated, closed trajectory. That is, neighbouring trajectories are not closed. It is an attractor if neighbouring trajectories spiral towards it, or a repeller if they spiral away.

Consider the following illustration of trajectories in a general two-variable system \( \dot{x} = f(x) \):

![Diagram of trajectories](image)
(note that we are not assuming that $\dot{x}_1 = x_2$, so there is no problem about the trajectory describing an anticlockwise loop in this case). We can identify a number of fixed points (i.e. those for which $f(x^*) = 0$).

Here A and B are saddle nodes, C is an unstable node (giving rise to the unstable spirals) and D is a limit cycle – that is, an isolated closed trajectory. Spirals internal and external to D are both attracted towards it.

**Example 14. A two-variable dynamic system – the Predator-Prey ecological model**

We consider the canonical case of interactive populations of Rabbits (no. = $R$) and Foxes (no. = $F$), (also Fishes and Shrimps).

**Assumptions:**

- Rabbits - have unlimited food supply and increase at rate $\dot{R} = aR$ but are eaten by foxes at a rate $\dot{R} = -bRF$
- Foxes - die naturally at a rate $\dot{F} = -cF$, but increase at a rate proportional to the food supply (rabbits!) $\dot{F} = dRF$

i.e. $\dot{R} = aR - bRF$; $\dot{F} = -cF + dRF$.

**Find the fixed point:**

For steady state, $\dot{R} = \dot{F} = 0$ gives $R_0 = c/d$ and $F_0 = a/b$.

**Linearize around this fixed point:**

Letting $r = R - R_0$ and $f = F - F_0$, evaluating partial derivatives at $R_0, F_0$, and substituting gives

$\dot{r} = -bR_0f$

$\dot{f} = dF_0r$

If we then eliminate (say) $f$ we find that

$\ddot{r} = -bdR_0F_0r$,

which is undamped SHM about the fixed point. It follows that if the system is perturbed, the populations will oscillate in quadrature indefinitely.

There are some very good interactive demos of this equation set available at [http://www.aw-bc.com/ide/Media/JavaTools/popltkvl.html](http://www.aw-bc.com/ide/Media/JavaTools/popltkvl.html)

You might like to try playing around to see that the oscillations become more sinusoidal as the deviation from the fixed point is decreased. However, no matter how large you make the original excursion, the population never dies out.

Note that these non-linear oscillations are not a limit-cycle, since, like the harmonic oscillator, it does not attract nearby points in phase-space.
9. Flows in three dimensions – the possibility of chaos

The introduction of a third variable brings with it increased dynamical possibilities. We begin by quickly reviewing the behaviour of linear systems, as we did in the previous section.

9.1. Linear trajectories

The roots of the third-order characteristic equation are either purely real, or consist of one real root and a complex pair. This gives a total of eight possibilities, plus a few more if we accept poles on the imaginary axis as special cases.

Three oscillatory cases, with a pair of complex conjugate poles, are shown in the next diagram. In each case, the top row shows the trajectory in “natural” coordinates, where the eigenvector is marked as a straight line, while the top row shows the same thing transformed to the frame of the eigenvectors. Depending on the relative time constants, the trajectory either spirals around and decays onto the eigenvector before reaching the origin, or continues to spiral in the plane normal to the eigenvector well after it has reached zero in the third coordinate.

Table 3-6  TRAJECTORY PATTERNS OF THIRD-ORDER OSCILLATORY SYSTEMS

<table>
<thead>
<tr>
<th>Poles</th>
<th>A in arbitrary form</th>
<th>1(a) x1: Eigenvector</th>
<th>2(a) x1: Eigenvector</th>
<th>3(a) x1: Eigenvector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1 \lambda_2 \lambda_3$</td>
<td>$A = \begin{bmatrix} \lambda_1 &amp; 0 &amp; 0 \ 0 &amp; \lambda_2 &amp; 0 \ 0 &amp; 0 &amp; \lambda_3 \end{bmatrix}$</td>
<td><img src="image1.png" alt="Diagram 1(a)" /></td>
<td><img src="image2.png" alt="Diagram 2(a)" /></td>
<td><img src="image3.png" alt="Diagram 3(a)" /></td>
</tr>
<tr>
<td>$\lambda_1 &lt; 0 &lt; \lambda_2$</td>
<td>$A = \begin{bmatrix} \lambda_1 &amp; 0 &amp; 0 \ 0 &amp; \lambda_2 &amp; 0 \ 0 &amp; 0 &amp; \lambda_3 \end{bmatrix}$</td>
<td><img src="image1b.png" alt="Diagram 1(b)" /></td>
<td><img src="image2b.png" alt="Diagram 2(b)" /></td>
<td><img src="image3b.png" alt="Diagram 3(b)" /></td>
</tr>
<tr>
<td>$\lambda_1 &lt; 0 &lt; \lambda_2$</td>
<td>$A = \begin{bmatrix} \lambda_1 &amp; 0 &amp; 0 \ 0 &amp; \lambda_2 &amp; 0 \ 0 &amp; 0 &amp; \lambda_3 \end{bmatrix}$</td>
<td><img src="image1c.png" alt="Diagram 1(c)" /></td>
<td><img src="image2c.png" alt="Diagram 2(c)" /></td>
<td><img src="image3c.png" alt="Diagram 3(c)" /></td>
</tr>
</tbody>
</table>
9.2. **Attractors in 3-D**

As with the 2-D case, we expect to find new behaviour in 3-space. Now that we have a third dimension, the requirement that trajectories cannot cross is not so restrictive. Of course, they still *can’t* cross, but they may appear to do so when viewed from any particular direction. In fact, while we still have all the same fixed points and limit cycles as we did in 2-D, we now add a further possibility, that of *chaos*. Chaotic behaviour was discovered mainly through numerical solutions of non-linear differential equations, the classic example being that of the Lorenz equations which characterize convective rolls in the atmosphere.

The Lorenz equations can be written in the form:\(^5\):

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz
\end{align*}
\]

There are two non-linear terms, in \(xy\) and \(xz\).

Lorenz discovered that the solutions were *sensitive to initial conditions*: that is, even an infinitesimal change in the initial conditions would lead to exponential divergence of the trajectory after long enough time (hence the notion that the flapping of a butterfly’s wings in Cambridge can lead to a hurricane in the US a month later).

These equations are *dissipative*, which is to say, the volume of 3-D phase-space occupied by any set of trajectories decreases monotonically with time (see detailed definition below). Dissipation implies the existence of an attractor of zero volume – a fixed point or a limit cycle for example.

Lorenz discovered the existence of aperiodic behaviour for large values of \(r\). So this is similar to a limit cycle, but not quite. The classic case is that in the figure, which shows the \(y\)-coordinate plotted against time for \(\sigma = 10\), \(b = 8/3\) and \(r = 28\).

Note that the trajectory is indeed aperiodic (and therefore *not* a limit cycle), and that it is constrained within some range of values (there are similar plots for both \(x\) and \(y\)). The object described thus is called a *strange attractor*.


---

\(^5\) Although it is not relevant to the discussion, \(\sigma\) is called the *Prandtl* number, and \(r\) is the *Rayleigh* number.
In order to make more progress we need first to define terms a bit more carefully.

\section*{Dissipation}

Let’s look at how volumes evolve with time. Take an arbitrary system $\dot{x} = f(x)$.

$f$ is the instantaneous velocity of points – hence $f\hat{n}$ is the outward normal component of velocity. A patch of area $dA$ sweeps out volume $f\hat{n} dt dA$ as shown.

Therefore the volume increases at a rate

$$\dot{V} = \int_S f\hat{n} dA.$$  \hfill 9.2

Making use of the Divergence theorem,

$$\dot{V} = \int_V \nabla \cdot f dV.$$  \hfill 9.3

For the Lorenz system, $\dot{V} = -\sigma - 1 - b < 0$, so the system is dissipative, as expected.

\section*{Definition of an attractor}

Loosely speaking, an attractor is any set to which all neighbouring trajectories converge, for example, stable fixed points and limit cycles. More rigorously, it is a closed set $A$ with the properties:

- $A$ is an invariant set: a trajectory $x(t)$ that starts in $A$ remains in $A$ for all time.
- $A$ attracts an open set of initial conditions: there is an open set $U$ containing $A$ such that if $x(0) \in U$ then the distance from $x(t)$ to $A$ tends to zero as $t \to \infty$. $A$ attracts all trajectories which start sufficiently close to it. The largest such $U$ is called the basin of attraction of $A$.
- $A$ is minimal: there is no proper subset of $A$ that satisfies the first two conditions.

\section*{Definition of a strange attractor}

A strange attractor is an attractor that exhibits sensitive dependence on initial conditions.
9.3. More on strange attractors

Another view of the Lorenz attractor is shown at right. This is a projection of the trajectory onto the $x-z$ plane. Remember, the trajectories don’t cross – the impression of crossing in the diagram is due simply to projecting the 3-D phase portrait onto the 2-D plane.

Strange attractors:

- Have zero volume in phase space (dissipative equations)
- Have infinite length (aperiodic trajectories).

The picture at the end of the section (scanned in from Strogatz) illustrates sensitive dependence on initial conditions. Unfortunately it only really works in colour, so you had better look at the PDF version of this document on the web if you want to see properly (or consult Strogatz, Plate 2). Or go to:

http://www.aw-bc.com/ide/Media/MainMenu/Part5.html and click on the “Lorenz equations” link to explore more fully.

9.4. Properties of the fixed points

The fixed points of the Lorenz system are obviously slightly strange. Qualitatively, they appear to be saddle points, in the sense that they attract the trajectory towards them along a line perpendicular to the “planes” of each of the two spirals (implying a negative eigenvalue in this direction), but the trajectory then spirals out from the centres, suggesting positive eigenvalues in the plane.

To pursue this more quantitatively, we first need to determine the location of the fixed points. This is straightforward: in equation (9.1) just set $\dot{x}, \dot{y}, \dot{z}$ to zero, and then solve algebraically for the fixed points. Doing this, we obtain:

\[
\begin{align*}
x &= \pm \sqrt{b(r - 1)} \\
y &= x \\
z &= \frac{x^2}{b}
\end{align*}
\]

Letting $k = \sqrt{b(r - 1)}$ we find that the fixed points are at $C^{\pm} = \{\pm k, \pm k, k^2 b\}$.

To determine the nature of the fixed points, we need to linearize around them, and then diagonalize the matrix to find the eigenvalues. Linearizing, we find that:

\[
\begin{bmatrix}
\delta \dot{x} \\
\delta \dot{y} \\
\delta \dot{z}
\end{bmatrix} =
\begin{bmatrix}
\sigma & \sigma & 0 \\
-1 & -1 & \mp k \\
\pm k & \pm k & -b
\end{bmatrix}
\begin{bmatrix}
\delta x \\
\delta y \\
\delta z
\end{bmatrix}
\]
This set of equations can be diagonalized using, for example, Maple, although for this particular system the answer is a bit messy. We do indeed find that for the values of the parameters used for this example, there are two positive and one negative eigenvalues.

The diagram shows the path of 10,000 different trajectories. They all start close together as the small red dot in the top left figure (just above and to the right of the rear intersection of the three axes). The trajectories diverge with time – the different plots are shown at times of 3, 6, 9 and 15 units. By the final plot the points are spread out all over phase-space, even though the initial conditions were almost the same.
10. Appendix 1: Table of Laplace Transforms

<table>
<thead>
<tr>
<th>f(t)</th>
<th>F(s)</th>
<th>s₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>c / s</td>
<td>0</td>
</tr>
<tr>
<td>ctⁿ</td>
<td>cn! / sⁿ⁺¹</td>
<td>0</td>
</tr>
<tr>
<td>sin bt</td>
<td>b / (s² + b²)</td>
<td>0</td>
</tr>
<tr>
<td>cos bt</td>
<td>s / (s² + b²)</td>
<td>0</td>
</tr>
<tr>
<td>eᵃᵗ</td>
<td>1 / (s - a)</td>
<td>a</td>
</tr>
<tr>
<td>tⁿeᵃᵗ</td>
<td>n! / (s - a)ⁿ⁺¹</td>
<td>a</td>
</tr>
<tr>
<td>sinh at</td>
<td>a / (s² - a²)</td>
<td>1+α</td>
</tr>
<tr>
<td>cosh at</td>
<td>s / (s² - a²)</td>
<td>1+α</td>
</tr>
<tr>
<td>eᵃᵗ sin bt</td>
<td>a / [(s - a)² + b²]</td>
<td>a</td>
</tr>
<tr>
<td>eᵃᵗ cos bt</td>
<td>(s - a) / [(s - a)² + b²]</td>
<td>a</td>
</tr>
<tr>
<td>t¹/²</td>
<td>1 / 2(π / s³)¹/²</td>
<td>0</td>
</tr>
<tr>
<td>t⁻¹/²</td>
<td>(π / s)¹/²</td>
<td>0</td>
</tr>
<tr>
<td>δ(t - t₀)</td>
<td>e⁻ˢᵗ₀</td>
<td>0</td>
</tr>
</tbody>
</table>
| H(t - t₀)   | \[\begin{cases} 
1 & \text{for } t \geq t₀ \\
0 & \text{for } t < t₀ 
\end{cases}\] e⁻ˢᵗ₀ / s | 0   |