Part IB Advanced Physics Answers to Electromagnetism Questions

These worked answers to the Electromagnetism problems are offered to students and supervisors as a guide, and I hope they contain some generally useful hints. In some cases, I have indulged in a little research that goes beyond the stated problem. Where possible I have included illustrative diagrams, and I have more planned for a later edition.

Health Warning

These solutions must be used with caution.

- The worked answers will be useless to you unless you have already made a very serious attempt to solve the problem yourself and have discussed it with your supervisor. If you consult the solutions earlier, they will just act as 'spoilers'.
- You must remember that there is often more than one way to solve a physics problem. For example, Q1 can be done by equally well by evaluating the potential, or by finding the electric field directly. I have sometimes suggested alternatives, but your supervisors will know many others.
- The most difficult part of a physics problem is knowing where to start. It is therefore quite possible that I taken too much for granted at the beginning of some problems. If so, please let me know.
- I have given the numerical answers to several more decimal places than is really justified, so that you can check your calculations (and mine) carefully. I have used values of the physical constants taken from the formula book that you will use in the Examination.

Please let me me know about any errors, typos and suggestions for improvement.

Steve Gull
`email: steve@mrao.cam.ac.uk`
`http://www.mrao.cam.ac.uk/~steve/electromagnetism`

Consider the E_x component

$$E_x = -\frac{\partial V}{\partial x} = \frac{q}{4\pi\epsilon_0}\left(\frac{x}{(x^2+y^2+(z-a/2)^2)^{3/2}} - \frac{x}{(x^2+y^2+(z+a/2)^2)^{3/2}}\right) . \tag{6}$$

Define $r^2 \equiv x^2+y^2+z^2$ and expand the denominators for $a \ll r$

$$\frac{1}{(x^2+y^2+(z\pm a/2)^2)^{3/2}} \approx \frac{1}{r^3}\left(1 \mp 3az/(2r^2)+\ldots\right) , \tag{7}$$

so that

$$E_x \approx \frac{3qaxz}{4\pi\epsilon_0 r^5} , \quad E_y \approx \frac{3qayz}{4\pi\epsilon_0 r^5} . \tag{8}$$

The E_z component is

$$E_z = \frac{q}{4\pi\epsilon_0}\left(\frac{z-a/2}{(x^2+y^2+(z-a/2)^2)^{3/2}} - \frac{z+a/2}{(x^2+y^2+(z+a/2)^2)^{3/2}}\right) , \tag{9}$$

which generates another term $-qa/(4\pi\epsilon_0 r^3)$. Putting these together, we get

$$E_z \approx \frac{qa(3z^2-r^2)}{4\pi\epsilon_0 r^5} . \tag{10}$$

- Q3. Consider the field due to dipole 1 at the position of dipole 2. Recall the formula for the field of a dipole parallel to and perpendicular to its axis. The parallel and perpendicular components of $\boldsymbol{E}$ are

$$E_{||} = \frac{2p_1\cos\theta_1}{4\pi\epsilon_0 d^3} , \quad E_{\perp} = \frac{-p_1\sin\theta_1}{4\pi\epsilon_0 d^3} . \tag{11}$$

Now consider the potential energy

$$U = -\mathbf{p}_2 \cdot \boldsymbol{E} \;=\; -\frac{p_1 p_2}{4\pi\epsilon_0 d^3}\;(2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) , \tag{12}$$

That's fine, but the answer needs the identity

$$3\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2) = 4\cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2 . \tag{13}$$

A plot of the potential surface (Figure 2) shows a minimum at $(0,0)$, a maximum at $(0,\pi)$ and saddle-points at $(\pi/2, \pm\pi/2)$. To investigate the stability properly we have consider all possible changes in θ_1 and θ_2, by looking at *all* the second derivatives. This is easier if we use the coordinates $\theta_1 \pm \theta_2$, since the matrix of second derivatives is then diagonal, but it is instructive to do it using θ_1, θ_2 anyway.

The formal procedure for doing this is not part of the electromagnetism course (links to dynamics course). For each of the points in turn:

  - find U;

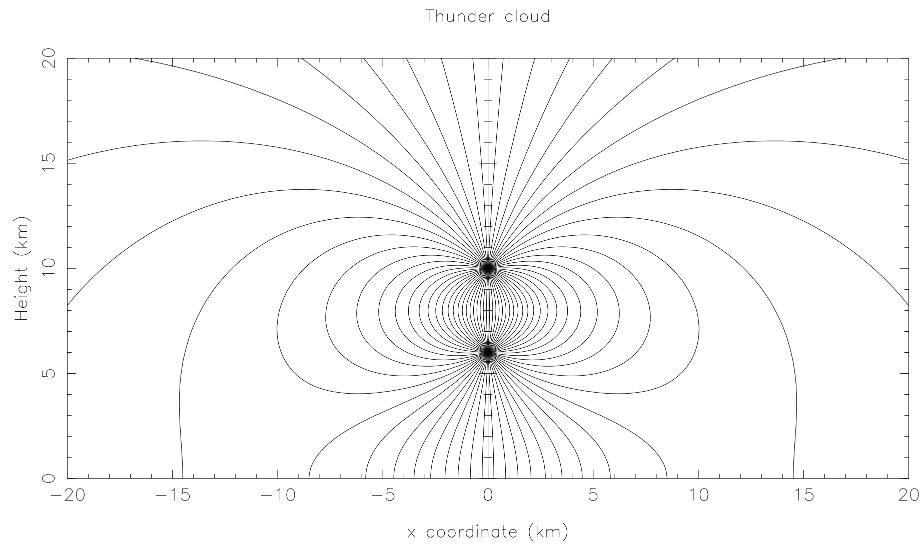

Figure 3: Field lines for the thundercloud of Q4. The reversal of the field at 11 km marks the boundary between field lines ending on the two different image charges.

- Q4. Electric field due to single charge Q at height z_1 and its image is

$$E_z = -\frac{Q}{4\pi\epsilon_0}\frac{2z_1}{(x^2+y^2+z_1^2)^{3/2}} . \tag{15}$$

The total is therefore

$$E_z = \frac{Q}{4\pi\epsilon_0}\left(\frac{2z_2}{(x^2+y^2+z_2^2)^{3/2}} - \frac{2z_1}{(x^2+y^2+z_1^2)^{3/2}}\right) . \tag{16}$$

Putting $z_1 = 10$ km, $z_2 = 6$ km, $Q = 40$ C gives $E_z = +12,782$ V m^{-1}, i.e. upwards. The field lines are plotted in Figure 3.

The next part asks us to solve

$$\frac{z_1}{(z_1^2+r^2)^{3/2}} = \frac{z_2}{(z_2^2+r^2)^{3/2}} \tag{17}$$

for $r^2 = x^2 + y^2$. Taking the two-thirds power of (17) and rearranging gives

$$r^2 = \frac{z_1^2 z_2^{2/3} - z_2^2 z_1^{2/3}}{z_1^{2/3} - z_2^{2/3}} . \tag{18}$$

For $z_1 = 10$ km, $z_2 = 6$ km this gives $r = 11.034$ km.

- Q5. The image charges have magnitude $\pm qa/d$ and are $\pm a^2/d$ from the centre of the sphere, making a dipole moment of $p = 2qa^3/d^2$. The field of a dipole at distance d along the axis is $2p/(4\pi\epsilon_0 d^3)$, hence the additional force (attractive) is $4q^2a^3/(4\pi\epsilon_0 d^5)$. The original force was $q^2/(16\pi\epsilon_0 d^2)$, hence result.

- Q6. Geometry: source at $(x,0)$; $-q$ image at $(-x,0)$ $-q$ images at $(x/2,\pm\sqrt{3}x/2)$; $+q$ images at $(-x/2,\pm\sqrt{3}x/2)$. Force is sum of 5 terms:

$$-2\times\frac{q^2}{4\pi\epsilon_0x^2}\times\frac{1}{2}\times\frac{1}{1^3}\ ;\quad +2\times\frac{q^2}{4\pi\epsilon_0x^2}\times\frac{3}{2}\times\frac{1}{\sqrt{3}^3}\ ;\quad -\frac{q^2}{4\pi\epsilon_0x^2}\times\frac{1}{2^2}\ ; \tag{19}$$

Hence result

$$F=-\frac{q^2}{4\pi\epsilon_0x^2}\left(\frac{5}{4}-\frac{1}{\sqrt{3}}\right)\equiv-\frac{A}{x^2}\ . \tag{20}$$

The orbit is an extreme ellipse with semi-major axis $a/2$. Kepler's law (dynamics course) gives periodic time $T=2\pi\sqrt{m(a/2)^3/A}$. We need 1/2 of this, i.e. $\pi\sqrt{ma^3/8A}$.
Alternatively, multiply the force equation by $\dot{x}$ and integrate to get the energy equation

$$\tfrac{1}{2}m\dot{x}^2=A\left(\frac{1}{x}-\frac{1}{a}\right)\ . \tag{21}$$

Separate this and complete the square to get

$$\int\frac{xdx}{\sqrt{(a/2)^2-(x-a/2)^2}}=\int\sqrt{\frac{2A}{ma}}\,dt \tag{22}$$

The substitution $x-a/2=(a/2)\cos\theta$ gives

$$\int(a/2)(1+\cos\theta)d\theta=\int\sqrt{\frac{2A}{ma}}\,dt\ . \tag{23}$$

This gives the 'cycloid' solution

$$\theta+\sin\theta=\sqrt{\frac{8A}{ma^3}}t\ . \tag{24}$$

The appropriate limits are $0<\theta<\pi$, giving the same result as before, but a lot more detail about the orbit.

- Q7. Gauss' theorem implies that the electric field is

$$E=\frac{Q}{4\pi\epsilon_0r^2}\quad(r>a);\quad E=\frac{Qr}{4\pi\epsilon_0a^3}\quad(r<a)\ . \tag{25}$$

Integrating, we get the potential

$$V=\frac{Q}{4\pi\epsilon_0r}\quad(r>a);\quad V=\frac{Q(3a^2-r^2)}{8\pi\epsilon_0a^3}\quad(r<a)\ . \tag{26}$$

The constant of integration for $r<a$ is needed to make the potential continuous at $r=a$. We now form $U=\frac{1}{2}\int d\tau\rho V$, with $\rho=3Q/(4\pi a^3)$:

$$U=\frac{3Q^2}{16\pi\epsilon_0a^6}\int_0^a dr\ r^2(3a^2-r^2)=\frac{3Q^2}{20\pi\epsilon_0a}\ . \tag{27}$$

Solving $U=m_ec^2$ for a yields 1.69076×10^{-15} m.

- Q8. The potential of a conducting sphere of radius r carrying a charge q is $V = q/(4\pi\epsilon_0 r)$ and its potential energy $U = qV/2$. The total charge of the system is conserved, so

$$Q = 4\pi\epsilon_0 \left(V_1 r_1 + V_2 r_2\right) = 4\pi\epsilon_0 V \left(r_1 + r_2\right) , \tag{28}$$

where V is the final potential of both spheres. The energy dissipated is thus

$$\Delta U = 2\pi\epsilon_0 \left(V_1^2 r_1 + V_2^2 r_2 - V^2 \left(r_1 + r_2\right)\right) , \tag{29}$$

This simplifies to

$$\Delta U = 2\pi\epsilon_0 \frac{(V_1 - V_2)^2 r_1 r_2}{r_1 + r_2} \tag{30}$$

and evaluates to 2.8164×10^{-6} J.

If the wire is not sufficiently resistive, the system oscillates, and the energy will eventually be radiated away.

- Q9. Consider constant charge Q on plates area A and distance x apart. The field is $Q/(A\epsilon_0)$ and the potential difference is $V = Qx/(A\epsilon_0)$. The potential energy is therefore

$$U = \frac{Q^2 x}{2\epsilon_0 A} . \tag{31}$$

The force between the plates can be evaluated as $F = -dU/dx = -Q^2/(2\epsilon_0 A)$. The force per unit area on a surface charge $\sigma = Q/A$ is therefore $\sigma^2/2\epsilon_0$ outwards.

Now do this a constant voltage, when the energy stored is

$$U = \frac{V^2 A\epsilon_0}{2x} . \tag{32}$$

This term by itself would give the opposite sign of the force, but we need to consider the work done by an external battery needed to keep the potential constant as we move the plates apart. This is $dW = dQV = V^2 A\epsilon_0 dx/x^2$, which is twice as large as the dU term and has the opposite sign. Hence we still get $p = \sigma^2/2\epsilon_0$ outwards.

- Q10. The electrostatic force between the plates a distance x apart is $V^2\pi r^2\epsilon_0/(2x^2)$. Consider force balance $k(a-b) = V^2 A\epsilon_0/b^2$. Then displace the upper disc slightly by dx, so new upward force is

$$-kdx + 2V^2 A\epsilon_0 dx/b^3 = kdx(2a - 3b)/b . \tag{33}$$

This is a restoring force only if $b < 2a/3$.

Here is an alternative way, which considers the weight of the upper disc as well, and does it using an effective potential (as in the dynamics course). The spring constant k must satisfy $k(x_0 - a) = mg$, where m is the mass of the upper disc and x_0 the natural length of the spring. The force is attractive so the stored electrostatic energy is *negative*, and the effective potential is

$$V_{\text{eff}}(x) = mgx + k(x - x_0)^2/2 - V^2\pi r^2\epsilon_0/(2x) . \tag{34}$$

Differentiating to find the new equilibrium gives

$$0 = \frac{dV_{\text{eff}}}{dx} = mg\frac{x-a}{x_0-a} + \frac{V^2\pi r^2\epsilon_0}{x^2} \ . \tag{35}$$

This new equilibrium at $x = b$ will be stable if the second derivative of the potential is positive.

$$\frac{d^2V_{\text{eff}}}{dx^2} = mg - \frac{2V^2\pi r^2\epsilon_0}{x^4} = \frac{mg}{x_0-a}\left(1 - \frac{2(a-x)}{x}\right) \ . \tag{36}$$

This evaluates to $mg(3b-2a)/(b(x_0-a))$. So the equilibrium is stable if $b > 2a/3$.

- Q11. The radius c of the inner dielectric evaluates nicely to 3 mm. The electric fields are

$$D = \frac{Q}{2\pi r}\ (a < r < b);\ \ E_1 = \frac{Q}{2\pi\epsilon_0\epsilon_1 r}\ (a < r < c);\ \ E_2 = \frac{Q}{2\pi\epsilon_0\epsilon_2 r}\ (c < r < b). \tag{37}$$

The total potential difference is

$$V = -\int_a^c dr\ E_1 - \int_c^b dr\ E_2, \tag{38}$$

so that the total capacity $C = Q/V$ is

$$C = \frac{2\pi\epsilon_0}{\log(c/a)/\epsilon_1 + \log(b/c)/\epsilon_2} \ . \tag{39}$$

The numerical value is 110.787 pF.

- Q12. A diagram would probably help here. The field E_0 is first decomposed into its components $E_{||}$ and $E_\perp$ relative to the slab. Here $E_{||} = E_\perp = E_0/\sqrt{2}$. The field in the slab $E'_{||} = E_{||}$ inside the slab, whereas $D'_\perp = D_\perp$ so that $\epsilon E'_\perp = E_\perp$. Resultant field in slab is $E_0(1/\sqrt{2}, 1/\sqrt{8})$, using $\epsilon = 2$.

Now resolve this relative to the cavity. The perpendicular component is $E_0(1/2 + 1/4)$ and the parallel component is $E_0(1/2 - 1/4)$, which become $3E_0/2$ and $E_0/4$ inside the cavity. Result is $E_1/E_0 = \sqrt{37}/4 = 1.5207$.

- Q13. We need the charge conservation equation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0 \tag{40}$$

and the constitutive relation for the conductivity

$$\boldsymbol{J} = \sigma\boldsymbol{E} \ . \tag{41}$$

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so that we have

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon\epsilon_0}\rho = 0 \ , \tag{43}$$

which has the general solution

$$\rho(\mathbf{x}, t) = \rho(\mathbf{x}, 0) \exp(-\sigma t/\epsilon\epsilon_0) \ . \tag{44}$$

- Q14. Place the charge q at $(a, 0, 0)$. In the vacuum half, we use an image charge q' at $(-a, 0, 0)$. In the dielectric, we use a charge $q''\epsilon$ at $(a, 0, 0)$. We match $E_{||}$ (or V) and $D_{\perp}$ at the plane $x = 0$. In detail, write the potential as

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{(x-a)^2+y^2+z^2)}} + \frac{q'}{\sqrt{(x+a)^2+y^2+z^2)}}\right) \qquad (x > 0) \ ; \\ V &= \frac{1}{4\pi\epsilon_0}\frac{q''}{\sqrt{(x-a)^2+y^2+z^2)}} \qquad (x < 0) \ . \end{aligned} \tag{45}$$

It suffices to match at $x = z = 0$; the E_y component gives

$$q + q' = q'' \tag{46}$$

and the D_x equation is

$$q - q' = \epsilon q'' \ . \tag{47}$$

This has solution

$$q' = \frac{q(1-\epsilon)}{(1+\epsilon)} \ , \quad q'' = \frac{2q}{(1+\epsilon)} \tag{48}$$

The image charge q' then gives an attractive force

$$F = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{2d}\right)^2 \frac{\epsilon - 1}{\epsilon + 1}. \tag{49}$$

- Q15. Dielectric sphere of radius a in external field E_0. Write the potential $V(r, \theta)$

$$V = \left(-E_0 r + \frac{p}{4\pi\epsilon_0 r^2}\right)\cos\theta \ \ (r > a); \quad V = -E_1 r\cos\theta \ \ (r < a). \tag{50}$$

Match V (or E_θ) and D_r at $r = a$:

$$\begin{aligned} E_1 &= E_0 - \frac{p}{4\pi\epsilon_0 a^3} \ ; \\ \epsilon E_1 &= E_0 + 2\frac{p}{4\pi\epsilon_0 a^3} \ . \end{aligned} \tag{51}$$

Eliminating E_1 gives the polarisation density

$$P = p/(4\pi a^3/3) = 3\epsilon_0 E_0 \frac{\epsilon - 1}{\epsilon + 2} \ . \tag{52}$$

Clausius-Mossotti argument says that the local field $E_{\rm loc}$ producing the polarisation is

$$E_{\rm loc} = E_0 + \frac{P}{3\epsilon_0} \ . \tag{53}$$

Hence the polarisation density produced by the fraction f of dielectric spheres is

$$P = 3\epsilon_0 E_{\rm loc} \frac{\epsilon - 1}{\epsilon + 2} f \ . \tag{54}$$

The macroscopic polarisation P is related to the bulk dielectric constant and the applied field E_0 by $\epsilon_{\rm m}$ by $P = (\epsilon_{\rm m} - 1)\epsilon_0 E_0$, so, by combining P from (54) with $E_{\rm loc}$ from (53), we obtain

$$(\epsilon_{\rm m} - 1)\epsilon_0 E_0 = 3\epsilon_0 \left(E_0 + \frac{(\epsilon_{\rm m} - 1)\epsilon_0 E_0}{3\epsilon_0} \right) \frac{\epsilon - 1}{\epsilon + 2} f \ . \tag{55}$$

This rearranges easily to give the relation

$$\frac{\epsilon_{\rm m} - 1}{\epsilon_{\rm m} + 2} = \frac{\epsilon - 1}{\epsilon + 2} f \ , \tag{56}$$

which can be solved for $\epsilon_{\rm m}$

$$\epsilon_{\rm m} = \frac{\epsilon + 2 + 2f(\epsilon - 1)}{\epsilon + 2 - f(\epsilon - 1)} \ . \tag{57}$$

- Q16. Essentials of Clausius-Mossotti argument.
  - Molecules have intrinsic polarisability α that relates their dipole moment $\mathbf{p}$ to the local field by $\mathbf{p} = \alpha E_{\rm loc}$.
  - The macroscopic polarisation density $\boldsymbol{P} = N\mathbf{p}$ is related to macroscopic field $\boldsymbol{P} = (\epsilon_{\rm r} - 1)\epsilon_0 \boldsymbol{E}$.
  - The macroscopic field contains a contribution from every molecule, so cannot be equal to the field that polarises any given molecule. We have to remove the contribution from the molecule itself.
  - Split the local field into three parts:
    * the molecule of interest itself (this is what we are trying to remove);
    * the effect of molecules in a spherical region surrounding the molecule of interest, sufficiently close that we have to sum up their effects carefully;
    * the effect of the molecules outside this sphere, sufficiently far away so we can calulate their effect macroscopically.
  - Lorentz showed that the contribution of the surrounding sphere was zero (think carefully why that is plausible).
  - The field inside a spherical cavity in a uniformly polarised medium is $E_{\rm loc} = \boldsymbol{E} + \boldsymbol{P}/3\epsilon_0$ (several ways of showing ths).

– Rearrange this to get the C-M relation.

Clausius-Mossotti relation for atomic polarisability α is

$$\alpha = \frac{3\epsilon_0(\epsilon - 1)}{N(\epsilon + 2)} , \tag{58}$$

rather than the more simple-minded $\alpha = \epsilon_0(\epsilon - 1)/N$. The data given for N_2 imply the following.

	density (kg m^{-3})	ϵ_r	α (F m^2) (C-M)	α (F m^2) (simple)
N_2 gas	4.50	1.002155	1.9701×10^{-40}	1.9715×10^{-40}
N_2 liquid	808	1.43616	1.9402×10^{-40}	2.2222×10^{-40}

Clearly, the Clausius-Mossotti estimate is better (consistent to 1.5%, rather than 19.7%). The actual value of α deserves comment, being well within an order of magnitude of the simplest estimate $2 \times 4\pi\epsilon_0 a_0^3$, where $a_0 \approx 10^{-10}$ m is the size of the N atom.

- Q17. Field is $\propto r$ inside uniform current and $\propto 1/r$ outside. Treat this case as two

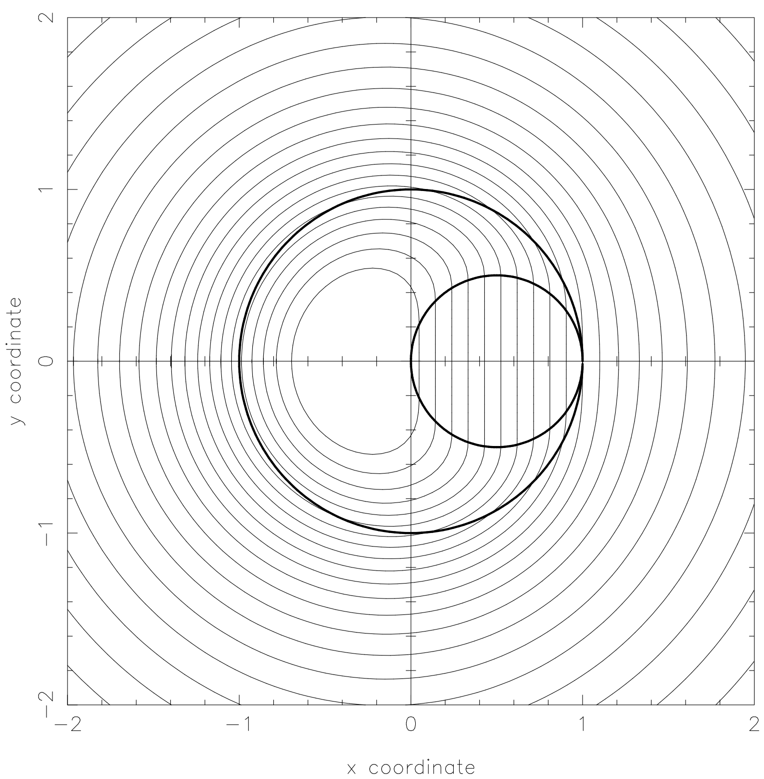

Figure 4: Field lines of **B** for the system of Q17.

displaced cylindrical currents of opposite sign. The field from the centre to the point A is actually constant and equal to half the value it would be at A for a complete wire, whereas the field at B is 5/6 of its value. The answer is $B : A = 5 : 3$.

- Q18. The field $B(r)$ inside the column is

$$B(r) = \frac{\mu_0 I r}{2\pi a^2} \,. \tag{59}$$

The force per unit volume on the current is $\boldsymbol{j}\times\mathbf{B}$, which is radially inward. This is balanced by the pressure gradient

$$\frac{dp}{dr} = -\frac{\mu_0 I^2 r}{2\pi^2 a^4} \,, \tag{60}$$

which integrates to give

$$p(r) = \frac{\mu_0 I^2 (a^2 - r^2)}{4\pi^2 a^4} + \text{ constant}, \tag{61}$$

which is the given answer if the constant of integration is zero. For $a = 5$ mm and $I = 100$ A, we get $p(a)$ =12.732 Pa.

However, the magnetic pressure (not part of the electromagnetism course) in the column is $B^2/(2\mu_0)$, so

$$p_{\text{mag}}(r) = \frac{\mu_0 I^2 r^2}{8\pi^2 a^4} \,, \tag{62}$$

which is *not* zero at the outside of the cylinder.

- Q19. The longitudinal magnetic field at x_0 along the axis of a solenoid with N turns per unit length is (Biot-Savart)

$$B_x = \frac{\mu_0 N I}{4\pi} \int_{x_0}^{\infty} dx \, \frac{2\pi a^2}{(a^2 + x_0^2)^{3/2}} \,. \tag{63}$$

Although we can evaluate this, we don't need to here, since we want the force on a dipole $\boldsymbol{F} = \boldsymbol{m}\cdot\boldsymbol{\nabla}\mathbf{B}$, which is just the value of the integrand of (63) at x_0. Hence

$$F = \frac{\mu_0 m N I a^2}{2(a^2 + x_0^2)^{3/2}} \,. \tag{64}$$

- Q20.

For the magnetised cylinder, the surface current is $M\hat{\boldsymbol{u}}_\phi$ over the cylindrical face and zero on the ends. There are magnetisation poles $\boldsymbol{\nabla}\cdot\boldsymbol{H} = \boldsymbol{\nabla}\cdot\boldsymbol{M} = \pm M$ on the ends. Points to note:

  - The field lines of $\mathbf{B}$ and $\boldsymbol{H}$ coincide outside the cylinder.
  - Lines of $\mathbf{B}$ are continuous, but the surface current makes sharp kinks in the field lines.
  - Lines of $\boldsymbol{H}$ are not kinked by the magnetisation currents, but they begin and end on the magnetisation poles.
  - In the middle of a long solenoid, the $\boldsymbol{H}$ field is low — $\boldsymbol{H}$ is an 'end effect'.

- Q22. For external field H_0, write the magnetic scalar potential $\boldsymbol{H} = -\boldsymbol{\nabla}\phi_{\rm m}$

$$\phi_{\rm m} = \left(-H_0 r + \frac{m}{4\pi r^2}\right)\cos\theta \quad (r > a). \tag{70}$$

The boundary condition is that $B_r = 0$ at $r = a$, so that the dipole moment is $m = -2\pi a^3 H_0$. This is indeed the same as the total moment of a sphere with uniform magnetisation $M = 3H_0/2$. The surface currents can be evaluated either as $\boldsymbol{M}\times\boldsymbol{n}$, where $\boldsymbol{n}$ is the outward normal to the sphere, or by looking at the discontinuity in B_θ at the surface. In either case we get

$$(J_{\rm s})_\phi = \frac{3}{2}H_0\sin\theta\ . \tag{71}$$

The longitudinal magnetic field at x_0 along the axis of a coil of radius b and with N turns is (Biot-Savart)

$$B_x = \frac{\mu_0 N I b^2}{2(b^2+x_0^2)^{3/2}}\ . \tag{72}$$

The force on a dipole $\boldsymbol{F} = \boldsymbol{m}\cdot\boldsymbol{\nabla}\mathbf{B}$, which is

$$F = \frac{3x_0\mu_0 N I b^2}{2(b^2+x_0^2)^{5/2}} \times \frac{4\pi a^3}{3} \times \frac{3}{2}\frac{NIb^2}{2(b^2+x_0^2)^{3/2}} = \frac{3\pi x_0 a^3 \mu_0 N^2 I^2 b^4}{2(b^2+x_0^2)^4}\ . \tag{73}$$

To support the sphere, we have $F = 4\pi\rho g a^3/3$, so that

$$I = \frac{2(b^2+x_0^2)^2}{3Nb^2}\sqrt{\frac{2\rho g}{\mu_0 x_0}}\ . \tag{74}$$

The steady (or r.m.s. alternating) current needed is 173.11 A.

Warning — the formula $\boldsymbol{F} = \boldsymbol{m}\cdot\boldsymbol{\nabla}\mathbf{B}$ is always correct. The alternative $\boldsymbol{F} = -\boldsymbol{\nabla}(\boldsymbol{m}\cdot\mathbf{B})$ assumes that the dipole moment is constant, not induced by the field. In the present case this alternative formula gives an answer which is a factor of two too large. However, it is correct (but no easier!) to evaluate the force as $\boldsymbol{F} = -\overset{*}{\boldsymbol{\nabla}}(\boldsymbol{m}\cdot\overset{*}{\mathbf{B}})$ (i.e. restrict the scope of the $\boldsymbol{\nabla}$ to act only on the $\mathbf{B}$). In suffix notation, you can use $F_i = -m_j\frac{\partial B_i}{\partial x_j}$ or $F_i = -m_j\frac{\partial B_j}{\partial x_i}$, but not $F_i = -\frac{\partial}{\partial x_i}(m_j B_j)$.

- Q23. Magnetic circuit has

$$\oint d\boldsymbol{l}\cdot\boldsymbol{H} = nI\ . \tag{75}$$

High permeability electromagnet means that all this appears 'shorted out' in the air gap, so the field there has $H_{\rm gap}\ s = NI = B_{\rm gap}\ s/\mu_0$. The force on the pole pieces per unit area is $B_{\rm gap}^2/2\mu_0$, so the force is

$$F = \frac{B_{\rm gap}^2 b^2}{2\mu_0} = \frac{n^2 I^2 b^2 \mu_0}{2s^2}\ . \tag{76}$$

The question is then the same as Q10. The force balance condition is

$$\frac{bt^3(s_0 - s)E}{8l^3} = \frac{n^2 I^2 b^2 \mu_0}{2s^2} , \tag{77}$$

and considering a small displacement ds we get a change in force tending to separate the blades

$$dF = -\frac{bt^3 E ds}{8l^3} + \frac{n^2 I^2 b^2 \mu_0 ds}{s^3} = -\frac{bt^3(2s_0 - 3s)E ds}{8l^3 s} , \tag{78}$$

which is unstable if $s < 2s_0/3$. Putting $s = 2s_0/3$ in (77) gives

$$I_{\mathrm{c}}^2 = \frac{s_0^3 t^3 E}{27 l^3 \mu_0 n^2 b} . \tag{79}$$

- Q24. Simple model given in lectures has perturbed electron orbit of radius r precessing at (Larmor) frequency $\omega_{\mathrm{L}} = eB/2m_{\mathrm{e}}$. The induced dipole moment is

$$dm = dI\ \pi r^2 = -\frac{e\omega_{\mathrm{L}}}{2\pi}\pi r^2 = -\frac{Be^2 r^2}{4m_{\mathrm{e}}} . \tag{80}$$

We also need the radius of the orbit, which we get from Bohr theory (revise):

$$J = n\hbar \text{ (quantised)}; \quad J^2 = \frac{e^2}{4\pi\epsilon_0} m_{\mathrm{e}} r \text{ (dynamics)}; \ \Rightarrow r = \frac{4\pi\epsilon_0 \hbar^2 n^2}{m_{\mathrm{e}} e^2} . \tag{81}$$

Final expression for the moment induced in the ground state ($n = 1$) is

$$dm = \frac{4\pi^2 \epsilon_0^2 \hbar^4 B}{m_{\mathrm{e}}^3 e^2} . \tag{82}$$

The Bohr magneton is $\mu_{\mathrm{B}} = e\hbar/2m_{\mathrm{e}}$ so ratio is

$$\frac{8\pi^2 B \epsilon_0^2 \hbar^3}{m_{\mathrm{e}}^2 e^3} = 2.1272 \times 10^{-6} . \tag{83}$$