Time-dependent electromagnetic fields

Electromagnetic induction

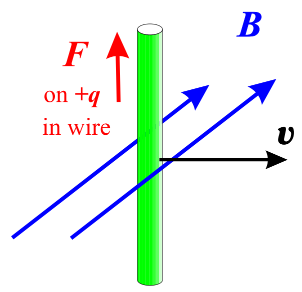
Faraday's Law

- When the magnetic flux through a circuit is changing, an electromotive force is induced in the circuit.
- The magnitude of the e.m.f. is proportional to the rate of change of magnetic flux.

Lenz' Law

- Induced e.m.f. is always is such a direction as to promote a current flow which creates a magnetic field that opposes the change in flux.

Explanation of Faraday's Law



- Consider force on a charge $+q$ in a wire moving at velocity $\boldsymbol{v}$ through a field $\boldsymbol{B}$

$$\boldsymbol{F} = q\,(\boldsymbol{v}\times\boldsymbol{B})$$

- If charge moves $d\boldsymbol{l}$ along the wire, work is done

$$dW = \boldsymbol{F}\cdot d\boldsymbol{l}$$

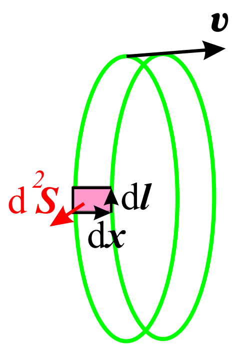
- Work done per unit charge is defined in terms of potential difference

$$\begin{aligned} dW &= q\,dV \\ \Rightarrow dV &= \boldsymbol{v}\times\boldsymbol{B}\cdot d\boldsymbol{l} \end{aligned}$$

Explanation of Faraday's Law

- Consider loop of wire moving through magnetic field
- Total e.m.f. around circuit

$$\mathcal{E} = \oint dV = -\oint \boldsymbol{B}\times\boldsymbol{v}\cdot d\boldsymbol{l} = -\oint \boldsymbol{B}\cdot(\boldsymbol{v}\times d\boldsymbol{l})$$



- But $\boldsymbol{v}\times d\boldsymbol{l} = \frac{d\boldsymbol{x}}{dt}\times d\boldsymbol{l} = \frac{d^2\boldsymbol{S}}{dt}$

  and $\oint \frac{d^2\boldsymbol{S}}{dt} = \frac{d\boldsymbol{S}}{dt}$

- $$\Rightarrow \mathcal{E} = -\oint \boldsymbol{B}\cdot(\boldsymbol{v}\times d\boldsymbol{l}) = -\int \boldsymbol{B}\cdot\frac{d\boldsymbol{S}}{dt} = -\frac{d\Phi}{dt}$$

- The formula $\mathcal{E} = -d\Phi/dt$ works for both cases:
  1. Moving loop cutting $\boldsymbol{B}$ stationary field lines (as here);
  2. Stationary loop with changing $\boldsymbol{B}$ field through it.

Faraday's Law

- Faraday's law:

$$\oint d\boldsymbol{l}\cdot\boldsymbol{E} = -\frac{d\Phi}{dt} = -\frac{\partial}{\partial t}\int d\boldsymbol{S}\cdot\boldsymbol{B}$$

- We can write Faraday's law in integral or differential form.
- To get the Maxwell equation we consider a fixed loop

  Integral form: $\oint d\boldsymbol{l}\cdot\boldsymbol{E} = -\int d\boldsymbol{S}\cdot\frac{\partial\boldsymbol{B}}{\partial t}$

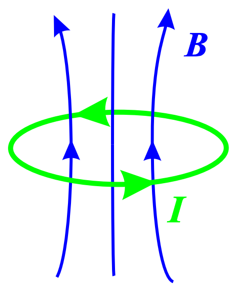
- Apply Stokes' theorem to get differential form:

$$\oint d\boldsymbol{l}\cdot\boldsymbol{E} = \int d\boldsymbol{S}\cdot\nabla\times\boldsymbol{E}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \quad \text{(Maxwell 2)}$$

Self Inductance

- A circuit carrying current I is linked by the magnetic field lines produced by its own current.



- If Φ = total self-linked magnetic flux

  Linearity $\Rightarrow$ $\Phi \propto I$

- Define self-inductance: $L \equiv \Phi / I$
- Unit of inductance (SI) is the Henry

$$1 \text{ H} \equiv 1 \text{ Wb A}^{-1}$$

- L is determined by geometry of circuit: current loops have larger L than straight wires.

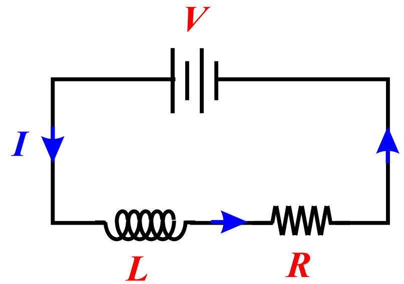
  (Examples: see handout and later.)

Energy Stored in Inductance

- If L is constant (rigid circuits) and I increases with time,

Fararday $\Rightarrow$ $\mathcal{E} = -\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial t}(LI) = -L\frac{\partial I}{\partial t}$

- Consider LR circuit + voltage source



$$V = RI + L\frac{\partial I}{\partial t}$$

↑ "back e.m.f"

- Rate of energy loss in voltage source = VI.

$$VI = RI^2 + LI\frac{\partial I}{\partial t} = RI^2 + \frac{\partial}{\partial t}\left(\tfrac{1}{2}LI^2\right)$$

↑ Dissipation in resistor

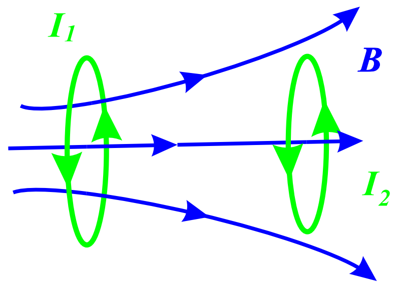
↑ Rate of gain of magnetic energy in inductance

- Energy stored in LR circuit: $U_{\mathrm{L}} = \frac{1}{2}LI^2$

  (c.f. capacitance $U_{\mathrm{C}} = \frac{1}{2}CV^2$)

Mutual Inductance

- Consider 2 circuits near each other



- Current I_1 produces a flux linkage Φ_2 in circuit #2.
- Linearity $\Rightarrow$ $\Phi_2 \propto I_1$

- Define *mutual inductance*:

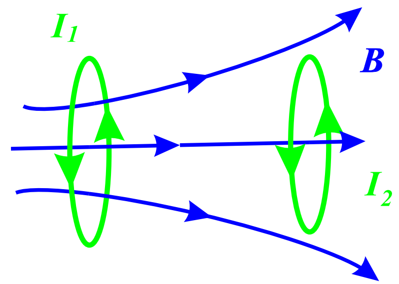
$$M_{12} \equiv \frac{\Phi_2}{I_1}$$

- Similarly, can define $M_{21} \equiv \Phi_1 / I_2$.
- But mutual inductance is symmetric:

$$M_{21} = M_{12}$$

- Example of very general *RECIPROCITY THEOREM*.
- Applicable whener there is a quadratic stored energy (e.g. cantilever — Dynamics Q.27).

Symmetry of Mutual Inductance



- Suppose $I_1 = I_2 = 0$ and I_1 is gradually turned on.
- There is a back e.m.f.

$$\frac{\partial \Phi_1}{\partial t} = L_1 \frac{\partial I_1}{\partial t}$$

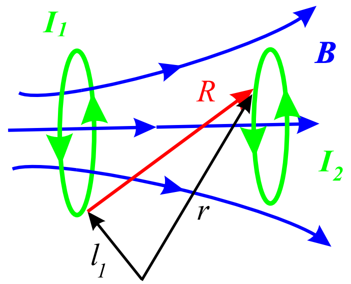
- I_1 does work against e.m.f. at rate $L_1 I_1 \frac{\partial I_1}{\partial t}$
- $\Rightarrow$ Energy stored in field $U = \frac{1}{2} L_1 I_1^2$
- Now suppose I_2 is turned on, keeping I_1 fixed.
- Extra energy due to self inductance is $\frac{1}{2} L_2 I_2^2$
- Back e.m.f. in first circuit due to mutual inductance is

$$\frac{\partial \Phi_1}{\partial t} = M_{21} \frac{\partial I_2}{\partial t}$$

- Steady current I_1 does work at rate $I_1 M_{21} \frac{\partial I_2}{\partial t}$
- Total energy $U = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{21} I_1 I_2$
- Must be the same if I_2 is switched on before I_1

$$\Rightarrow M_{21} = M_{12}$$

Symmetry of Mutual Inductance II



- High-Tech Direct Proof

  Not for Exam

- $R = \boldsymbol{r} - \boldsymbol{l}_1$

  ($\boldsymbol{r} = \boldsymbol{l}_2$ on loop #2)

- Magnetic field $\boldsymbol{B}(\boldsymbol{r})$ due to I_1

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I_1}{4\pi}\oint \frac{d\boldsymbol{l}_1 \times \boldsymbol{R}}{R^3} = \frac{\mu_0 I_1}{4\pi}\oint d\boldsymbol{l}_1 \times \nabla\left(-\frac{1}{R}\right)$$

- Flux linkage through circuit #2

$$\Phi_2 = \int d\boldsymbol{S}_2 \cdot \boldsymbol{B} = \frac{\mu_0 I_1}{4\pi}\oint d\boldsymbol{l}_1 \times \int \nabla\left(-\frac{1}{R}\right) \cdot d\boldsymbol{S}_2$$

$$\Rightarrow M_{12} = \frac{\mu_0}{4\pi}\oint d\boldsymbol{l}_1 \cdot \int (d\boldsymbol{S}_2 \times \nabla)\frac{1}{R}$$

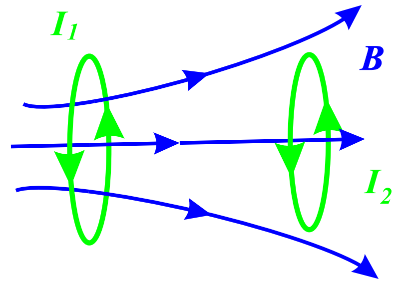
- Using generalised Stokes' theorem:

$$M_{12} = \frac{\mu_0}{4\pi}\oint d\boldsymbol{l}_1 \cdot \oint d\boldsymbol{l}_2 \frac{1}{R} = M_{21}$$

$R = |\boldsymbol{l}_2 - \boldsymbol{l}_1|$ (manifestly symmetric)

Inductance

- Mutual Inductance: $M_{12} = M_{21} = M$
- Self Inductance: L



- Signs:

  currents in same sense

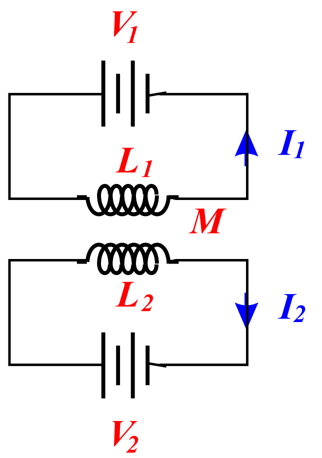
  $\Phi_2 = L_2 I_2 + M I_1$

- Mutual inductance produces e.m.f. in circuit #2 if the current in circuit #1 changes.

$$-\mathcal{E}_1 = L_1\frac{\partial I_1}{\partial t} + M\frac{\partial I_2}{\partial t}$$

$$-\mathcal{E}_2 = L_2\frac{\partial I_2}{\partial t} + M\frac{\partial I_1}{\partial t}$$

Energy in Coupled Circuits



$$V_1 = L_1 \frac{\partial I_1}{\partial t} + M \frac{\partial I_2}{\partial t}$$

$$V_2 = L_2 \frac{\partial I_2}{\partial t} + M \frac{\partial I_1}{\partial t}$$

- Energy loss from voltage sources = $V_1 I_1 + V_2 I_2$

$$= \frac{\partial}{\partial t}\left(\tfrac{1}{2} L_1 I_1^2\right) + \frac{\partial}{\partial t}\left(\tfrac{1}{2} L_2 I_2^2\right) + \frac{\partial}{\partial t}\left(M I_1 I_2\right)$$

$\uparrow$ energy gain in L_1 — $\uparrow$ energy gain in L_2 — $\uparrow$ energy gain in mutual field

- Total energy: $U_{\text{M}} = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$
- First 2 terms are > 0, but 3rd can take either sign.

The Ideal Transformer

- $U_{\text{M}} = \frac{1}{2}(L_1 I_1^2 + 2 M I_1 I_2 + L_2 I_2^2)$

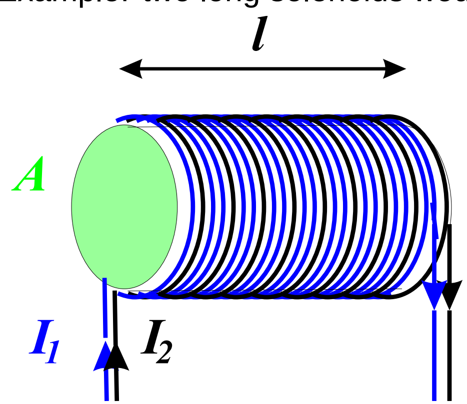
$$= \frac{1}{2} \begin{pmatrix} I_1 & I_2 \end{pmatrix} \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

- Total U_{M} must be positive $\Rightarrow$ restriction on value of M.

  $M^2 \leq L_1 L_2$ define $M = k(L_1 L_2)^{1/2}$ $(0 \leq k \leq 1)$

  k = coefficient of coupling; $k = 1 \Rightarrow$ perfect coupling.

- Example: two long solenoids wound over each other.



turns (n_1, n_2)

(Same A, same l)

$B_1 = \mu_0 n_1 I_1$

Flux linking #1 $= B_1 A n_1$

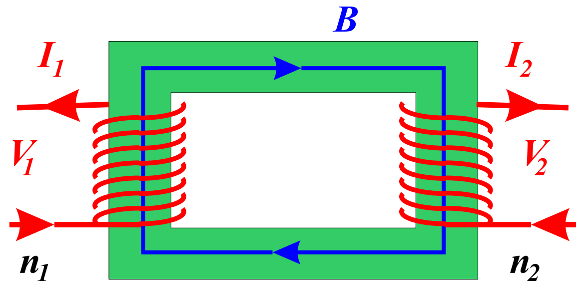
$\Rightarrow L_1 = \mu_0 n_1^2 A$

Flux linking #2 $= B_1 A n_2$

$\Rightarrow M = \mu_0 n_1 n_2 A$

- Similarly $L_2 = \mu_0 n_2^2 A \Rightarrow M = (L_1 L_2)^{1/2}$ for this case.

The Ideal Transformer



- Perfect coupling

  No flux losses

  $k = 1$.

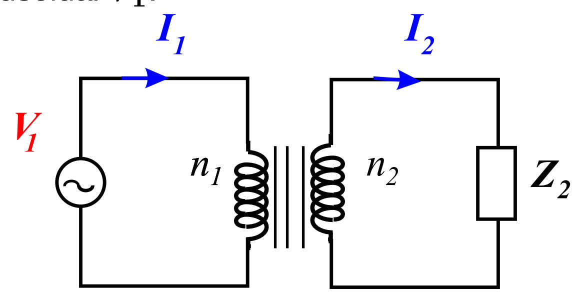
- No energy losses in wires; no hysteresis.
- Flux of each turn of coil #1 links each turn of coil #2.

$$\Phi_1 = n_1\Phi\ ; \quad \Phi_2 = n_2\Phi$$
$$\Phi = \text{flux linkage per turn}$$
$$V_1 = -\frac{\partial\Phi_1}{\partial t} = -n_1\frac{\partial\Phi}{\partial t}$$
$$V_2 = -\frac{\partial\Phi_2}{\partial t} = -n_2\frac{\partial\Phi}{\partial t}$$

- Transformer: $$\frac{V_1}{V_2} = \frac{n_1}{n_2}$$
- Self inductance: $\propto n^2$ $$\frac{L_1}{L_2} = \left(\frac{n_1}{n_2}\right)^2$$

Transformer — Example

- Perfect transformer driven by voltage generator giving sinusoidal V_1.



- Connected to load impedance Z_2.

(primary) $$V_1 = L_1\frac{\partial I_1}{\partial t} - M\frac{\partial I_2}{\partial t}$$
(secondary) $$V_2 = 0 = L_2\frac{\partial I_2}{\partial t} - M\frac{\partial I_1}{\partial t} + I_2 Z_2$$

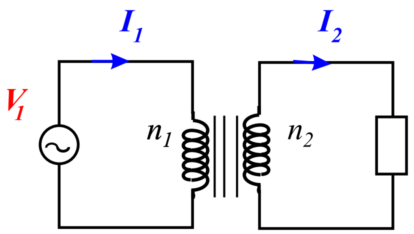
- Time dependence like $e^{j\omega t}$

$$\Rightarrow \frac{\partial V_1}{\partial t} = j\omega V_1\ ; \quad \frac{\partial I_1}{\partial t} = j\omega I_1\ ; \text{ etc.}$$

- Rearranging:

$$V_2 = 0 \Rightarrow \quad (j\omega L_2 + Z_2)I_2 = j\omega M I_1$$
$$V_1 = j\omega L_1 I_1 - j\omega M I_2 = j\omega L_1 I_1 - \frac{(j\omega M)^2 I_1}{Z_2 + j\omega L_2}$$

Transformer — Example II



$$\frac{V_1}{I_1} = j\omega L_1 - \frac{(j\omega M)^2}{Z_2 + j\omega L_2}$$

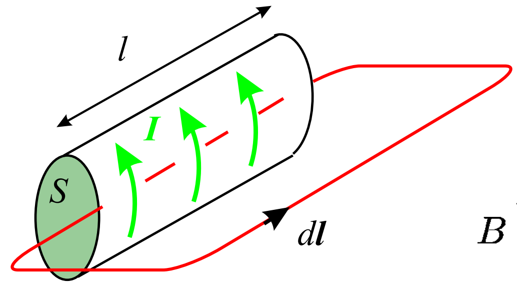
- Substitute $M^2 = L_1 L_2$; $\frac{L_1}{L_2} = \left(\frac{n_1}{n_2}\right)^2$

$$\Rightarrow \frac{V_1}{I_1} = \frac{j\omega L_1\, Z_2(n_1/n_2)^2}{j\omega L_1 + Z_2(n_1/n_2)^2}$$

- Also find $I_2 Z_2 = V_1 \frac{n_2}{n_1}$ as expected.
- V_1/I_1 is relationship for impedances $Z_2(n_1/n_2)^2$ and $j\omega L_1$ in parallel.
- Transformer 'looks like' inductance L_1 and a 'reflected impedance' $Z_2(n_1/n_2)^2$.
- Transformer can match a low impedance device with a high impedance device if the turns ratio is chosen appropriately.
- Usually $j\omega L_1 \gg Z_2(n_1/n_2)^2$, so $Z_{\mathrm{i}} \approx Z_2(n_1/n_2)^2$.

Self Inductance — Calculating

- Self inductance of long solenoid (ignore end effects).



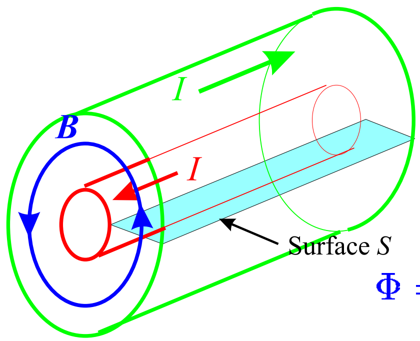
N turns altogether.
$B = 0$ outside solenoid.

$$\oint d\boldsymbol{l}\cdot\boldsymbol{H} = NI \Rightarrow B_{\mathrm{i}} = \mu_0 NI/l$$

Flux linked: $\Phi = NB_{\mathrm{i}}S = N^2 I\mu_0 S/l$

$\Rightarrow$ Self inductance: $L = \Phi/I = \mu_0 N^2 S/l$.

- Coaxial cylinders. Length l, inner and outer radii a and b.



$$\oint d\boldsymbol{l}\cdot\boldsymbol{H} = I$$

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

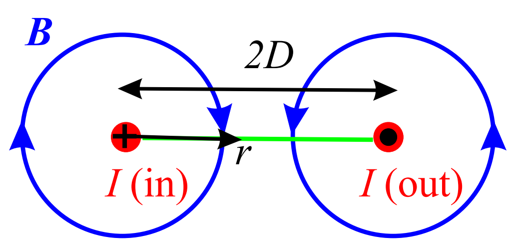
$$\Phi = l\int_a^b dr\, B(r)$$

$$\Phi = LI = \frac{\mu_0 I l}{2\pi}\log\left(\frac{b}{a}\right)$$

$\Rightarrow$ Self inductance per unit length $= \frac{\mu_0}{2\pi}\log\left(\frac{b}{a}\right)$

Self Inductance of Pair of Wires

- Length l, radius of wires $a \ll D$.



- Assume currents uniformly distributed.

  $\Rightarrow$ Magnetic field due to one wire

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

  $\Rightarrow$ Flux due to one wire

$$\Phi = \frac{\mu_0 I l}{2\pi} \int_a^{2D-a} \frac{dr}{r} \approx \frac{\mu_0 I l}{2\pi} \log\left(\frac{2D}{a}\right) \quad (a \ll D)$$

- Flux due to both wires is $2\times$ this.

- $\Rightarrow$ Self inductance per unit length $= \dfrac{\mu_0}{\pi} \log\left(\dfrac{2D}{a}\right)$

Magnetic Energy

- Current I flowing in inductance L

  $\Rightarrow$ energy $W = \frac{1}{2} L I^2$.

- $\Rightarrow \quad L = \Phi / I, \quad \Phi =$ flux through circuit.

- $\Rightarrow \quad W = \frac{1}{2} \Phi I$

- Many circuits $\Rightarrow W = \sum_i \frac{1}{2} \Phi_i I_i$.

- But $\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{S}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$

- $\Rightarrow \quad \Phi = \int d\boldsymbol{S} \cdot \nabla \times \boldsymbol{A} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

- $\Rightarrow \quad W = \frac{1}{2} \sum_i \left(\oint \boldsymbol{A} \cdot (I d\boldsymbol{l}) \right)_i$

- Go to distributed limit: $I d\boldsymbol{l} \to \boldsymbol{J} d\tau$

$$W = \frac{1}{2} \int d\tau \, \boldsymbol{A} \cdot \boldsymbol{J}$$

Magnetic Energy II

- Ampère: $\nabla\times\boldsymbol{H} = \boldsymbol{J}$ (for now. . .)
- $W = \frac{1}{2}\int d\tau\, \boldsymbol{A}\cdot\boldsymbol{J} = \frac{1}{2}\int d\tau\ \boldsymbol{A}\cdot\nabla\times\boldsymbol{H}$
- To prove the following handy theorem, note that the ∇ acts on both $\boldsymbol{A}$ and $\boldsymbol{H}$, generating two terms (Leibniz); then use rearrangements of the scalar triple product:

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{H}) = \boldsymbol{H}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{H})$$

- Use theorem to form divergence, convert to surface integral

$$\begin{aligned} W &= -\tfrac{1}{2}\int d\tau\ \nabla\cdot(\boldsymbol{A}\times\boldsymbol{H}) + \tfrac{1}{2}\int d\tau\ \boldsymbol{H}\cdot\nabla\times\boldsymbol{A} \\ &= -\tfrac{1}{2}\oint d\boldsymbol{S}\cdot\boldsymbol{A}\times\boldsymbol{H} + \tfrac{1}{2}\int d\tau\ \boldsymbol{H}\cdot\boldsymbol{B} \end{aligned}$$

- Take integral over large surface of radius R;
  $d\boldsymbol{S} \propto R^2$; $\boldsymbol{A} \propto R^{-1}$; $\boldsymbol{H} \propto R^{-2}$;
  $\Rightarrow$ surface integral $\to 0$ as $R \to \infty$.

$$W = \int d\tau\ \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}$$

Magnetic energy density: $U_{\mathrm{M}} = \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}$

Electromagnetic Energy (Not for Exam)

- $U_{\mathrm{E}} = \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D}\ ;\qquad U_{\mathrm{M}} = \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}$
- $\int d\tau U_{\mathrm{E}} = \frac{1}{2}\int d\tau\,\rho V\ ;\ \int d\tau U_{\mathrm{M}} = \frac{1}{2}\int d\tau\,\boldsymbol{J}\cdot\boldsymbol{A}$
- But we haven't assembled all of Maxwell's equations yet — caution!

$$\nabla\times\boldsymbol{H} = \boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial t}$$

$$\boldsymbol{E} = -\nabla V - \frac{\partial\boldsymbol{A}}{\partial t}$$

- The total energy density (in the laboratory frame) is indeed

$$U = U_{\mathrm{E}} + U_{\mathrm{M}} = \tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H})$$

- But the relation to ρ, $\boldsymbol{J}$ is

$$\frac{1}{2}\int d\tau(\boldsymbol{E}\cdot\boldsymbol{D} - \boldsymbol{B}\cdot\boldsymbol{H}) = \frac{1}{2}\int d\tau(\rho V - \boldsymbol{J}\cdot\boldsymbol{A})$$

- N.B. Signs (Relativistic scalar quantities)

Time-varying Electric Fields

- Electromagnetic equations so far are NOT CONSISTENT with *charge conservation*

$$\oint d\boldsymbol{S}\cdot\boldsymbol{J} + \int d\tau \frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \nabla\cdot\boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

- We have

$$\nabla\times\boldsymbol{H} = \boldsymbol{J} \quad \Rightarrow \nabla\cdot\boldsymbol{J} = 0 \quad (!)$$

- So we must add *Maxwell's Displacement Current*

(Maxwell 4) $$\nabla\times\boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

- Take divergence:

$$0 = \nabla\cdot\boldsymbol{J} + \frac{\partial}{\partial t}\nabla\cdot\boldsymbol{D}$$

- But $\nabla\cdot\boldsymbol{D} = \rho$ (Maxwell 1), so now OK.