Part IB Advanced Physics Answers to Electromagnetism Questions

This second set of worked answers to the Electromagnetism problems are offered to students and supervisors as a guide, and I hope they contain some generally useful hints. I have sometimes indulged in a little research that goes beyond the stated problem, and a few extra results are derived in an Appendix.

Introduction

- Although you have now had plenty of time to look at the questions, I must remind you that these worked answers will only be of value if you have already made a very serious attempt to solve the problem yourself and have discussed it with your supervisor.
- You must remember that there is often more than one way to solve a physics problem, so don't necessarily be alarmed if the worked answer is different from yours. If you think your answer is better, please tell me — I'd like to know about it.
- The most difficult part of a physics problem is knowing where to start. It is therefore quite possible that I might have taken too much for granted at the beginning of some problems. If so, please let me know.
- I have given the numerical answers to several more decimal places than is really justified, so that you can check your calculations (and mine) carefully. I have used values of the physical constants taken from the formula book that you will use in the Examination.
- Apologies in advance for typos and other mistakes in these solutions. For the 2001 edition I have corrected several major numerical errors in Q41 and Q42.
- Thanks are due to many people for helping me with these solutions, particularly Peter Duffet-Smith, Howard Hughes and Anthony Challinor.

Please let me me know about any errors, typos and suggestions for improvement.

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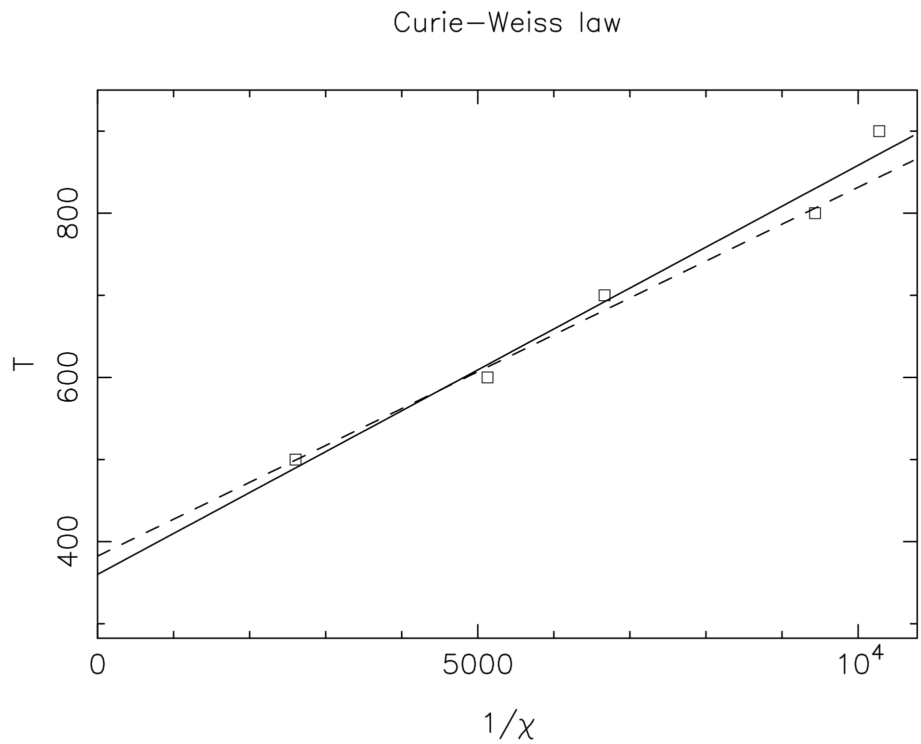

Figure 1: Temperature and inverse susceptibility values for Nickel. The intercept on the temperature axis is the Curie temperature and the slope is the Curie constant. The dotted line ignores the datapoint at $T = 900$ K.

- Q25. The Curie-Weiss law can be written as

$$T = T_\mathrm{c} + C(1/\chi) \ . \tag{1}$$

Figure 1 shows the plot of T against $1/\chi$ and a fitted line using the formulae you were given in the Experimental Methods course. The fitted parameters are $T_\mathrm{c} = 633 \pm 30$ K, $C = 0.50 \pm 0.04$ K, though it's not a great fit, and the values don't agree particularly well with the answers given in the problems book. The reason my answer does not agree with that given is that the datapoint at $T = 900$ K was ignored in the previous analysis. I have plotted both lines so you can judge for yourselves.

We need the number density n, which is 9.08×10^{28} m^{-3}, using an atomic mass of 58.7 and a density of 8.85 kg m^{-3}. From $C = \mu_0 m_0^2 n/3k$ we calculate $m_0 = 4.26 \times 10^{-24}$ J T^{-1}, which is 0.46 Bohr magnetons.

- Q26. The external current is zero, so Ampère's law gives

$$\oint d\boldsymbol{l} \cdot \boldsymbol{H} = 0 = H_\mathrm{i} L + H_\mathrm{g} l \ , \tag{2}$$

where l is the gap and L the length of the magnet. If the gap is small we have $B_\mathrm{g} = \mu_0 H_\mathrm{g} = B_\mathrm{i}$, giving

$$H_\mathrm{i} = -\frac{l}{L}\frac{B_\mathrm{g}}{\mu_0} \ . \tag{3}$$

The elliptical part of the hysteresis curve can be expressed as

$$\frac{H^2}{H_{\rm c}^2} + \frac{B^2}{B_{\rm r}^2} = 1 \ , \tag{4}$$

so that

$$\frac{l^2 B_{\rm i}^2}{L^2 \mu_0^2 H_{\rm c}^2} + \frac{B_{\rm i}^2}{B_{\rm r}^2} = 1 \ , \tag{5}$$

which can be rearranged to give

$$B_{\rm g} = \frac{B_{\rm r}}{\sqrt{1 + Al^2}} \tag{6}$$

where $A = B_{\rm r}^2/(L^2\mu_0^2 H_{\rm c}^2)$. Using $B_{\rm r} = 1.4$ T and $\mu_0 H_{\rm c} = 0.0017$ T from the diagram, together with $L = 0.6$ m, gives $A = 1.89 \times 10^6$ m^{-2}.

To reduce the field $\boldsymbol{B}$ to zero we have $NI = -LH_{\rm c}$, so with $N = 100$ we find $I = 8.12$ A.

- Q27. The pendulum equation in the absence of the B field can be written as

$$ml^2 \frac{d^2\theta}{dt^2} + mgl\theta = 0 \ , \tag{7}$$

for small angular displacements θ. The wire cuts the magnetic field, and the rate of change of flux is given by $Bl^2\dot{\theta}/2$. The electromotive force is then given by Faraday's law and is equal to the potential on the sphere

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\tfrac{1}{2} l^2 B \frac{d\theta}{dt} = V \ . \tag{8}$$

The capacity of the sphere is $C = 4\pi\epsilon_0 r$, so that it must carry a charge $Q = CV$. A current $I = \dot{Q}$ must therefore flow in the wire, so that there is a couple on the pendulum. The force on any part of the wire is IB per unit length, opposing the motion, and the total couple (integrating down the wire) is $G = \frac{1}{2} IBl^2$. Adding this couple to (7) and substituting for I gives a modified equation of motion

$$\left(ml^2 + \pi\epsilon_0 B^2 r l^4\right) \frac{d^2\theta}{dt^2} + mgl\theta = 0 \ . \tag{9}$$

Note that the effective inertia has changed, so that the period of the SHM is longer, but there is no damping. We are asked to calculate the fractional change in period $T = 2\pi/\omega$, which we can do since

$$\omega^2 = mgl \left(ml^2 + \pi\epsilon_0 B^2 r l^4\right)^{-1} \ . \tag{10}$$

Assuming the perturbation is small, we expand to get

$$\frac{\delta T}{T} = -\frac{\delta\omega}{\omega} = \frac{\pi\epsilon_0 B^2 r l^2}{2m} = \frac{3\epsilon_0 B^2 l^2}{8\rho r^2} \ , \tag{11}$$

where $m \equiv 4\pi\rho r^3/3$.

- Q28. Take the x-axis to be along the magnetic field and let the loop rotate at angular velocity $\omega = \dot{\phi}$ about the z-axis. The flux linkage at angle ϕ is

$$\Phi = B\pi a^2 \cos\phi \ . \tag{12}$$

There is an electromotive force

$$\mathcal{E} = -\dot{\Phi} = B\pi a^2 \omega \sin\phi \tag{13}$$

around the loop, which has total resistance $2\pi a/(\sigma A)$, where A is the cross-sectional area of the wire. The current flowing in the loop is thus

$$I = \tfrac{1}{2} B\sigma A a\omega \sin\phi \ . \tag{14}$$

The current produces a magnetic moment $|\boldsymbol{m}| = I\pi a^2$, which experiences a couple $\boldsymbol{G} = \boldsymbol{M}\times\boldsymbol{B}$ opposed to the rotation where

$$|\boldsymbol{G}| = \tfrac{1}{2}\pi B^2 \sigma A a^3 \dot{\phi} \sin^2\phi \ . \tag{15}$$

The moment of inertia of the loop is $\frac{1}{2}$ (mass)$\times a^2 = \rho\pi a^3 A$, so the equation of motion is

$$\rho\pi a^4 A\dot{\omega} = -\tfrac{1}{2}\pi B^2 a^3 A\omega \sin^2\phi \ . \tag{16}$$

The $\sin^2\phi$ term averages to 1/2 over a cycle, provided that the decay is not too rapid, so the equation for the rate of change of ω becomes

$$\dot{\omega} = -\frac{B^2\sigma}{4\rho}\ \omega \ , \tag{17}$$

showing that the loop slows with a time constant

$$\tau = \frac{4\rho}{B^2\sigma} \ . \tag{18}$$

The next part can be made as difficult as you like. The easy way is to say that the rotating sphere is just like a lot of wire loops, and plug the numbers into the formula above. With $B = 1.2\times10^{-7}$ T, $\rho = 2.7\times10^3$ kg m^{-3} and $\sigma = 3.8\times10^7\ \Omega^{-1}$ m^{-1}, we find $\tau = 1.97\times10^{10}$ s. That's 1 part in 626 per year.

This simple estimate has, however, ignored two effects:

1. The current distribution inside the sphere is not that same as in the wire (this doesn't change the answer — see Appendix).
2. The magnetic field is dragged around by the rotating conductor, so does not penetrate the sphere uniformly. For realistic values of the satellite radius and rotation rates, the field is confined to a shallow region on the surface, with a depth equal to the skin depth $\delta = \sqrt{2/(\sigma\mu_0\omega)}$. This reduces the torque on the satellite.

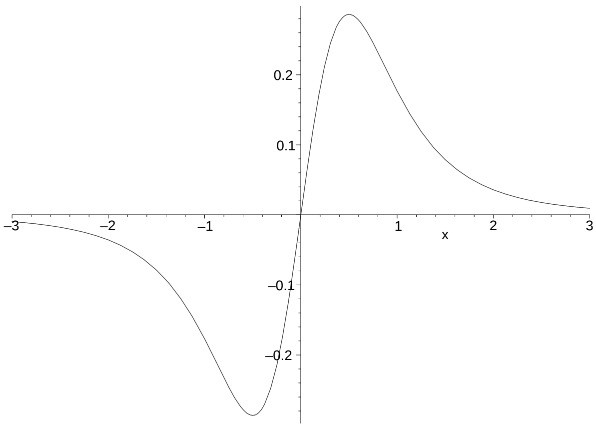

Figure 2: Signal induced in loop as a magnetic dipole travels through it. The x-axis is in units of r.

A detailed calculation of the torque on a rotating sphere is well beyond the level of the course, but I give two attempts in an Appendix for your amusement.

- Q29. The varying electric field between the plates generates an azimuthal magnetic field. From $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J} + \partial/\boldsymbol{D}\partial t$ and $\boldsymbol{J} = 0$ we get the integral form

$$\oint d\boldsymbol{l}\cdot\boldsymbol{B} = -i\omega\mu_0\epsilon_0 \int d\boldsymbol{S}\cdot\boldsymbol{E} \ . \tag{19}$$

Taking a loop of radius r in the interior, we have

$$2\pi r B(r) = -i\omega\mu_0\epsilon_0\pi r^2 V/d \ \Rightarrow |B(r)| = \frac{\omega V r}{2dc^2}, \tag{20}$$

where d is the separation of the plates and we have used $\mu_0\epsilon_0 = c^{-2}$. The field $B(r)$ is proportional to radius, just as for a uniform current. For $r > a$ we get $|B(r)| = \omega V a^2/(2dc^2 r)$, so that the magnetic field reaches a maximum at the edge of the plates.

It is instructive to calculate the magnetic field *outside* the capacitor generated by the currents bringing the charges to the plates $I = C\dot{V}$. At radial distance r from the wire, this field is $\mu_0 I/2\pi r$ so, using $C = \pi a^2\epsilon_0/d$, we see that the field generated by the currents is exactly equal to the exterior field generated by the displacement current.

For $V = 300$ V, $\omega = 10^4$ rad s^{-1}, $a = 0.3$ m and $d = 1$ mm we find a maximum field of 5×10^{-9} T, taking $c = 3 \times 10^8$ m s^{-1}.

The field of the Earth is 10^4 times higher than this value. The oscillating magnetic field in turn generates an induced electric field, and so on, but these secondary effects are negligible in this case.

- Q30.

The field at distance x on the axis of the larger coil (radius r) carrying a current I_1 is (Biot-Savart)

$$B(x) = \frac{\mu_0 I_1 r^2}{2(r^2 + x^2)^{3/2}} \ . \tag{21}$$

If the second coil is small ($r' \ll r$), the flux linked is just $B(x)\pi(r')^2$ so that the mutual inductance is

$$M = \frac{\mu_0 \pi (r')^2 r^2}{2(r^2 + x^2)^{3/2}} \ . \tag{22}$$

If the small coil now carries a current I_2, it is equivalent to a magnetic dipole of strength $m = I_2\pi(r')^2$. The flux linking the large coil is then

$$\Phi_1 = MI_2 = \frac{\mu_0 r^2 m}{2(r^2 + x^2)^{3/2}} \ . \tag{23}$$

Using Faraday's law, we find for the e.m.f. induced in the loop when the dipole moves at $\dot{x} = u$

$$\mathcal{E} = -\frac{\mu_0 r^2 m}{2}\frac{d}{dt}(r^2 + x^2)^{-3/2} = \frac{3\mu_0 r^2 m}{2(r^2 + x^2)^{5/2}} x\dot{x} = \frac{3\mu_0 r^2 m x u}{2(r^2 + x^2)^{5/2}} \ . \tag{24}$$

Discussion of feasibility should include:

- maximum signal/size of detector;
- noise in apparatus (Johnson etc.);
- use of superconductors;
- stability required.

- Q31. The reflected impedance of the load Z is $Z(n_1/n_2)^2$, determining the ratio $(n_1/n_2)^2 = 75/5$. We are also told that the impedance $j\omega L_1$ of the primary should be 150 Ω at 20 Hz. We calculate L_1 from Ampère's theorem

  $$2\pi r H = n_1 I \Rightarrow \Phi_1 = n_1 \mu\mu_0 A H = \mu\mu_0 A n_1^2 I/(2\pi r) = L_1 I \ . \tag{25}$$

  With $r = 0.05$ m, $\mu = 500$, $A = 0.003$ m^2, we get $n_1 > 466.031, n_2 = 115.165$.

  The input impedance may easily be calculated from scratch. You should find that the impedance of the voltage generator is in series with the effective impedance of the transformer, which in turn is made up from the primary inductance and the 'reflected' secondary impedance in parallel.

- Q32. This is another difficult question that has caused grief to generations of supervisors, never mind students. Bleaney and Bleaney give a pretty good discussion of this problem, and it's well worth a look.

  This problem is probably best done by using an energy method, but you have to be extremely careful to include the work done by the induced back e.m.f. on the currents maintaining the external field. Fortunately, there is a general argument which shows that, if the currents are constant, the work done against the magnetic forces is *the negative* of the change in the stored energy. Consider a system of currents I_i with linked fluxes Φ_i. Let the fluxes change as a result of small movements $\boldsymbol{r}_i$ under the action of forces $\boldsymbol{F}_i$. Then the total work done must be equal to the change in stored energy $W = \frac{1}{2}\sum_i I_i\Phi_i$. Using Faraday's law, the work done by the back e.m.f. is $\sum_i I_i \; d\Phi_i$. Thus we have

$$-\sum_i d\boldsymbol{r}\cdot\boldsymbol{F} + \sum_i I_i \; d\Phi_i = \tfrac{1}{2}\sum_i d(\Phi_i I_i) = \tfrac{1}{2}\sum_i I_i \; d\Phi_i \qquad (26)$$

because the currents are constant. Hence result.

To calculate the stored energy, again assuming that the external currents are held constant, we can use another handy theorem proved in the Appendix that the change in magnetic energy can be written as

$$\Delta W = \int d\tau \tfrac{1}{2}\mu_0 \boldsymbol{M}\cdot\boldsymbol{H}_0 \; , \qquad (27)$$

which involves the unperturbed $\boldsymbol{H}_0$ field, and the magnetisation $\boldsymbol{M}$. The advantage of this formula is that it only involves the region of space occupied by the paramagnetic body, and certainly shows that stored energy always increases when the body is introduced. The internal field of the cylinder is related to the applied field by

$$H_{\mathrm{i}} = \frac{1}{1+\chi/2} H_0 \; , \qquad (28)$$

so that

$$\Delta W = \tfrac{1}{2}\mu_0 H_0^2 A x \frac{\chi}{1+\chi/2} \; , \qquad (29)$$

where A is the cross-section area and x is the length inserted into the field. This ignores end effects. Using our earlier result we see that the force attracts the specimen into the field and, for a cross-section area of $\pi d^2/4$ with $d = 0.005$ m and a force of 0.015 g N in a field of 1 T, we get $\chi/(1+\chi/2) = 1.8829 \times 10^{-3}$.

In the Appendix I give an alternative direct force argument (together with the electrostatic version) that leads to the formula

$$\boldsymbol{F} = \oint d\boldsymbol{S} \; \tfrac{1}{2}\chi(1+\chi)\mu_0 |\boldsymbol{H}|^2 \; . \qquad (30)$$

If the paramagnetic rod is partially immersed in the field, there is an unbalanced contribution to the force from the surface at the end of the cylinder. The basic reason is that $\boldsymbol{J}\times\boldsymbol{B}$ force at the bottom of the cylinder (where the $\boldsymbol{B}$ field is essentially constant) is larger than the corresponding force at the part of the cylinder where the magnetic field reduces to zero (hence the factor of two). This shows clearly that the force is an end effect, but that makes it difficult to be sure that we are calculating the fields and forces accurately. The energy method allows us to calculate the force using the constant field in the middle of the cylinder.

- Q33. This is another infamous question; beware of many ancient misconceptions that fester in the darker recesses of Cambridge Colleges. First, the energy method. The stored energy of two inductances is (notation as in lectures) $\frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2$. If you increase the separation of the two solenoids whilst keeping the currents constant, the mutual inductance must fall, so the energy stored *goes down*. Nevertheless, the force is attractive, because of the work done by the back e.m.f. on the currents I_1 and I_2. We again use the theorem proved in Q32 that the total work done is the negative of the change in stored energy, provided that the external currents are held constant.

  To do the problem we need the change in mutual inductance when the solenoids are moved a distance dz apart. This is the same as taking ndz turns of one end of the solenoid, and transferring them to the other end where the flux linkage is small. So $dM = ndz\Phi/I$, where Φ is the magnetic flux emerging from one end of a long solenoid. A symmetry argument says that, when two solenoids are placed end to end, the field is the same as in the interior of a long solenoid $B = \mu_0 nI$. So the longitudinal field at the end of a single solenoid must be half this, and the flux emerging is $\Phi = \mu_0 nIA/2$, where A is the cross section area. Putting all these bits together yields an attractive force

  $$F = \tfrac{1}{2}\mu_0 n^2 I^2 A \; . \tag{31}$$

  For $A = \pi a^2$, $n = 2000$ m^{-1}, $a = 0.02$ m, $I = 1$ A, we get $F = 3.1583 \times 10^{-3}$ N.

  It's worth pointing out that, although we assumed the currents were held strictly constant in order to calculate the energy correctly, the force would be the same anyway.

  The alternative method calculates the force directly, as the flux emerging from one solenoid crosses the currents flowing in the other. Suppose that the perpendicular magnetic field at distance z from the end is $B_\perp$. The force on current element dz is attractive and equal to (current)$\times$(length)$\times B_\perp$

  $$dF = (nIdz) \times (2\pi a) \times B_\perp \; . \tag{32}$$

  The total force is then

  $$F = \int dz(2\pi a B_\perp)\; nI. \tag{33}$$

  But $\int dz(2\pi a B_\perp)$ is just the flux leaving the solenoid, which has to be equal to the flux $\Phi\frac{1}{2}\pi a^2 nI$ that entered from the other solenoid. Again we get

  $$F = \tfrac{1}{2}\mu_0 \pi a^2 n^2 I^2 \; . \tag{34}$$

- Q34. The transmitter will be constructed to emit preferentially into directions around the horizon. Suppose the effective solid angle of the transmitter is Ω and estimate $\Omega \approx \pi$ steradians. The Poynting vector at distance r has magnitude $|N| = P/(\Omega r^2)$, where P is the transmitted power. The electric field is then calculated from $|N| = |E|^2/Z_0$ to give $|E| \approx 35$ mV m^{-1}.

- Q35. The flux emitted per unit area of the Sun's surface is $|N| = \sigma T^4$. Using the values given in the question $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4} and $T = 6000$ K we get $|N| = 74$ MW m^{-2} just above the surface. The total solar luminosity is $L_\odot = 4\pi R_\odot^2 \sigma T^4$, so the Poynting flux at the Earth is $|N| = \sigma T^4 (R_\odot/R)^2$, where R is the Earth-Sun distance. The Sun subtends only half a degree at the Earth, so we can find the energy density U and electric field using the formulae developed for plane waves: $U = |N|/c$; $|N| = |E|^2/Z_0$. We find $|N| = 1.399$ W m^{-2}; $U = 4.667 \times 10^{-6}$ J m^{-3}; $|E| = 726$ V m^{-1} r.m.s.

  Returning to the surface of the Sun, we have to correct for the fact that the radiation is coming from 2π steradians, so there are waves coming from many different directions at once. They all add to the energy density, but their Poynting flux averages out. The net flux away from the Sun can be calculated by taking an average of the outward component $\cos\theta$ over solid angles from $0 < \theta < \pi/2$. We find

$$\frac{|N|}{Uc} = \frac{\int_0^{\pi/2} d\theta \ \sin\theta \cos\theta}{\int_0^{\pi/2} d\theta \ \sin\theta} = \frac{1}{2} \tag{35}$$

  Using $|N|$ as calculated earlier, we then find $U = 0.49$ J m^{-3}; $|E| = 235$ kV m^{-1} r.m.s.

- Q36. Forward waves:

$$\begin{aligned} E_y &= E_1 \cos(kx - \omega t) \ ; \qquad Z_0 H_z = E_1 \cos(kx - \omega t) \\ E_z &= E_2 \cos(kx - \omega t) \ ; \qquad Z_0 H_y = -E_2 \cos(kx - \omega t) \end{aligned} \tag{36}$$

  Backward waves:

$$\begin{aligned} E_y &= E_3 \cos(-kx - \omega t) \ ; \qquad Z_0 H_z = -E_3 \cos(-kx - \omega t) \\ E_z &= E_4 \cos(-kx - \omega t) \ ; \qquad Z_0 H_y = E_4 \cos(-kx - \omega t) \end{aligned} \tag{37}$$

  We are given the form of the wave at $x = 0$, and this yields the relations

$$\begin{aligned} E_y &= E_1 + E_3 = 3E_0 \ ; \quad Z_0 H_z = E_1 - E_3 = E_0 \\ &\Rightarrow E_1 = E_0 \ ; \quad E_3 = 2E_0 \\ E_z &= E_2 + E_4 = E_0 \ ; \quad Z_0 H_y = -E_2 + E_4 = -E_0 \\ &\Rightarrow E_2 = E_0 \ ; \quad E_4 = 0 \end{aligned} \tag{38}$$

  The forward wave is $E_1, E_2 = E_0$, and the resultant has $\boldsymbol{E}$ of amplitude $\sqrt{2}E_0$ plane polarised at $\pi/4$ to the y and z-axes. The backward wave has amplitude $2E_0$, with $\boldsymbol{E}$ polarised along the y-axis. The Poynting vector has net flux $-E_0^2/Z_0$ in the x-direction.

- Q37. The gravitational attraction at distance R due the Sun on a grain of radius r, density ρ is

$$F_{\rm g} = \frac{4\pi G M \rho r^3}{3R^2} \ . \tag{39}$$

The energy flux per unit area is $|\boldsymbol{N}| = L/(4\pi R^2)$, so that the force (assuming absorption) due to the radiation pressure on cross-sectional area πr^2 is

$$F_{\rm r} = \pi r^2 \frac{|N|}{c} = \frac{Lr^2}{4R^2c} \ . \tag{40}$$

Comparing $F_{\rm g}$ and $F_{\rm r}$, we see that the grains will be expelled if

$$r < \frac{3L}{16\pi GM\rho c} \ . \tag{41}$$

Using the values in the question gives $r = 597$ nm.

- Q38. The frequency used to communicate with the satellite must be greater than the plasma frequency $\omega_{\rm p} = (Ne^2/m\epsilon_0)^{1/2}$ for propagation through the ionosphere. For $N = 1.5 \times 10^{12}$ m^{-3}, this gives $\nu > \omega_{\rm p}/2\pi = 10.9966$ MHz. In practice the frequencies used are *very* much higher than this.

- Q39. The Lorentz force on the electron (charge $-e$) is $\boldsymbol{F} = -e(\boldsymbol{B} + \boldsymbol{v}\times\boldsymbol{B})$. An electromagnetic wave in free space has $c|\boldsymbol{B}| = |\boldsymbol{E}|$, so the magnetic field is negligible if the velocity $v \ll c$. Newton's law for an electron at the origin, oscillating in an applied transverse electric field $Ee^{-i\omega t}$, is

$$m_{\rm e}\ddot{\boldsymbol{x}} = -eE\exp(-i\omega t) \ , \tag{42}$$

which has steady-state solution

$$\boldsymbol{x} = \frac{eE}{m_{\rm e}\omega^2}\exp(-i\omega t) \ , \tag{43}$$

giveing an oscillating dipole moment $\boldsymbol{p} = -e\boldsymbol{x}$. Summing over all the electrons (number density n) we get a polarisation density

$$\boldsymbol{P} = \frac{-ne^2}{m_{\rm e}\epsilon_0\omega^2}\epsilon_0\boldsymbol{E} \ . \tag{44}$$

This shows that the dielectric constant of the plasma is $\epsilon = 1 - \omega_{\rm p}^2/\omega^2$, where $\omega_{\rm p}^2 = ne^2/m_{\rm e}\epsilon_0$.

The refractive index is $n = \sqrt{\epsilon}$, so the wave (phase) velocity is

$$\frac{\omega}{k} = \frac{c}{n} = \frac{c}{\sqrt{\left(1 - \omega_{\rm p}^2/\omega^2\right)}} \ . \tag{45}$$

Squaring this equation, we get the convenient form of the dispersion relation

$$\omega^2 = \omega_{\rm p}^2 + k^2c^2 \ . \tag{46}$$

Differentiating,

$$2\omega d\omega = 2c^2kdk \quad \Rightarrow \frac{\omega}{k}\frac{d\omega}{dk} = v_\phi v_{\rm g} = c^2 \ . \tag{47}$$

We have therefore shown that the group velocity can be written as

$$v_{\rm g} = c\left(1 - \frac{\omega_{\rm p}^2}{\omega^2}\right)^{1/2} . \tag{48}$$

For a pulsar at distance r the pulses will arrive after a time $r/v_{\rm g}$, the higher frequencies arriving first. For $\omega = 2\pi\nu \gg \omega_{\rm p}$ we can expand this, so that the difference of arrival times between frequencies ν_1 and ν_2 is

$$\Delta t \approx \frac{r\ \omega_{\rm p}^2}{8\pi^2 c}\left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2}\right) . \tag{49}$$

With $N = 3 \times 10^4\ {\rm m}^{-3}$ and $\Delta t = 4$ s, we find $r = 5.3 \times 10^{19}$ m.

The pulsar can be observed through the ionosphere provided $\nu > 11$ MHz (directly overhead), but it's awfully difficult until $\nu > 50$ MHz. We used $\nu = 81$ MHz to discover them.

- Q40. We start from Maxwell 4, using the constitutive relations $\boldsymbol{J} = \sigma\boldsymbol{E}$, $\boldsymbol{D} = \epsilon\epsilon_0\boldsymbol{E}$:

$$\boldsymbol{\nabla}\times\boldsymbol{H} = \sigma\boldsymbol{E} + \epsilon\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} . \tag{50}$$

For a plane wave proportional to $\exp i(kz - \omega t)$ this gives

$$-ikH_y = \sigma E_x - i\omega\epsilon\epsilon_0 E_x = -i\omega\epsilon_0(\epsilon + \frac{i\sigma}{\omega\epsilon_0})E_x . \tag{51}$$

This means that the effective dielectric constant is $\epsilon' = \epsilon + \dfrac{i\sigma}{\omega\epsilon_0}$, so that the wave impedance is

$$Z = \frac{E_x}{H_y} = \sqrt{\frac{\mu\mu_0}{\epsilon'\epsilon_0}} . \tag{52}$$

For a good conductor $\epsilon' \approx i\sigma/\omega\epsilon_0$, so

$$Z = \sqrt{\frac{\omega\mu\mu_0}{i\sigma}} = \pm\sqrt{\frac{\omega\mu\mu_0}{2\sigma}}(1 - i) \equiv \pm a(1 - i) , \tag{53}$$

as required.

The power reflection coefficient is (use $+Z$)

$$R = \frac{|Z - Z_0|^2}{|Z + Z_0|^2} = \frac{(Z_0 - a + ia)(Z_0 - a - ia)}{(Z_0 + a - ia)(1 + a + ia)} = \frac{Z_0^2 - 2a + 2a^2}{Z_0^2 + 2Z_0 a + 2a^2} . \tag{54}$$

If $a \ll Z_0$, we can expand to get

$$R \approx 1 - 2\sqrt{\frac{2\mu\epsilon_0\omega}{\sigma}} \tag{55}$$

and for a non-magnetic material such as Aluminium we can set $\mu \approx 1$. With $1 - R = 0.03$ at $\lambda = 5\ \mu$m, we get $\sigma = 2.965 \times 10^7\ \Omega^{-1}\ {\rm m}^{-1}$.

- Q41. Continuing the analysis from Q40, we note that the wave velocity is

$$\frac{\omega}{k} = \frac{c}{\sqrt{\epsilon' \mu}} \Rightarrow k = \sqrt{i\sigma\omega\mu\mu_0} = \pm\sqrt{\frac{\sigma\omega\mu\mu_0}{2}}(1+i) \equiv \pm\delta^{-1}(1+i) \ , \qquad (56)$$

where $\delta = \sqrt{2/(\sigma\omega\mu\mu_0)}$ is the skin depth and we have used $\epsilon_0\mu_0 = c^{-2}$.
The electric field of the wave propagating to $+z$ is thus

$$E_x = E_0 \cos(x/\delta - \omega t)\ e^{-z/\delta} \ . \qquad (57)$$

Recalling the wave impedance (53), we get the magnetic field

$$H_y = E_0\sqrt{\frac{\sigma}{2\omega\mu\mu_0}}(\cos(x/\delta - \omega t) + \sin(x/\delta - \omega t))\ e^{-z/\delta} \ . \qquad (58)$$

The magnetic field lags the electric field by $\pi/4$ in phase, so there is a net energy flow $\boldsymbol{N} = \boldsymbol{E}\times\boldsymbol{H}$ and

$$\langle N_z \rangle = \tfrac{1}{4}E_0^2\sigma\delta\ e^{-2z/\delta} \ . \qquad (59)$$

This falls off exponentially, satisfying

$$\frac{d\langle N_z\rangle}{dz} = \sigma\left\langle E_x^2\right\rangle \qquad (60)$$

as it should to account for Ohmic losses.

For sea water $\epsilon = 80, \mu \approx 1, \sigma = 4\ \Omega^{-1}\ \mathrm{m}^{-1}$, we check that $\epsilon \ll \sigma/(2\pi\nu\epsilon_0) \approx 4.5 \times 10^6$ at $\nu = 16$ kHz. The skin depth $\delta = 1.9894$ m. The allowed depth is $\delta \log 100 = 9.1617$—m.

- Q42. Part (a) follows the lecture notes, but uses only real numbers, calculating the total current I of the 'unwrapped' cylinder for $a \gg \delta$ as

$$I = 2\pi a \int_0^\infty dz\ J_x(z) = 2\pi J_0 a \int_0^\infty dz \cos(z/\delta - \omega t)\ e^{-z/\delta} \ , \qquad (61)$$

which can be done by parts (or complexification) to give

$$I = \pi a J_0 \delta(\cos(\omega t) + \sin(\omega t)) \ , \qquad (62)$$

so that

$$\left\langle I^2 \right\rangle = \pi^2 a^2 J_0^2 \delta^2 \ . \qquad (63)$$

The power dissipated per unit volume is J_x^2/σ, so the total is

$$P = 2\pi a\frac{J_0^2}{\sigma}\int_0^\infty dz \cos^2(z/\delta - \omega t)\ e^{-2z/\delta} = \pi a\frac{J_0^2\delta}{2\sigma}[1 + \tfrac{1}{2}\cos(2\omega t) + \tfrac{1}{2}\sin(2\omega t)] \ . \qquad (64)$$

This averages to $P = \pi a J_0^2 \delta / 2\sigma = I^2 R$ where R is the resistance. so that

$$R = \frac{1}{2\pi a\delta\ \sigma} \qquad (65)$$

per unit length, as required.

For part (b) we can use the answer to part (a) by noting that

$$|H_0| = |E_0|\sqrt{\frac{\sigma}{\omega\mu\mu_0}} \tag{66}$$

so that the energy dissipation $\langle\sigma E_x^2\rangle$ is given by ($\cos^2(z/\delta - \omega t)$ averages to $\frac{1}{2}$)

$$\left\langle\sigma E_x^2\right\rangle = \tfrac{1}{2}H_0^2\omega\mu\mu_0\, e^{-2z/\delta} . \tag{67}$$

Integrating this to find the total power dissipated per unit area yields

$$\int_0^\infty dz\, \left\langle\sigma E_x^2\right\rangle = \frac{H_0^2\delta}{4}\omega\mu\mu_0 = \frac{H_0^2}{2\delta\sigma} . \tag{68}$$

For part (c), we have already seen that

$$\langle N_z\rangle = \tfrac{1}{4}E_0^2\sigma\delta\, e^{-2z/\delta} , \tag{69}$$

so, using (66), we obtain the required result at $z = 0$

$$\langle N_z\rangle = \tfrac{1}{4}H_0^2\omega\mu\mu_0\delta = \frac{H_0^2}{2\sigma\delta} . \tag{70}$$

- Q43. Recall that the impedance of a coaxial cable is $Z = Z_0\log(b/a)/2\pi$, and that its capacity is $C = 2\pi\epsilon_0/\log(b/a)$ per unit length. The electric field inside the cable is $|E| = Q/(2\pi\epsilon_0 r)$, where Q is the charge per unit length, so the field has its maximum value at the inner radius $r = a$. We can therefore express the maximum field $E_{\rm max}$ in terms of the voltage $V = Q/C$ as

$$E_{\rm max} = \frac{V}{a\log(b/a)} = \frac{VZ_0}{2\pi aZ} . \tag{71}$$

The power flow is $P = V^2/Z$, and we must include a factor of 2 to get the r.m.s power

$$P = \frac{(2\pi aE_{\rm max})^2Z}{2Z_0^2} . \tag{72}$$

The breakdown voltage given in this question should read 10^5 V, not 105 V — apologies. We then get $P = 104.311$ W (using an accurate value for Z_0).

- Q44. Let the incident, reflected and transmitted waves have voltages $V_0\exp i(kz - \omega t)$, $V_0 r\exp i(-kz - \omega t)$, $V_0 t\exp i(kz - \omega t)$ respectively. Suppose the shunt resistance R is at $z = 0$, so the condition that the voltage is continuous is

$$1 + r = t . \tag{73}$$

The boundary condition for the current is that the difference in currents at z_- and z_+ must be equal to the current $V(z=0)/R$ flowing into the shunt resistor. Hence, remembering that the impedance is negative for the reflected wave, we have

$$\frac{1}{Z} - \frac{r}{Z} = \frac{t}{Z} + \frac{t}{R} \,. \tag{74}$$

Solving (73) and (74) for r, t gives

$$r = \frac{-Z}{2R+Z} \,; \quad t = \frac{2R}{2R+Z} \,. \tag{75}$$

With $\rho = 0.5\ \Omega$ m and $t = 1$ mm, we get $r = -0.27364$ (using an accurate value for Z_0).

Alternative: You can solve this one using the $r = (Z_2 - Z_1)/(Z_2 + Z_1)$ formula if you are careful, because the first part of the line just sees the shunt resistor R and the rest of the line Z as impedances in parallel. That is, you set $Z_1 = Z$ and

$$Z_2 = \frac{1}{1/Z + 1/R} = \frac{ZR}{Z+R} \Rightarrow r = \frac{-Z}{2R+Z} \tag{76}$$

as before.

- Q45. We need the formula for the input impedance of a terminated line developed in the lectures

$$\frac{Z_{\text{in}}}{Z} = \frac{Z_{\text{T}}\cos(ka) - iZ\sin(ka)}{Z\cos(ka) - iZ_{\text{T}}\sin(ka)} \,. \tag{77}$$

When the line is short-circuited $Z_{\text{T}} = 0$, so that $Z_{\text{in}} = Z_1 = -iZ\tan(ka)$; when the line is open-circuit $Z_{\text{T}} = \infty$, so that $Z_{\text{in}} = Z_2 = iZ\cot(ka)$. Hence $Z_1 Z_2 = Z^2$.

- Q46. We again need the formula for the input impedance of a terminated line (77) When the line is open-circuit $Z_{\text{T}} = \infty \Rightarrow Z = -i\cot(ka)$. The line is said to be resonant when its input impedance is zero, so that $ka = \pi/2$ or has a length that is an odd multiple of $\lambda/4$. We are told in the question that $a = \lambda/4 = 0.25$ m.

The capacitor has impedance $1/(-i\omega C)$, rather than $1/(i\omega C)$ that we were taught earlier, because we are now using the physicists' convention for waves $\exp i(kz - \omega t)$. This makes the input impedance

$$Z_{\text{in}} = \frac{i\cos(ka)/(\omega C) - iZ\sin(ka)}{Z\cos(ka) + \sin(ka)/(\omega C)} \tag{78}$$

For the same wavelength, the line now resonates when $a = 0.125$ m $= \lambda/8$, so that $ka = \pi/4$ and $\cos(ka) = \sin(ka) = 1/\sqrt{2}$. We therefore have $1/(\omega C) = Z$. For $C = 1$ pF and $\lambda = 1$ m, we calculate $Z =$ 530.884 Ω.

- Q47. The impedance is $Z = Z_0\sqrt{\mu/\epsilon}$ and the refractive index is $n = \sqrt{\epsilon\mu}$. Taking $\mu = 1$, this means $Z = Z_0/n$.

This question is easier than it looks, because of the properties of the 1/4-wavelength transformer of impedance Z, which can match an inpedance $Z_{\rm in}$ to a medium of impedance Z_0 provided $Z^2 = Z_0 Z_{\rm in}$. The medium of impedance Z_0 coated with a 1/4-wave layer thus presents an effective impedance $Z_{\rm eff} = Z^2/Z_0$, so the reflection coefficient is $r = (Z_{\rm eff} - Z_{\rm in})/(Z_{\rm eff} + Z_{\rm in})$. The power reflection coefficient from a crown glass ($n_0 = 1.52$)/vacuum interface is $|r^2| = ((1/n_0 - 1)/(1/n_0 + 1))^2 = 0.04258$. With a layer of Magnesium Fluoride ($n_1 = 1.38$) the effective refractive index is $n_{\rm eff} = n_1^2/n_0 = 1.2529$ so that $|r|^2 = 0.01260$.

The first part analyses an antireflective coating, but we can make a highly reflective surface by coating this with another layer of material (Zinc Sulphide) with a high refractive index $n_2 = 2.35$. We find a new effective refractive index $n'_{\rm eff} = n_2^2/n_{\rm eff} = (n_2^2/n_1^2)n_0 = 4.4078$ and $|r|^2 = 0.39711$. Adding more layers in a similar fashion gives $n_{\rm eff} = (n_2/n_1)^{2x} n_0$ for an even number of layers (so that n_2 is on top) and $n_{\rm eff} = (n_1/n_2)^{2x} n_1^2/n_0$ for an odd number (x is the number of complete pairs). We are asked to show the result for an even number of layers, for which $n_{\rm eff} \to \infty$ and $|r|^2 \to 1$ as x increases, but $|r|^2 \to 1$ for the odd number case as well. The reflection coefficient for $x = 10$ is 0.999937, as required, but the odd number case doesn't get there until $x = 11$.

All this 1/4-wave trickery won't help you at all in the general case, though.

- Q48. The TM_{11} mode propagating in the x-direction has components $(0, B_y, B_z)$ and E_x, E_y, E_z. Take the B_y component as

$$B_y = B_0 \frac{\pi}{b} \cos\left(\frac{\pi z}{b}\right) \sin\left(\frac{\pi y}{a}\right) e^{i(k_x x - \omega t)} \tag{79}$$

and note that it satisfies the boundary condition $B_\perp = 0$ at $y = 0$ and $y = a$. Find the component B_z from $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$:

$$B_z = -B_0 \frac{\pi}{a} \sin\left(\frac{\pi z}{b}\right) \cos\left(\frac{\pi y}{a}\right) e^{i(k_x x - \omega t)} \ . \tag{80}$$

The components of $\boldsymbol{E}$ follow from $\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + \partial\boldsymbol{D}/\partial t \Rightarrow c^2 \boldsymbol{\nabla}\times\boldsymbol{B} = \partial\boldsymbol{E}/\partial t$,

$$\begin{aligned}
E_x &= iB_0 \frac{c^2}{\omega}\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right) \sin\left(\frac{\pi z}{b}\right) \sin\left(\frac{\pi y}{a}\right) e^{i(k_x x - \omega t)} \\
E_y &= -B_0 \frac{\pi c^2 k}{\omega a} \sin\left(\frac{\pi z}{b}\right) \cos\left(\frac{\pi y}{a}\right) e^{i(k_x x - \omega t)} \\
E_z &= -B_0 \frac{\pi c^2 k}{\omega b} \cos\left(\frac{\pi z}{b}\right) \sin\left(\frac{\pi y}{a}\right) e^{i(k_x x - \omega t)}
\end{aligned}$$

Note the following points:

  - the longitudinal electric field is in quadrature with transverse components and vanishes at the walls;

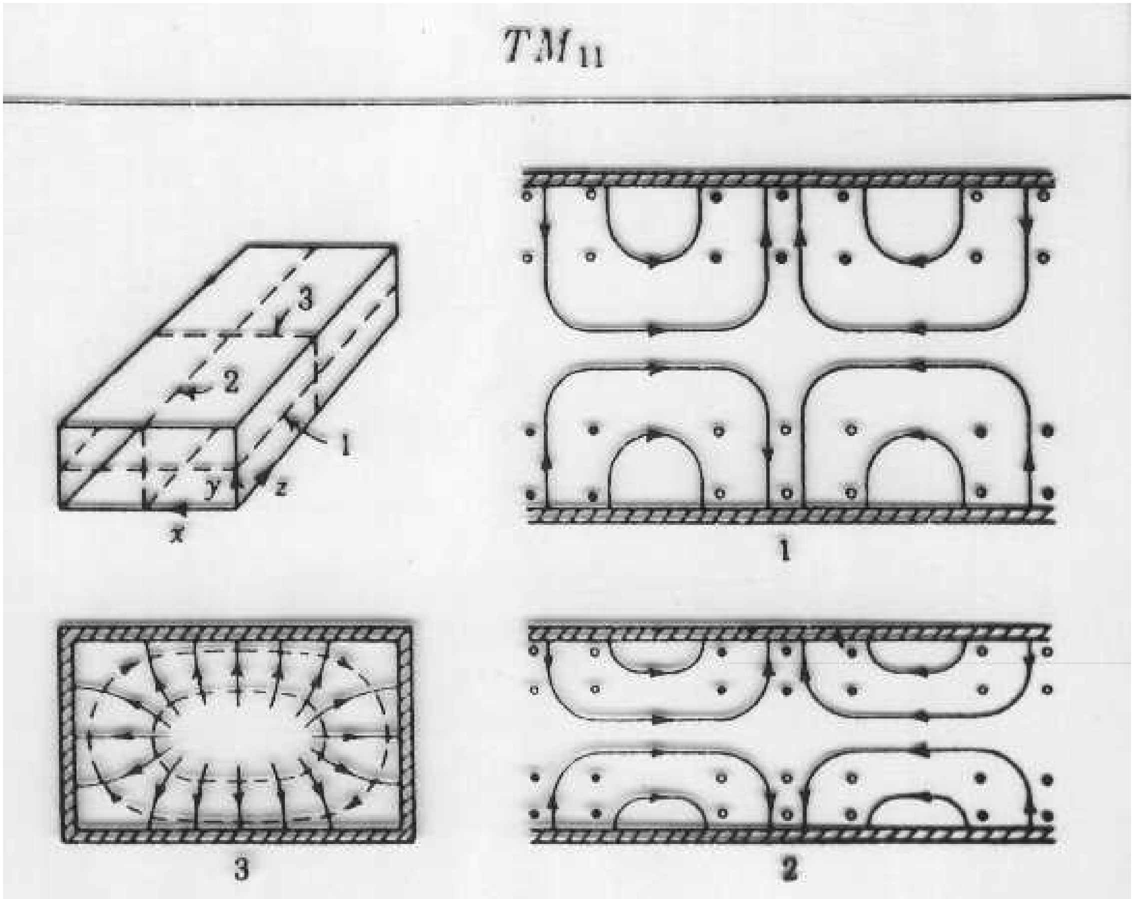

Figure 3: Electric fields (solid lines) and magnetic fields (dotted) for the TM_{11} waveguide mode.

- there are charges on the surface of the waveguide. We can calculate these by looking at the normal component of the electric field and using Gauss' theorem. For example, the charge density ρ_S on the surface $y = 0$ is

$$\rho_\mathrm{S} = \epsilon_0 E_y = -B_0 \frac{\pi k}{\mu_0 \omega a} \sin\left(\frac{\pi z}{b}\right) e^{i(k_x x - \omega t)} . \tag{81}$$

- There are currents flowing on the surface of the conductors. We can calculate these from the longitudinal components of the $\boldsymbol{B}$ field by using Ampère's law. For example, on the $y = 0$ surface we get a current J_S in the z-direction

$$J_\mathrm{S} = \frac{B_z}{\mu_0} = -B_0 \frac{\pi}{a} \sin\left(\frac{\pi z}{b}\right) e^{i(k_x x - \omega t)} . \tag{82}$$

- Q49. Waveguide equation for the m, n mode

$$\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} + k_\mathrm{g}^2 = \frac{4\pi^2}{\lambda^2} . \tag{83}$$

For a square waveguide of side 110 mm, operating at $\lambda = 100$ mm, to have a propagating mode ($k_{\rm g}^2 > 0$), we must have

$$\frac{m^2+n^2}{110^2} < \frac{1}{50^2} \;; \quad \lambda_{\rm g} = \frac{2\pi}{k_{\rm g}} \;. \tag{84}$$

Enumerating the possible modes, we have

Mode	$\lambda_{\rm g}$
$\rm TE_{10}$,$\rm TE_{01}$	0.11227 m
$\rm TE_{11}$	0.13055 m
$\rm TE_{20}$,$\rm TE_{02}$	0.24004 m
$\rm TM_{11}$	0.13055 m

Note that $\rm TM_{11}$ is the only TM mode possible on a conducting (non-magnetic) guide.

For sketches, see the lecture notes.

- Q50. Manipulate the dispersion relation to get

$$v_\phi = c\left(1-\frac{\omega_{\rm c}^2}{\omega^2}\right)^{-1/2} \;; \quad v_{\rm g} = c\left(1-\frac{\omega_{\rm c}^2}{\omega^2}\right)^{1/2} = c\left(1-\frac{\nu_{\rm c}^2}{\nu^2}\right)^{1/2} \;. \tag{85}$$

For $\nu = 15$ GHz and $\nu_{\rm c} = 12$ GHz we get the central group velocity $v_{\rm g} = 3c/5$. Differentiating (85), we obtain

$$\frac{dv_{\rm g}}{d\nu} = c\left(1-\frac{\nu_{\rm c}^2}{\nu^2}\right)^{-1/2} \frac{\nu_{\rm c}^2}{\nu^3} \approx \frac{\delta v_{\rm g}}{\delta\nu} \;. \tag{86}$$

With $\delta\nu = 8$ MHz, this gives $\delta v_{\rm g} = 1.70066 \times 10^5$ m s^{-1}.

Appendix

A Force on dielectric and paramagnetic bodies

This is a response to question (Q32) about the paramagnetic cylinder on the example sheet. We will calculate the force directly, rather than attempting to use an energy argument. There is an equivalent electrostatic problem, which is worth solving first, since the vector manipulations are easier.

A.1 Force on dielectric medium

Recall the expressions for surface charge $\rho_{\mathrm{S}} = \boldsymbol{P}\cdot\boldsymbol{n}$ on a dielectric and for the polarisation charge density $\rho_{\mathrm{P}} = -\boldsymbol{\nabla}\cdot\boldsymbol{P}$. Taken together they express the fact that there is no net polarisation charge:

$$\oint d\boldsymbol{S}\cdot\boldsymbol{P} - \int d\tau\ \boldsymbol{\nabla}\cdot\boldsymbol{P} = 0\ . \tag{87}$$

The force can now be calculated as the sum $\sum_i q_i\boldsymbol{E}_i$ over all the charges:

$$\boldsymbol{F} = \oint \boldsymbol{E}\ d\boldsymbol{S}\cdot\boldsymbol{P} - \int d\tau\ \boldsymbol{E}\ \boldsymbol{\nabla}\cdot\boldsymbol{P}\ . \tag{88}$$

Using the generalised divergence theorem on the first term, we get

$$\oint \boldsymbol{E}\ d\boldsymbol{S}\cdot\boldsymbol{P} = \int d\tau\ \boldsymbol{E}\ \overset{\leftrightarrow}{\boldsymbol{\nabla}}\cdot\boldsymbol{P} = \int d\tau\ \left(\boldsymbol{E}\ \boldsymbol{\nabla}\cdot\boldsymbol{P} + \boldsymbol{P}\cdot\boldsymbol{\nabla}\boldsymbol{E}\right), \tag{89}$$

where the $\overset{\leftrightarrow}{\boldsymbol{\nabla}}$ operator operates on both $\boldsymbol{E}$ and $\boldsymbol{P}$. The expression for the force then simplifies to

$$\boldsymbol{F} = \int d\tau\ \boldsymbol{P}\cdot\boldsymbol{\nabla}\boldsymbol{E}\ , \tag{90}$$

as expected. For a linear dielectric $\boldsymbol{P} = \chi\epsilon_0\boldsymbol{E}$ and we can write the expression for the force as

$$\boldsymbol{F} = \int d\tau\ \left(\boldsymbol{\nabla}(\tfrac{1}{2}\chi|\boldsymbol{E}|^2) - \tfrac{1}{2}|\boldsymbol{E}|^2\boldsymbol{\nabla}\chi\right) = \oint d\boldsymbol{S}\ \tfrac{1}{2}\chi|\boldsymbol{E}|^2 - \int d\tau\ \tfrac{1}{2}|\boldsymbol{E}|^2\boldsymbol{\nabla}\chi\ . \tag{91}$$

The second term makes a finite contribution only on the surface of the dielectric, and ensures that the formula is valid for any shape of surface, whether or not it coincides with the surface of the body. If the surface S is in fact the surface of the body, we have just

$$\boldsymbol{F} = \oint d\boldsymbol{S}\ \tfrac{1}{2}\chi|\boldsymbol{E}|^2\ . \tag{92}$$

A.2 Force on magnetisable medium

For a magnetisable body things are slightly more complicated, because the surface current is $J_{\mathrm{S}} = \boldsymbol{M}\times\boldsymbol{n}$ and the magnetisation current is $J_{\mathrm{M}} = \boldsymbol{\nabla}\times\boldsymbol{M}$. This means that the total magnetisation current is zero:

$$\oint \boldsymbol{M}\times d\boldsymbol{S} + \int d\tau\ \boldsymbol{\nabla}\times\boldsymbol{M} = 0\ . \tag{93}$$

We find the force as the sum $\sum_i \boldsymbol{J}_i \times \boldsymbol{B}$ over all the currents:

$$
\begin{aligned}
\boldsymbol{F} &= \oint (\boldsymbol{M} \times d\boldsymbol{S}) \times \boldsymbol{B} + \int d\tau \ (\boldsymbol{\nabla} \times \boldsymbol{M}) \times \boldsymbol{B} \\
&= \int d\tau (\boldsymbol{M} \times \overset{\leftrightarrow}{\boldsymbol{\nabla}}) \times \boldsymbol{B} + \int d\tau \ (\boldsymbol{\nabla} \times \boldsymbol{M}) \times \boldsymbol{B} \\
&= \int d\tau \ (\boldsymbol{M} \times \boldsymbol{\nabla}) \times \boldsymbol{B} \ ,
\end{aligned}
\tag{94}
$$

noting again a pleasing cancellation of terms. Expanding the triple product and using $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, we find

$$
\boldsymbol{F} = \int d\tau \ \boldsymbol{M} \cdot \boldsymbol{\nabla} \boldsymbol{B} \ , \tag{95}
$$

which looks very similar to the dielectric case, except that it is $\boldsymbol{B}$ rather than $\boldsymbol{H}$ that determines the force, changing the form of the expression involving the susceptibility. For a linear medium $\boldsymbol{M} = \chi \boldsymbol{H}$, and we get

$$
\boldsymbol{F} = \oint d\boldsymbol{S} \ \tfrac{1}{2} \mu_0 \chi (1 + \chi) |\boldsymbol{H}|^2 - \int d\tau \ \tfrac{1}{2} \mu_0 (1 + \chi) |\boldsymbol{H}|^2 \boldsymbol{\nabla} \chi \ . \tag{96}
$$

Again, if the surface S is in fact the surface of the body, we get the simpler result

$$
\boldsymbol{F} = \oint d\boldsymbol{S} \ \tfrac{1}{2} \mu_0 \chi (1 + \chi) |\boldsymbol{H}|^2 \ . \tag{97}
$$

Of course, we still have to calculate $\boldsymbol{H}$ inside the medium...

A.3 Calculation of magnetic energy

There is a nice argument that shows that, when a paramagnetic body is inserted in a magnetic field, the magnetic energy change can be expressed as an integral over the paramagnetic body itself, rather than over the whole of space. Let the initial fields be $\boldsymbol{H}_0$ and $\boldsymbol{B}_0$ and the perturbed fields $\boldsymbol{H}$ and $\boldsymbol{B}$. The change in magnetic energy is

$$
\Delta W = \int d\tau \tfrac{1}{2} \left(\boldsymbol{B} \cdot \boldsymbol{H} - \boldsymbol{B}_0 \cdot \boldsymbol{H}_0 \right) \ , \tag{98}
$$

where the integral is taken over the whole region where the fields are perturbed. We can rewrite this integral in what at first sight appears to be a rather unlikely looking form

$$
\Delta W = \int d\tau \tfrac{1}{2} \left(\boldsymbol{B} \cdot \boldsymbol{H}_0 - \boldsymbol{B}_0 \cdot \boldsymbol{H} + (\boldsymbol{B} + \boldsymbol{B}_0) \cdot (\boldsymbol{H} - \boldsymbol{H}_0) \right) \ . \tag{99}
$$

If the external currents generating the magnetic fields are held constant when the body is inserted, then from $\boldsymbol{\nabla} \times \boldsymbol{H} = J_{\text{free}}$ we must have

$$
\boldsymbol{\nabla} \times (\boldsymbol{H} - \boldsymbol{H}_0) = 0 \ . \tag{100}
$$

This means that we can write $\boldsymbol{H} - \boldsymbol{H}_0$ as the gradient of a scalar: $\boldsymbol{H} - \boldsymbol{H}_0 = \boldsymbol{\nabla} \phi$. Furthermore, since $\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla} \cdot \boldsymbol{B}_0 = 0$, we have

$$
\int d\tau \tfrac{1}{2} (\boldsymbol{B} + \boldsymbol{B}_0) \cdot (\boldsymbol{H} - \boldsymbol{H}_0) = \int d\tau \tfrac{1}{2} (\boldsymbol{B} + \boldsymbol{B}_0) \cdot \boldsymbol{\nabla} \phi = \int d\tau \tfrac{1}{2} \boldsymbol{\nabla} \cdot (\phi (\boldsymbol{B} + \boldsymbol{B}_0) \ . \tag{101}
$$

This term is a total divergence and can be converted to an integral over a surface sufficiently far away that the fields are negligible. We thus are left with

$$\Delta W = \int d\tau \tfrac{1}{2} \left(\boldsymbol{B} \cdot \boldsymbol{H}_0 - \boldsymbol{B}_0 \cdot \boldsymbol{H} \right) \ , \tag{102}$$

where the integrand now vanishes except over the volume of the paramagnetic body.

Using $\boldsymbol{H}$ and $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$ and $\boldsymbol{B}_0 = \mu_0 \boldsymbol{H}_0$ we obtain the result

$$\Delta W = \int d\tau \tfrac{1}{2} \mu_0 \boldsymbol{M} \cdot \boldsymbol{H}_0 \ , \tag{103}$$

which involves the unperturbed $\boldsymbol{H}_0$ field, but we still have to find the actual magnetisation $\boldsymbol{M}$.

See if you can find a similar result for a dielectric body in an electric field.

B More on Q28.

The rotating sphere in a magnetic field threatened to become a major research topic, and what follows is for amusement only. We look at a simple limit first, then plunge into the general case.

B.1 The limit of low conductivity

We start with the case where the conductivity is sufficiently low that the magnetic field penetrates the sphere fully. The current in the sphere is given by (non-relativistic!)

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \ , \tag{104}$$

where $\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{\omega}$. The electric field has to be non-zero, because the $\boldsymbol{v} \times \boldsymbol{B}$ term generates a current that crosses the surface of the sphere, which isn't allowed. Rather, a surface charge distribution is set up that drives the current around the sphere (in small circles — see if you can visualise this).

Take the x-axis along the $\boldsymbol{B}$ field and the let the sphere rotate at $\boldsymbol{\omega}$ about the z-axis. We need to guess the form of the electric potential, I guess it's

$$V(r, \theta, \phi) = V_0 \frac{r^2}{a^2} \sin\theta \cos\theta \cos\phi \ , \tag{105}$$

which is a solution of Laplace's equation with the appropriate symmetry. The condition that there is no radial component of the current at $r = a$ sets the value of $V_0 = -\frac{1}{2} B \omega a^2$ (J_r is then zero everywhere).

We now have the current, so find the force from

$$\boldsymbol{F} = \boldsymbol{J} \times \boldsymbol{B} = \left(0, \tfrac{1}{2} \sigma \omega B^2 r \sin\theta \cos\phi, 0\right) \tag{106}$$

and the couple from $\boldsymbol{G} = \boldsymbol{r} \times \boldsymbol{F}$ and integrate. Only the z-component of the couple survives the integration:

$$G_z = \frac{2\pi}{15} \sigma \omega B^2 a^5 \ . \tag{107}$$

The moment of inertia of the sphere is $(8\pi/15)\rho a^5$, so the time constant is again $\tau = 4\rho/(\sigma\omega B^2)$.

The vector potential $\boldsymbol{A}(\boldsymbol{r})$ can probably be written as

$$\boldsymbol{A}(\boldsymbol{r}) = A(r)\ \boldsymbol{X}_1^1(\theta,\phi)\ , \tag{112}$$

where $A(r)$ is complex. We then find

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{A} = \frac{\sqrt{2}}{r}A(r)\ i\sigma_r Y_1^1 + \frac{1}{r}\partial_r(rA(r))\ \sigma_r\times\boldsymbol{X}_1^1\ . \tag{113}$$

The velocity field is proportional to $\boldsymbol{X}_1^0$, specifically

$$\boldsymbol{v} = \omega\times\boldsymbol{r} = \frac{-2i\sqrt{2\pi}}{\sqrt{3}}\ r\Omega\boldsymbol{X}_1^1\ . \tag{114}$$

The $\boldsymbol{v}\times\boldsymbol{B}$ term then generates some interesting terms, and we need the theorems

$$\begin{aligned} \boldsymbol{X}_1^0\times(\sigma_r\times\boldsymbol{X}_1^1) &= \frac{-i\sqrt{15}}{20\sqrt{\pi}}\ i\sigma_r Y_2^1\ , \\ \boldsymbol{X}_1^0\times\ i\sigma_r Y_1^1 &= \frac{-3i\sqrt{5}}{20\sqrt{\pi}}\ \sigma_r\times\boldsymbol{X}_2^1 + \frac{\sqrt{3}}{4\sqrt{\pi}}\boldsymbol{X}_1^1\ . \end{aligned} \tag{115}$$

The $\boldsymbol{X}_1^1$ term generates the necessary circulating currents, and the other two terms might represent the electric field, provided that they have zero curl. If they don't, then there is an additional $\boldsymbol{X}_2^1$ component to $\boldsymbol{A}$ and we have a very nasty problem on our hands. But the fates are kind, and we find a suitable electrostatic potential

$$V(\boldsymbol{r}) = \frac{-i\Omega}{\sqrt{10}}rA(r)Y_2^1(\theta,\phi)\ . \tag{116}$$

The remaining equation for A is

$$A'' + 2\frac{A'}{r} - \frac{2A}{r^2} - i\sigma\mu_0\Omega A = 0\ , \tag{117}$$

and the solution regular at the origin is

$$\boldsymbol{A}(\boldsymbol{r}) \propto j_1\left(\sqrt{\sigma\mu_0\Omega i}\ r\right)\ \boldsymbol{X}_1^1(\theta,\phi)\ . \tag{118}$$

That's as far as I have got with the problem so far, I now need to match the internal and external fields and write a short routine to generate a spherical Bessel funtion $j_1(z)$ of complex argument (the power series is OK for modest values $|z| < 40$), but I can illustrate the effect with a simulation I completed for a rotating cylinder (which is actually just as difficult).