ASTROPHYSICAL FLUID DYNAMICS

Astrophysical processes frequently involve fluids and plasmas in motion. We will need to study fluid dynamics to understand the following.

- The internal structure of the major planets, including the Earth.
- The structure and evolution of the Sun and other stars.
- The formation and dynamics of the solar wind and its interaction with the planets.
- The properties and dynamics of the various components of the interstellar medium
  - The cold (100 K, $10^6$ particles m$^{-3}$) neutral hydrogen.
  - Bright nebulae ionised by hot, young stars.
  - Dark, dense clouds of molecular hydrogen and dust (10 K, $10^{12}$ particles m$^{-3}$).
- Accretion discs and jets in protostars and exotic binary star systems.
- The dynamics of supernovae and their remnants.
- The overall dynamics of the gas in spiral galaxies.
- The properties of hot gas in clusters of galaxies (10$^8$ K, $10^3$ particles m$^{-3}$)
- The dynamics of the jets in radio galaxies and quasars.

DYNAMICS OF PARTICLES, FLUIDS AND PLASMAS

- We will need to study:
  a) systems of particles moving under gravity;
  b) incompressible and compressible fluids with viscosity;
  c) behaviour of plasmas;
  d) magnetohydrodynamics.
- The following approaches all provide insight, depending on the circumstances:
  - **The single particle approach** This gives useful hints if interactions are few. It can be extended by computer simulation to the $N$-particle case, where it provides powerful constraints on theoretical approximations.
  - **Statistical approach** When there are very many particles, we can keep a track on the number of particles in any given “energy level" (the particle velocity distribution function) and work out how it varies as a function of space and time. Some details of the individual dynamics will be lost as a result of the approximation.
  - **Fluid approach** When there are even more particles we may not care at all about microscopic properties and just use the average or macroscopic properties: density; velocity; pressure and temperature.
**STATISTICAL TREATMENT — THE DISTRIBUTION FUNCTION**

- Distribution function \( f(x, v, t) \) describes the density of particles in 6-dimensional phase space \((x, v)\).

- The probability of finding a particle (assumed all of equal mass \(m\)) in a volume element is \(f(x, v, t) \, d^3x \, d^3v\).

- The normalisation is such that \( \int d^3v f(x, v, t) = n(x, t) \), where \(n(x, t)\) is the particle density.

- Kinetic theory (statistical treatment) tries to predict the evolution of the distribution function.

- There is a flux of particles \((f \, v, f \, a)\) through any region of phase space, where \(a\) is the acceleration acting on the particles.

- Conservation of particles and use of the 6-dimensional divergence theorem leads to the governing equation, the Boltzmann equation:

\[
\frac{df(x, v, t)}{dt} = \nabla \cdot (v \, \nabla f) + a \cdot \nabla f = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}
\]

- The collision term on the RHS will often be ignored.

- The Boltzmann equation is a perfectly horrible approximation to the physics of an \(N\)-body system, but still provides very useful insights (and is still very difficult to solve).

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**REVISION: THE PERFECT GAS**

- The interiors of stars, the interstellar medium and cluster gas are all collisional, since the mean free paths of particles are much smaller than the overall scale of interest. We can usually treat the as an inviscid perfect gas. We now recall some results from previous years.

- A perfect gas obeys Sir Edward Boyle's law\(^a\). The volume \(V\), pressure \(P\) and temperature \(T\) of one mole of a perfect gas has the equation of state \(PV = RT\), where the gas constant is \(R = N_A k = 8.31 \text{ J K}^{-1}\).

- The First Law of Thermodynamics

\[
dU = TdS - PdV
\]

- **Specific heats**

At constant volume, no work done: \(C_V \, dT = T \, dS = dU\)

At constant pressure: \(C_P dT = T \, dS = dU + P \, dV = dU + R \, dT\)

\[\Rightarrow C_P - C_V = R \equiv (\gamma - 1)C_V\]

- Important property of gas: ratio of specific heats \(\gamma = C_P / C_V\)

- Can rearrange: \(U = C_V T = \frac{RT}{\gamma - 1} = \frac{P}{\gamma - 1} \, V\)

- Specific internal energy \(u: \, U = \int dV \, u \Rightarrow u = \frac{P}{\gamma - 1}\)

\(^a\)“The greater the external pressure, the greater the volume of hot air.”
Adiabatic compression of perfect gas

- **Adiabatic compression** No heat — no change in entropy \((dS = 0)\).

\[
\begin{align*}
\frac{dU}{dU} &= -PdV \\
\frac{dU}{dU} &= d\left(\frac{PV}{\gamma - 1}\right) = \frac{1}{\gamma - 1}(PdV + VdP)
\end{align*}
\]

\[
\Rightarrow \frac{dP}{P} = \frac{-\gamma dV}{V} \Rightarrow P \propto V^{-\gamma}
\]

- For an adiabatic fluid \(P \propto \rho^\gamma\).

- This is true as long as the flow is adiabatic. However, even if the thermal conductivity is negligible, the entropy can increase catastrophically at shock fronts.

- The speed of sound in a gas is \(v_s = \left(\frac{\gamma P \rho}{\rho}\right)^{1/2} = \left(\frac{\gamma kT}{m}\right)^{1/2}\)

\((m)\) is the mass of an individual molecule).

[Air: \(\gamma = 1.4\); non-relativistic monatomic gas: \(\gamma = 5/3\); relativistic gas: \(\gamma = 4/3\)]

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Pressure and hydrostatic equilibrium

- Apparently, Archimedes knew something about pressure and hydrostatic equilibrium.

- The Mercury barometer was invented by Evangelista Torricelli in 1643.

- Atmospheric pressure can balance a column of mercury of height \(\sim 760\) mm.

- Mean atmospheric pressure at sea level on Earth is about \(1.0 \times 10^5\) Pa.

- The weight of the above us per unit area is equal to this pressure, so the mass of air we support is \(\approx 1.0 \times 10^4\) kg m\(^{-2}\).

- The pressure needed to support the weight \(\rho g dz\) per unit area is \(dP = \rho g dz\) (\(g\) negative).

- Generalising, differential equation of hydrostatic equilibrium is

\[
\nabla P = \rho g
\]
Abell 2029 is a dynamically relaxed cluster of galaxies at $z = 0.0779$ (300 Mpc). It contains some active galaxies, but none that disturb the hot cluster gas very much. It is seen here in X-rays emitted by gas that fell into the potential well. The spatial extent is about 300 kpc.

- The temperature of the gas drops from 9 keV (almost exactly $10^8$ K) in the outer regions to $4 \times 10^7$ K in the centre, where there is a massive elliptical galaxy that is eating everything that comes close.
- The density of the gas is about $10^4$ particles m$^{-3}$ and spectroscopy shows that the gas near the centre has been enriched in metals (metals are anything with $Z > 2$) due to supernovae.

Fluid dynamics and plasma physics

**FLUID DYNAMICS**

- When the mean free path $\lambda$ of particles in a gas or plasma is small compared to the scales of interest, we can treat the system as a hydrodynamical fluid.
- We imagine the fluid composed of fluid elements, which are regions bigger than $\lambda$, large enough to have well-defined values of macroscopic properties: density $\rho(x, t)$, velocity $\mathbf{v}(x, t)$; pressure $P(x, t)$; energy density $u(x, t)$ and anything else that is relevant, e.g. magnetic field.
- A fluid element is acted on by gravity (and other body forces), by pressure forces and other stresses on the surface.
- These fluid elements stay more or less well-defined (on scales $\gg \lambda$), but the surface of an element is moving at the local fluid velocity, so elements distort as they move around. The distortions can become very large — in the presence of turbulent motions the fluid elements can be dispersed.
- There are still some microscopic physical processes occurring on scales of $\lambda$ that determine important properties of a fluid: heat conduction; viscosity. These are transport processes resulting from interpenetration of the of the velocity distributions from nearby points. The macroscopic properties can also change discontinuously on scale so $\lambda$ at shock fronts.
- We generalise the equation of hydrostatics: $0 = -\nabla P + \rho g$ to include the acceleration of a fluid element:

$$\frac{D\mathbf{v}}{Dt} = -\nabla P + \rho g$$
THE CONVECTIVE DERIVATIVE

- A material element moves with the velocity of the fluid, which is a function of space and time: \( \mathbf{v}(x, t) \).
- For any change of \( dt, dx \), the change in velocity is (in components)
  \[
d\mathbf{v} = dt \frac{\partial \mathbf{v}}{\partial t} + dx \cdot \nabla \mathbf{v} = dt \frac{\partial \mathbf{v}}{\partial t} + dx_i \frac{\partial \mathbf{v}}{\partial x_i}
\]
- An element of fluid moves with velocity \( \mathbf{v} \), so in time \( dt \), the position change is \( dx = \mathbf{v} dt \).
- Hence for changes at position of fluid elements:
  \[
d\mathbf{v} = dt \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right)
\]
- Hence define convective derivative \( \frac{D \mathbf{v}}{Dt} = \frac{d\mathbf{v}}{dt} \) moving with fluid

LAGRANGE OR EULER?

- In order to work out the accelerations, we need to keep track of movement of the fluid elements.
- Here we have a choice: we can watch the fluid flowing through space (Eulerian viewpoint), so that we work directly with the \( \rho(x, t), \mathbf{v}(x, t) \) and \( P(x, t) \) variables and work how things are changing at points fixed in space.
- Alternately, (Lagrangian formulation) we assign time-independent labels \( \xi \) to the fluid elements and track their positions \( \mathbf{x}(\xi, t) \).
- In the Lagrangian view the mass of fluid in an element is fixed so \( d^3 \xi \rho(\xi) \), is not a function of time.
- The dynamics is easier in Lagrange’s view: the velocity is \( \mathbf{v}(\xi, t) = \left( \frac{\partial \mathbf{x}}{\partial t} \right)_\xi \) and \( \frac{D \mathbf{v}}{Dt} = \left( \frac{\partial \mathbf{v}}{\partial t} \right)_\xi \).
- From Lagrange’s viewpoint we keep track of the whole history of the flow, but the surfaces of constant \( \xi \) can become horribly scrambled as the flow progresses.
- The Lagrangian viewpoint is preferable in principle, but it is really only possible for one-dimensional flows, but this case includes some very important applications.
The equations of hydrodynamics

- The equation of continuity is only needed in the Eulerian formulation:
  \[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]

- The Lagrangian force equation (can divide by \( \rho(x) \)):
  \[ \frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho(x)} \nabla P(x) + \mathbf{g}(x) \]

The Eulerian force equation:

\[ \rho \frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \rho \mathbf{g} \]

- We also need an energy equation: for Lagrangian hydrodynamics we can use \( dU = TdS - PdV \). For adiabatic flow we can often get away with using \( P \propto \rho^\gamma \). The energy equation in Eulerian form is complicated by the fact that the volume of fluid elements is changing. The following equation is only given for completeness (\( \epsilon \) is the heat input per unit volume):

\[ \frac{Du}{Dt} = \epsilon + (Ts - u - P) \nabla \cdot \mathbf{v} \]

- To make a realistic model of any fluid we also need to consider the effects of viscosity, and thermal conduction, and the any anisotropic component to the pressure of a plasma arising from any magnetic field present.

Bernoulli equation for compressible flow

- There is an important theorem which is often useful for cases involving steady flow, expressing the conservation of energy as it transported along a streamline: the Bernoulli equation.

- For steady flow, consider streamlines connecting areas \( A_1 \) and \( A_2 \):

\[ \frac{v_1}{\rho_1} \frac{P_1}{u_1} = \frac{v_2}{\rho_2} \frac{P_2}{u_2} \]

- Energy flowing in: \( A_1 v_1 (\rho \phi_1 + \frac{1}{2} \rho_1 v_1^2 + u_1) \)
  \((u) \) is internal energy per unit volume).

- Pressure does work: \( A_1 v_1 P_1 \).

- Similarly at area \( A_2 \), but mass flow is the same: \( A_1 v_1 \rho_1 = A_2 v_2 \rho_2 \).

- Bernoulli equation: along a streamline the quantity

\[ \frac{u + P}{\rho} + \frac{1}{2} v^2 + \phi = \text{constant} \]

- For a perfect gas specific enthalpy \( u + P = \frac{\gamma}{\gamma - 1} P \)

- For incompressible flow \( \gamma = \infty \) set \( u = 0 \).
Approximations to Hydrodynamics

- The equations of compressible hydrodynamics (already a massive simplification of the microscopic picture) are clearly very complicated.

- They are also notoriously difficult to integrate numerically — in 3-D computing time scales like \( (\text{grid size})^4 \). Accuracy also a problem due to advection through grid, which will blur density contrasts.

- Very useful approximations that yield insight:
  - **Incompressible flow**: \( \nabla \cdot \mathbf{v} = 0 \). Clearly a good model for liquids, but also a surprisingly good approximation for gases provided the flow is subsonic.
  
  - **Irrotational flow**: this is the case when there is no vorticity \( \nabla \times \mathbf{v} = 0 \). The vorticity field \( \Omega = \nabla \times \mathbf{v} \) moves with the fluid, and vorticity is usually generated at boundaries, so often the bulk of a flow is irrotational.

If \( \nabla \times \mathbf{v} = 0 \) the velocity field can be generated from a scalar potential \( \mathbf{v} = \nabla \Phi \) (sorry, but fluid dynamics has a plus sign here . . . ). If the fluid is both incompressible and irrotational, the velocity potential satisfies Laplace's equation \( \nabla^2 \Phi = 0 \).

- In this important case \(^{a}\) we can use potential theory to find \( \mathbf{v} \) and Bernoulli’s theorem to find the pressure.

\(^{a}\) Feynman calls it the "flow of dry water" (Lectures: Volume 2)

Hydrostatic Equilibrium of the Sun

- The equation of hydrostatic equilibrium for a spherically-symmetric body of radius \( R \) and mass \( M \):
  \[
  \frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}
  \]
  where \( m(r) \) is the mass interior to \( r \).

- Multiply by \( 4\pi r^3 \) and integrate:
  \[
  \left[4\pi r^3 P(r)\right]_0^R - 3 \int_0^R dr 4\pi r^2 P(r) = -\int_0^R dr 4\pi r^2 \rho(r) \frac{Gm(r)}{r}
  \]
  The integrated part is zero if the pressure falls to zero at \( r = R \), and the integral on the LHS is \(-3\) times the integral of the pressure over the volume. The RHS is the stored gravitational potential energy, i.e. the total energy needed to disassemble the sphere.

- The volume-averaged pressure is therefore \( \langle P \rangle = -\frac{E_{\text{grav}}}{3V} \)

- The gravitational stored energy depends on the density distribution \( E_{\text{grav}} = -f \frac{GM^2}{R} \), where \( f \) is of order unity. For a uniform sphere \( f = 3/5 \), but it would be greater if the density increases towards the centre.

- For \( M = 2 \times 10^{30} \) kg and \( R = 7 \times 10^8 \) m, we find a stored energy of \( 4 \times 10^{41} \) J and an average pressure of \( \langle P \rangle = 9 \times 10^{13} \) Pa

- Assuming that the Sun is made of hydrogen, we calculate that it contains \( \approx 10^{57} \) protons and the same number of electrons. Using \( PV = NkT \) we estimate the temperature as \( \approx 5 \times 10^6 \) K, three orders of magnitude greater than its surface temperature (6000 K)
THE SUN AND BERNOULLI’S THEOREM

- We have concluded that the Sun’s surface is very cool compared to its interior (the central temperature is $1.56 \times 10^7$ K).

- We can get many insights into solar physics by applying Bernoulli’s theorem to (hypothetical) streamlines connecting the Sun’s surface to infinity.

- In Bernoulli’s theorem $\frac{u + P}{\rho} + \frac{1}{2}v^2 + \phi = \text{constant}$ set
  $v_\infty = 0, P_\infty = u_\infty = 0$, so the constant is zero. Then at the surface:
  - if the material is cold ($P = u = 0$) we conclude the velocity (inwards or outwards) is 600 km s$^{-1}$.
  - alternatively, if the velocity is low, the temperature must be (approximately)
  \[
  \frac{P}{\rho} = kT = \frac{\gamma - 1}{\gamma} \frac{GM_\odot}{R_\odot} \Rightarrow T = 5 \times 10^6 \text{ K}
  \]

- In fact, both alternatives happen: in quiescent conditions there is a hot solar corona, which produces a slow solar wind 400 km s$^{-1}$ by heat conduction to radius of about 2 $R_\odot$.

- In solar flares and coronal ejections, hot material is thrown upwards, where it escapes as a faster solar wind at $\approx 700$ km s$^{-1}$.

BERNOULLI EQUATION FOR INCOMPRESSIBLE FLOW

- Flow from water tank or bathtub.
  - The pressure is the same at the top of the tank at point A where $v = 0$ and at point B where the water escapes.
  - Use Bernoulli:
    \[
    P + \frac{1}{2}v^2 + gh = \text{constant}.
    \]

  - This says the outflow velocity is $\sqrt{2gh}$.

  - The actual draining rate might not be equal to this velocity times the area of the hole, since the flow can still be converging as the water leaves the tank.

  - Efflux coefficient (effective area of hole divided by geometric area) varies between 0.5 to 1. For a simple hole in the side of a tank it is 0.62.

- Cylindrical jet hitting a wall
  - Ignore gravity for this problem.

  - The stagnation point at A must have pressure $P_0$ + $\frac{1}{2}\rho v_0^2$.

  - The pressure at B (large radius compared to jet radius) must be $P_0$ again, so Bernoulli says that the velocity is $v_0$ there.
**Collisional processes: viscosity and thermal conductivity**

- Viscosity and thermal conductivity in fluids and plasmas are determined by the extent to which the particle distributions at different spatial position can interpenetrate; i.e. it depends on the mean free path $\lambda$.

- For collisional fluids this is related to the thermal velocity $v_T$ and collision frequency $\nu_c$ by $\lambda \approx \frac{v_T}{\nu_c}$.

- Consider the transport of a quantity $Q$ which varies macroscopically with position.

- $Q$ might be the specific energy $u$ (thermal conductivity) or a component of the momentum $\rho v$ (viscosity).

- Transport of a $Q$ then occurs by exchange of “blobs” of amount $\Delta Q$, travelling a distance $\lambda$ at velocity $v_T$.

- This random walk leads to a diffusion equation for property $Q$, in the frame moving with the fluid element:

\[
\frac{1}{3} \lambda v_T \nabla^2 Q = \frac{DQ}{Dt}
\]

- The factor of $\frac{1}{3}$ in the diffusion coefficient accounts for the fact that the blob is free to move in any of the three dimensions.

**Viscosity in gases**

- Viscous stresses arise from momentum transport $Q = \rho v_i$ and we will expect a diffusive term $\frac{1}{3} \rho \lambda v_T \nabla^2 v$ to appear in the force equation.

- Suppose the velocity varies as in a shear layer: $v_x \propto y$. The tangential stress (force per unit area) on the surface at A is due to the material coming from B where the flow is faster by $\lambda \frac{dv_x}{dy}$.

- In a time $\lambda/v_T$ a volume $\frac{1}{3} \lambda$ is transported per unit area, with momentum density $\rho \lambda \frac{dv_x}{dy}$.

- The tangential viscous stress is the momentum per unit area per unit time $\tau_{yx} = \rho \lambda \frac{dv_x}{dy} \times \frac{1}{3} \lambda/(\lambda/v_T) = \frac{1}{3} \rho \lambda v_T \frac{dv_x}{dy} = \eta \frac{dv_x}{dy}$, where $\eta$ is the viscosity.

- In a uniform shear layer the viscous shear stress (force per unit area) is given by $\tau_{yx} = \eta V/d$.

- The direction of $\tau_{yx}$ indicates the direction of the stress acting on the inside of the plates at A and B.
The mean free path $\lambda$ is related to the number density of particles $n$ and the collision cross-section $\sigma$ by $n\sigma\lambda = 1$.

The cross-section is related to the size of the atoms and various other assumptions about the collisions, e.g. hard spheres: $\sigma = 4\pi a^2$.

The viscosity is given by $\eta = \frac{1}{3} \rho v T \sim \frac{1}{3} mn \times \frac{1}{n\sigma} \sqrt{\frac{3kT}{m}}$, where $m$ is the particle mass.

Gas viscosity is independent of density, and proportional to the square root of temperature: $\eta \approx \sqrt{\frac{mkT}{3\sigma}}$. (Analysis of the hard sphere model suggests a numerical factor of $5\sqrt{\pi}/16$ (4% different. . .).)

Viscosity of air at STP is $1.8 \times 10^{-5}$ Pa s, suggesting $\sigma = 8 \times 10^{-19}$ m$^2$ and $a = 2 \times 10^{-10}$ m. The mean free path implied is 4 microns.

For atomic hydrogen in the interstellar medium, we can scale the measurements from He$_4$ ($46 \mu$Pa s at 100 K), to get an estimate of $\sigma \approx 10^{-19}$ m$^2$. At interstellar densities of $10^6$ particles m$^{-3}$ this gives $\lambda \approx 10^{13}$ m, which is larger than the solar system, but small on interstellar scales.

The solar wind has a similar density $\sim 10^6$ particles m$^{-3}$, but is ionised and its mean free path will be smaller, due to plasma processes and magnetic fields.

A proper derivation yields a viscous force term $\eta(\nabla^2 v + \nabla \nabla \cdot v)$ which is the gradient of a symmetric stress tensor: $\eta (\nabla_j v_i + \nabla_i v_j)$.

Compressible fluids also have a resistance to compression — a bulk viscosity $\eta'(\nabla \nabla \cdot v)$.

The force equation now becomes

$$\frac{\rho Dv}{Dt} = -\nabla P + \rho g + \eta \nabla^2 v + (\eta + \eta') \nabla \nabla \cdot v$$

There are two interesting limits depending on the ratio of the viscous and inertial stresses:

- if the viscosity is large we get laminar flow (streamlines are smooth).
- if the inertial stresses $\rho v_i v_j$ are greater than the viscous stress the flow may become unstable and turbulence will result.

We will now do some examples on fluid flow to help you with Problem Sheet 2.

- Sources and sinks. Illustrating potential flow and Bernoulli’s equation.
- Potential flow past sphere. Illustrates Bernoulli and problems with “dry water” approach.
- Viscous flow.
Potential Flow — Sources and Sinks

- Sources/sinks: analogue of charges in electrostatics ($\nabla \cdot \mathbf{v} \neq 0$).
- Flow rate $Q$ from an isotropic nozzle has velocity potential and flow
  \[ \Phi = -\frac{Q}{4\pi \mathbf{R}}; \]
  \[ \mathbf{v} = \frac{Q}{4\pi \mathbf{R}^2} \hat{u}_R. \]

- Use the same methods to solve problems as in electrostatics.
- Source at $(0, 0, d)$, image at $(0, 0, -d)$.
- Potential at point $P$ at $x, y, z$ is
  \[ \Phi = -\frac{Q}{4\pi} \left( \frac{1}{|R_1|} + \frac{1}{|R_2|} \right). \]
- The velocity field is then given by $\mathbf{v} = \nabla \Phi$.
- We then use Bernoulli to find the pressure on the plate.

- At the plate the velocity potential is
  \[ \Phi = -\frac{Q}{4\pi} \frac{2}{(d^2 + r^2)^{1/2}} \]
  where $r = (x^2 + y^2)^{1/2}$ is the cylindrical radial coordinate.
- The radial velocity at the plate is
  \[ \frac{Q}{2\pi} \frac{r}{(d^2 + r^2)^{3/2}}. \]
- Apply Bernoulli to streamline AB: $P = P_0 - \frac{Q^2 \rho}{8\pi^2} \frac{r^2}{(d^2 + r^2)^3}$
- Integrate the pressure defect $\delta P = P - P_0$ to get the total force
  \[ F = \int_0^\infty 2\pi r dr \delta P = \frac{Q^2 \rho}{16\pi d^2} \]
- Two sources (or two sinks) distance $D$ apart are attracted to each other by force $\frac{Q^2 \rho}{4\pi D^2}$ (c.f. electrostatics). A source and a sink repel each other (net momentum transfer).
Steady flow past sphere of radius $a$.

At large distances flow is uniform velocity in $z$ direction

\[ \Phi = V_0 z = V_0 r \cos \theta \]

Boundary condition: $v_r = 0$ at $r = a$.

Cylindrically-symmetric solutions of $\nabla^2 \Phi = 0$ which tend to zero at $r = \infty$ are

\[ \Phi = \frac{A}{r} + \frac{B}{r^2} \cos \theta + \frac{C}{r^3} \frac{1}{2} (3 \cos^2 \theta - 1) \ldots \]

Present case needs dipole at centre: $\Phi = V_0 r \cos \theta + \frac{B}{r^2} \cos \theta$

Radial velocity $v_r = \frac{\partial \Phi}{\partial r} = \cos \theta \left( V_0 - \frac{2B}{r^2} \right)$.

Boundary conditions gives $B = \frac{1}{2} V_0 a^3$.

Velocity potential for flow past sphere

\[ \Phi = V_0 \cos \theta \left( r + \frac{a^3}{2r^2} \right) \]
**Boundary Layers**

- Boundary condition for surface of solid body: no radial or tangential velocity (no slip).
- Flow past a solid body must develop a boundary layer with a velocity gradient. This is a region with vorticity ($\nabla \times \mathbf{v} \neq 0$), which can then enter the fluid flow.
- The ratio of the inertial stress $\rho v^2$ to the viscous stress $\eta v/L$ is an important dimensionless quantity known as the Reynolds number $N_R = \frac{\rho vL}{\eta}$.
- If the inertial stress is too high in a flow random transverse motions will cause turbulent mixing, which in turn increases the effective “eddy viscosity” $\left(\frac{\eta}{\rho}\right)_{\text{effective}} \sim v_{\text{eddy}} L_{\text{eddy}}$.
- Turbulence transports energy to smaller scales (and to larger scales) by splitting up of eddies on a timescale $L_{\text{eddy}}/v_{\text{eddy}}$.
- Energy is rapidly transported to the smallest scales, where viscous dissipation can occur.

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**Viscous Shear Layer**

- Equation of motion $\rho \frac{Dv}{Dt} = -\nabla P + \rho g + \eta \nabla^2 v$
- Simple case — shear flow between two flat plates at $y = 0$ and $y = d$.

\[ v = V \]

- Steady flow: $0 = -\nabla P + \rho g + \eta \nabla^2 v$
- The equation of motion is a vector equation:
  \[ \frac{dP}{dy} = -\rho g \]
  \[ \eta \frac{d^2 v_x}{dy^2} = 0 \]
- Vertical forces imply $P(y) = P(0) - \rho gy$ (as expected).
- Horizontal forces give: $v_x = A + By$, where $A$ and $B$ are constants.
- Boundary condition: no slip at either plate $v_x = \frac{V y}{d}$
- In steady flow the horizontal forces on a fluid element are zero, so the stresses on either side of the element must balance: $\tau_{yx} = \eta \frac{dv_x}{dy} = \eta V/d$.
- The stress $= \eta V/d$ is constant in $y$ and is transmitted to both plates.
POISEUILLE FLOW — DRAINING PLATE

- Turn plate to vertical with free edge (keeping $x$ longitudinal and $y$ transverse) (Situation models oil draining from plate.).
- Pressure is equal to $P_0$ everywhere (ignoring density of air).
- The net viscous force on an element must now balance gravity:
  \[ d\tau_{yx} = -\rho g dy, \]
  so that
  \[ \frac{d\tau_{yx}}{dy} = \eta \frac{d^2 v_x}{dy^2} = -\rho g \]
- Solution: 
  \[ \eta \frac{dv_x}{dy} = A - \rho gy, \]
  where $A$ is a constant.
- Boundary condition: the stress must vanish at $y = d \Rightarrow A = \rho gd$
- Integrating and applying the boundary condition $v_x = 0$ at $y = 0$ we get 
  \[ v_x = \frac{\rho \eta}{\eta} (yd - \frac{1}{2}y^2). \]
- Characteristic parabolic (Poiseuille) flow profile. The surface flow rate is 
  \[ \frac{g\rho d^2}{2\eta} \]
- The total volume flow rate per unit area is 
  \[ \int_0^d dy v_x = Q = \frac{g\rho d^3}{3\eta}. \]

POISEUILLE FLOW IN PIPES — TURBULENCE

- Flow in pipe of circular cross-section with longitudinal pressure gradient.
- Balance of forces on element
  \[ \frac{dP}{dz} 2\pi r dr = d \left( 2\pi r \eta \frac{dv_z}{dr} \right) \]
  \[ \eta \frac{dv_z}{dr} = \frac{1}{2} r^2 \frac{dP}{dz} \]
  (constant of integration is zero since no stress at $r = 0$).
- Integrating and setting $v_z = 0$ at $r = a$ we get 
  \[ v_z = \frac{1}{4} \frac{dP}{dz} \left( a^2 - r^2 \right) \] and flow rate 
  \[ Q = \frac{\pi a^4}{8\eta} \frac{dP}{dz} \]
- Friction factor $f$: 
  \[ \frac{dP}{dz} = f \frac{\rho \bar{v}^2}{2a} \]
- Lamina flow: $f = 32/N_R$ where Reynolds number 
  \[ N_R = \rho (2a) \bar{v} / \eta. \]
- Flow becomes turbulent at Reynolds number $N_R \approx 2000$.
- After that $f \approx 0.03$ for turbulent flow.
- In turbulent flow the velocity profile across bulk of pipe is irregular, but of approximately constant mean velocity.
- Lamina flow: 
  \[ Q \propto \frac{dP}{dz}. \]
- Turbulent flow: 
  \[ Q \propto \left( \frac{dP}{dz} \right)^{1/2}. \]
Method of dimensions: drag force on sphere: \( F = \rho v^2 d^2 f(N_R) \) (\( d \) is diameter, \( N_R = \rho v d/\eta \), \( f \) is drag coefficient).

- Low Reynolds number: ignore \( F \approx \eta d v (f \approx 1/N_R) \).

- Drag force on sphere in laminar flow is \( 6\pi \eta dv \) (Solved by Stokes).

- High Reynolds number \( (> 10^6) \) ignore \( \eta \Rightarrow F \approx \rho v^2 d^2 (f \text{ constant}) \).

- Boundary layer swept round to low-pressure region at \( \theta = \pi/2 \).

- Thereafter pressure predicted by potential solution increases (adverse pressure gradient).

- Boundary layer detaches and eddies form behind sphere, decreasing the pressure and giving a drag force.

- Laminar boundary layer around sphere is more likely to detach than a turbulent one.

- For smooth sphere the drag coefficient suddenly decreases by factor of two at \( N_R \approx 2 \times 10^6 \) due to turbulence.

- Golf balls exploit this by having dimples that cause the transition to occur earlier at \( N_R \approx 10^6 \).
Entrainment of external material slows the jet as it expands.

- Assumption: turbulent velocity $w$ is proportional to average central velocity $v$: $w = \alpha v$.

- Turbulent motions entrain external fluid into element $dz$: $\frac{Dr}{Dt} = \alpha v$

- Element $dz$ of jet shares its forward momentum with entrained fluid, so that $\pi r^2 dz \rho v$ is constant along the jet.

- Although the centre of the jet is moving faster than the edges, the shape of the velocity profile remains the same at all distances, so we set $dz = v dt$. In terms of the fluid element injected per unit time, we have $\pi r^2 \rho v^2$ is constant along the jet.

- The jet expands sideways linearly with distance travelled. The constant $\alpha \approx 0.08$ for turbulent jets in external medium of equal density. The jet slows down $v \propto 1/r$. In time: $v \propto 1/r^{1/2}$.

- High density jets can have lower spreading ratio. Have also assumed jet was fully turbulent from the outset.

- The average velocity profile $v(r)$ is approximately Gaussian.

Waves in Compressible Fluids

- Sound waves in a gas travel at $v_s = \sqrt{\gamma P / \rho}$.

- The sound waves carry “information” around the fluid about moving objects, etc.

- If a fluid is moving subsonically past a source of waves (e.g. a body), the waves are carried slowly downstream, but can still propagate upstream.

- Subsonic movement of fluid is qualitatively different from supersonic flow.

- Waves get carried away quickly downstream, leaving an undisturbed flow of fluid upstream of the caustic at the Mach angle $\sin \alpha = v_s / v$.

- The Mach number $M = v / v_s$ where $v_s$ is the upstream sound speed.
- Supersonic flows are characterised by the presence of shock fronts, with a sudden discontinuity of density, pressure and velocity. This is accompanied by dissipation leading to an increase of entropy.

- In a one-dimensional shock tube a piston moves into a fluid at a speed \( v_p \) greater than the initial sound speed \( v_s \) of the gas in the tube.

- A shock advances into the colder, stationary gas, and hot, shocked gas then moves with the velocity of the piston.

- Look at the process in the frame moving with the shock. Upstream of the shock, the particles of the fluid have an ordered, supersonic motion. The downstream particles have a much more disordered motion (hotter), and a lower bulk velocity.

- Collisional gases/plasmas: thickness \( \sim \lambda \) collisional mean free path.

- Collisionless plasma: thickness \( \sim v/\lambda_{Di} \) due to plasma waves.

\[ v_2 = \frac{\rho_1 v_1}{\rho_2} = \frac{\gamma - 1}{\gamma + 1} \left( 1 + \frac{2}{(\gamma - 1)M^2} \right) \]

\[ P_2 = P_1 + \frac{2\rho_1 v_s^2}{\gamma + 1} \left( 1 - \frac{1}{M^2} \right) \]

where \( M \) is the upstream Mach number \( \frac{v_1}{v_s} \) and \( v_s = \sqrt{\frac{\gamma P_1}{\rho_1}} \).

(These formulae are not for examination, but very much easier to prove than appears at first sight.)

- Density compression of \( 4 \) for the usual case of a strong shock in a monatomic non-relativistic gas (\( \gamma = \frac{5}{3} \)).
Plasma is a state of matter containing significant ionisation. Plasma is very different from other states of matter (solid, liquid, gas). Why?

A plasma is a soup of oppositely charged particles: electrons/ions (or $e^-$/$e^+$). These particles interact via strong, long-range electromagnetic forces. In gases the interparticle forces are short range and only important in collisions.

Plasmas show a wide range of new, collective phenomena (plasma waves). These can interact with each other and the particles, producing plasma turbulence and particle acceleration.

Most important is the plasma frequency $\omega_{pe} = \left(\frac{n_e e^2}{\varepsilon_0 m_e}\right)^{1/2}$. The whole plasma undergoes longitudinal oscillations at this frequency.

The extent to which particles behave individually or collectively is determined by the Debye length — the distance a thermal particle travels in time $1/\omega_{pe}$. Electric fields are screened on scales larger than $\lambda_{De} = v_T/\omega_p = \left(\frac{\varepsilon_0 kT_e}{n_e e^2}\right)^{1/2}$.

Influence of magnetic field is hugely important:

- Particles circulate the field lines at the gyro frequency $\omega_{ge} = \frac{eB}{m_e}$
  This limits the mobility of individual particles to the Larmor radius, which is tiny on astrophysical scales. Particles can still stream along field lines, but will scatter off magnetic irregularities.
- the anisotropic magnetic pressure can be very important dynamically, particularly near compact objects (stars/neutron stars).

Almost all material (normal, star stuff) in the universe is ionised (plasma). A tiny fraction of the universe is made up of ordinary solids, liquids and gases (0.1% of the solar system).

There is a very great range of densities and temperatures of plasma found in Nature.
Motion of particles in electromagnetic fields:

- **Electric field** gives uniform acceleration, leading to currents opposing $E$. This means that plasmas are highly conducting so there are no large-scale $E$ fields.

- Plasmas are usually electrically neutral (on scales greater than the Debye length).

- **Magnetic field** Determines the basic behaviour of particles in plasma: motion is helix of constant pitch angle.

  - Velocity parallel to field $v_{||}$ is constant (no force).
  - Moves in circle perpendicular to field.

- Balance of forces: $\frac{mv_{\perp}^2}{r} = ev_{\perp}B$ gives gyro (Larmor) frequency:

  $$\omega_g = \frac{eB}{m} \quad r = \frac{mv_{\perp}}{eB}$$
**PLASMA FREQUENCY AND LONGITUDINAL WAVES**

- Electrons in cold plasma can oscillate at the plasma frequency.
- Consider a slab of plasma. The ions are too heavy to move far in the period of a plasma oscillation, so consider them as a fixed, neutralising background.
- Suppose the electrons are displaced from the ions by a distance \( \xi \).
- This generates an electric field \( E = \frac{\sigma}{\varepsilon_0} = \frac{n_e e \xi}{\varepsilon_0} \).
- Field tends to restore electrons to original position:
  \[
m_e \ddot{\xi} = eE = -\frac{n_e e^2}{\varepsilon_0} \xi \Rightarrow \ddot{\xi} + \omega_p^2 \xi = 0
\]
  where \( \omega_p^2 = \frac{n_e e^2}{m_e \varepsilon_0} \).
- Simple harmonic motion at the electron plasma frequency
  \[
  \omega_p = \sqrt{\frac{n_e e^2}{m_e \varepsilon_0}}
  \]
- Fundamental frequency of plasma oscillations.
- Determines electromagnetic properties; interaction of particles with plasma waves; acceleration of particles.

**COLLECTIVE BEHAVIOUR — DEBYE LENGTH**

- Consider a plasma of fixed ions and warm electrons of temperature \( T_e \) and average density \( n_{0e} \).
- The electron density depends on the electrostatic potential \( \Phi(r) \) via Boltzmann factor:
  \[
  n_e(r) = n_{0e} \exp \left( -\frac{e\Phi(r)}{kT_e} \right)
  \]
  for small \( e\Phi/kT_e \) we approximate \( \exp(x) \approx 1 + x \) so that the net charge is \( \rho_e = -\frac{n_{0e} e^2}{kT_e} \Phi \) (the ions cancel out the average charge).
- Equation for electrostatic potential:
  \[
  \nabla^2 \Phi = -\frac{\rho_e}{\varepsilon_0} = \frac{n_{0e} e^2}{\varepsilon_0 kT_e} \Phi
  \]
  The potential satisfies \( \nabla^2 \Phi - \frac{1}{\lambda_D^2} \Phi = 0 \), where \( \lambda_D^2 = \frac{\varepsilon_0 kT_e}{n_{0e} e^2} \) is the Debye length.
- Charges move to neutralise the applied field on scales \( > \lambda_D \).
- Consider plasma between charged plates.
  In one dimension \( \frac{d^2 \Phi}{dx^2} + \frac{1}{\lambda_D^2} \Phi = 0 \) has solutions \( \Phi \propto \exp(\pm x/\lambda_D) \)
- Excess charge confined to sheath of size \( \lambda_D \) near plates.
- Spherically symmetric field: \( \Phi(r) \propto \frac{e^{-r/\lambda_D}}{r} \).
  Screened Coulomb potential (Yukawa).
- Every ion is surrounded by shielding a "coat" of excess electrons.
**Debye Number**

- Consider a sphere of radius the Debye length $\lambda_D$. It contains $N_D \equiv \frac{4}{3} \pi \lambda_D^3 n_e$ electrons: the Debye number.
- The Debye number is the number of electrons in the “coat” shielding any ion in the plasma.
- The Debye number is a measure of the importance of collective effects in the plasma.
- If $N_D < 1$ there are no collective effects. The “plasma” is merely a collection of individual particles.
- If $N_D > 1$ it is a true plasma and cooperative effects are important.
- Usually $N_D \gg 1$, with $N_D$ ranging from $10^4$ (laboratory) to $10^{32}$ (cluster of galaxies).

**Collisions in Plasma**

- Mean free paths in plasma are determined by long-range interactions between ions and electrons. The collision process is almost exactly the same as we have seen for gravitational interactions in a galaxy.
- **Electron-ion collisions.** Consider an electron moving at $v$ going past a stationary ion with impact parameter $b$.
- We assume that the interaction is so weak that we can approximate the trajectory as a straight line, moving at constant velocity.

\[
\begin{align*}
\text{e}^- & \quad \text{v} \\
Ze^+ & \quad b
\end{align*}
\]

- Maximum lateral force \( \frac{Ze^2}{4\pi\epsilon_0 b^2} \) applied for a time \( \approx \frac{2b}{v} \)
- Momentum transfer per “collision” \( \Delta p = \frac{2Ze^2}{4\pi\epsilon_0 bv} \) (exact result).
- The lateral momentum changes have random directions, but the effect on \( \langle \Delta p^2 \rangle \) is cumulative:
  \( \langle \Delta p^2(t) \rangle = \Delta p^2 \times N_{\text{coll}} \).
- Number of collisions with impact parameter $b \rightarrow b + db$ per unit time is $2\pi b\,db\,n_i\,v$, where $n_i = n_e/Z$. 

The rate of increase of $\langle \Delta p^2(t) \rangle$ per unit time is

$$\approx \int db \, 2\pi b \, n_i v \left( \frac{2Ze^2}{4\pi\epsilon_0 bv} \right)^2 = n_e \frac{Ze^4}{2\pi\epsilon_0^2 v} \log(b_{\text{max}}/b_{\text{min}})$$

The cumulative effects of the small collisions build up and are significant when the overall $\langle \Delta p^2 \rangle \approx m^2 n^2$

Setting $v = \bar{v} \approx \sqrt{kT_e/m_e}$ and expressing everything in terms of $n_e$ and $T_e$ we get a mean free path $\lambda_{ei} = \tau_{\text{coll}} \bar{v} = \bar{v}/\nu_{ei}$

$$\lambda_{ei} \approx \frac{m^2 \pi^2 \epsilon_0^2}{n_e Ze^4 \log \Lambda_e} \approx \frac{k^2 T_e \epsilon_0^2}{Z n_e e^4 \log \Lambda_e} = \frac{N_D \lambda_D}{Z \log \Lambda_e}$$

where $\log \Lambda_e = \log(b_{\text{max}}/b_{\text{min}})$.

The largest impact parameter is $b_{\text{max}} \approx \lambda_D$ since the ions are screened beyond this distance.

The largest amount of momentum that can be transferred is $\Delta p \approx m_e \bar{v}$, so $b_{\text{min}} = \frac{Ze^2 m_e}{4\pi \epsilon_0 \bar{v}^2} \approx \frac{Ze^2 m_e}{\epsilon_0 kT} = \frac{Z}{n_e \lambda_D^2}$.

The logarithm $\log \Lambda_e$ is therefore simply $\log(N_D/Z)$, showing how all aspects of the collision process involve collective effects.

$\nu_{ei} \approx \nu_{ei}$, but ions are heavier so $\frac{v_{\text{ii}}}{\nu_{\text{ee}}} \approx \left( \frac{T_e}{\bar{T}_i} \right)^{3/2} \left( \frac{m_e}{m_i} \right)^{1/2}$

A plasma is a highly conducting fluid. In a stationary medium we have the constitutive relation (Ohm’s law) $j = \sigma E$.

In a (non-relativistic) medium this becomes $j = \sigma(E + v \times B)$. If the medium is highly conducting and we let $\sigma \to \infty$, so $E = -v \times B$.

Taken together with the induction equation $\nabla \times E = -\frac{\partial B}{\partial t}$, this gives $\frac{\partial B}{\partial t} = \nabla \times (v \times B)$.

One can show that this means that the magnetic field lines are “frozen in” and move with the fluid velocity.

In the force equation of fluid dynamics we have to add a term $j \times B$. For fluid motions we ignore the displacement current and set $\nabla \times B = \mu_0 J$ to get

$$\rho \frac{Dv}{Dt} = -\nabla P + \rho g - \frac{1}{\mu_0} (B \times (\nabla \times B))$$

We can write $\left[ \frac{1}{\mu_0} (B \times (\nabla \times B)) \right]_i = \nabla j P_{ij}^\text{mag}$ where

$$P_{ij}^\text{mag} = \left( \frac{B^2 \delta_{ij}}{2\mu_0} - \frac{B_i B_j}{\mu_0} \right) \text{ (used } \nabla \cdot B = 0)$$

There is an additional anisotropic magnetic pressure, which has a magnitude $\frac{B^2}{2\mu_0}$. Magnetic field provides a pressure force perpendicular to the field, but a tension in the direction of the field lines.

The tension leads to a new form of magnetohydrodynamic wave — Alfvén waves, which are like waves on magnetic strings.
**Properties of Magnetic Pressure**

- Magnetic pressure: \( P_{ij}^{\text{mag}} = \left( \frac{B^2 \delta_{ij}}{2 \mu_0} - \frac{B_i B_j}{\mu_0} \right) \).

- Take \( B = (0, 0, B) \), so that
  \[
P_{ij} = \begin{pmatrix}
    \frac{B^2}{2 \mu_0} & 0 & 0 \\
    0 & \frac{B^2}{2 \mu_0} & 0 \\
    0 & 0 & -\frac{B^2}{2 \mu_0}
  \end{pmatrix}
\]

- Magnetic pressure is positive (i.e. pushes) perpendicular to \( B \) but has tension parallel to \( B \).

- Magnetic pressure produces waves, just like sound waves: \( \frac{\omega}{k} = \sqrt{\frac{\gamma P}{\rho}} \) (\( \gamma \) is adiabatic index).

- Parallel to \( B \): (vertical stretch) \( B \) stays the same. \( P \propto B^2 \propto V^0 \) \( \Rightarrow \gamma_\parallel = 0 \).

- Perpendicular to \( B \): (horizontal stretch) \( B \propto V^{-1} \). \( P \propto B^2 \propto V^{-2} \) \( \Rightarrow \gamma_\perp = 2 \).

**Types of MagnetoHydroDynamic Wave**

- (1) \( k \) parallel to \( B \)
  - (A) \( v \) parallel to \( k \)
  - (B) \( v \) perpendicular to \( k \)

  \[ \frac{\omega}{k} = v_b = \sqrt{\frac{\gamma P}{\rho}} \]
  \[ v_a = \sqrt{\frac{B^2}{\mu_0 \rho}} \]

  Sound wave (x1)
  
  Alfven waves (x2)

- (2) \( k \) perpendicular to \( B \)
  - (A) \( v \) parallel to \( k \)
  - (B) \( v \) perpendicular to \( k \)

  \[ v_m = \sqrt{v_b^2 + v_a^2} \]

  Magnetosound (x2)
  
  No restoring force (x4)

- For arbitrary directions of \( k \) and \( v \) the magnetohydrodynamic modes are mixtures of the above.
**General Features of Turbulence**

- The spatial power spectrum $E(k)$ of the velocity distribution in hydrodynamical turbulence shows the general features of all types of turbulence.

1. Energy generation at some spatial scale ($k$-number).
2. Energy flow from this region due to nonlinear effects, through a region with a stationary spectrum ($E(k) \propto k^{-5/3}$).
3. A region where energy is dissipated in the form of heat.

- In hydrodynamical turbulence energy is generated on large scales and is dissipated by viscosity at the scale of the mean free path $k \sim 1/\lambda$.

- In collisionless plasma turbulence the eventual dissipation will be at low $k$-number (again at $k \sim 1/\lambda$), so the energy flows the other way.

**Summary of Important Plasma Parameters**

- **Plasma frequency** $\omega_{pe}$ (and $\omega_{pi}$) = \( \left( \frac{n_e e^2}{\epsilon_0 m_e} \right)^{1/2} \)  
  Fundamental frequency of plasma oscillations.  
  Determines electromagnetic properties; interaction of particles with plasma waves; acceleration of particles.

- **Debye length** $\lambda_{De}$ (and $\lambda_{Di}$)  
  $\lambda_D = \frac{v_T}{\omega_p} = \left( \frac{\epsilon_0 k T_e}{n_e e^2} \right)^{1/2}$
  Debye number $N_{De} = \frac{4}{3} \pi n_e \lambda_{De}^3$
  Determine importance of collective effects; screening; Landau damping of plasma waves.
  Debye number is the figure of merit for a plasma.

- **Collision frequencies** $\nu_{ee}, \nu_{ei}, \nu_{ii}$  
  $\nu_{ee} = \Lambda_e \frac{\omega_{pe}}{N_{De}}$
  Ions are heavier so $\frac{\nu_{ii}}{\nu_{ee}} \approx \left( \frac{T_{ei}}{T_i} \right)^{3/2} \left( \frac{m_e}{m_i} \right)^{1/2}$
  Determines conductivities; viscosity; damping of non-resonant waves; thermal equalisation time. ($\Lambda_e \approx \log(N_{De})$)

- **Gyro frequency** $\omega_{ge}$ (and $\omega_{gi}$)  
  $\omega_{ge} = \frac{eB}{m_e}$
  Usually $\ll \omega_{pe}$. Determines polarisation of electromagnetic waves; anisotropy; mobility of particles in collisionless plasma.