MAGNETOSTATICS

- Moving charges $\rightarrow$ currents (Units: A).
- Currents are sources of **magnetic field**.

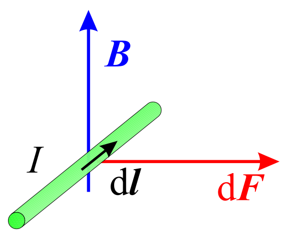
Magnetostatics: No variation of currents with time; no free charges; no electric field.

Maxwell's equations:

$$\Rightarrow \quad \begin{array}{ll} \text{(M3)} & \nabla \cdot \boldsymbol{B} = 0 \\ \text{(M4)} & \nabla \times \boldsymbol{H} = \boldsymbol{J}_{\text{free}} \end{array}$$

- M3 is simpler than M1 in electrostatics.
- M4 is more complicated than M2 — **vector** equation.

Starting point: 1820's Ampère, Oersted, Biot, Savart.
Force between currents: due to magnetic fields.

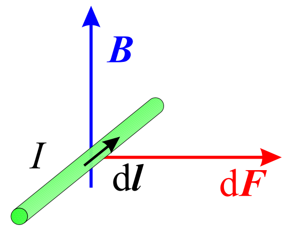


- Force on elementary current element $I\mathrm{d}\boldsymbol{l}$

$$\mathrm{d}\boldsymbol{F} = I\mathrm{d}\boldsymbol{l}\times\boldsymbol{B}$$

$\boldsymbol{B}$ is the **magnetic flux density** (units: T).

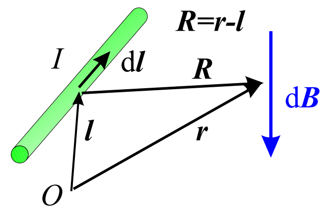
MAGNETOSTATICS II



$$\mathrm{d}\boldsymbol{F} = I\mathrm{d}\boldsymbol{l}\times\boldsymbol{B}$$

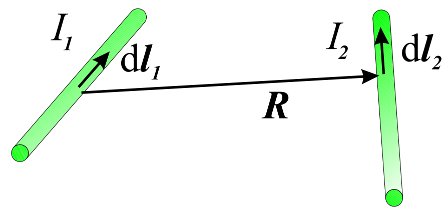
- N.B. consistent with the Lorentz force on a moving charge.

- Put $I\mathrm{d}\boldsymbol{l} = \left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)\boldsymbol{v}\mathrm{d}t = \mathrm{d}q\,\boldsymbol{v} \Rightarrow \mathrm{d}\boldsymbol{F} = \mathrm{d}q\,\boldsymbol{v}\times\boldsymbol{B}$
- Magnetic field due to current element $I\mathrm{d}\boldsymbol{l}$ is given by the Biot-Savart law: $(\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{l})$



$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0 I}{4\pi R^3}\mathrm{d}\boldsymbol{l}\times\boldsymbol{R}$$

- Force between two current elements



$$\mathrm{d}\boldsymbol{B}_2 = \frac{\mu_0 I_1}{4\pi R^3}\mathrm{d}\boldsymbol{l}_1\times\boldsymbol{R}$$

$$\mathrm{d}\boldsymbol{F}_2 = I_2\mathrm{d}\boldsymbol{l}_2\times\mathrm{d}\boldsymbol{B}_2$$

$$\Rightarrow \mathrm{d}\boldsymbol{F}_2 = \frac{\mu_0 I_1 I_2}{4\pi R^3}\mathrm{d}\boldsymbol{l}_2\times(\mathrm{d}\boldsymbol{l}_1\times\boldsymbol{R})$$

Force Between Currents

- The force between two parallel wires is used as the fundamental definition of the unit of current (and hence charge).
- The Ampère is the current which, when flowing in each of two parallel, straight conductors, of infinite length and negligible cross-section, placed 1 m apart in vacuum, produces a force of 2×10^{-7} N m^{-1} per unit length.
- (We shall see that this result is consistent with the formula for the force between current elements.)
- It follows that μ_0 is thereby **defined** to be

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$$

Gauss' Theorem for Magnetic Field

- There are NO sources of the $\boldsymbol{B}$ field (magnetic monopoles; equivalent to magnetic charges).
- The $\boldsymbol{B}$ field is always caused by electric currents.
- Field lines of $\boldsymbol{B}$ must form closed loops.
- As many field lines of $\boldsymbol{B}$ leave any closed surface as enter

$$\Rightarrow \oint \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B} = 0$$

$$\Rightarrow \boxed{\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0}$$

Aside

- Any field generated by the Biot-Savart law is divergence-free.

$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\boldsymbol{l}\times\boldsymbol{R}}{R^3} \qquad (\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{l})$$

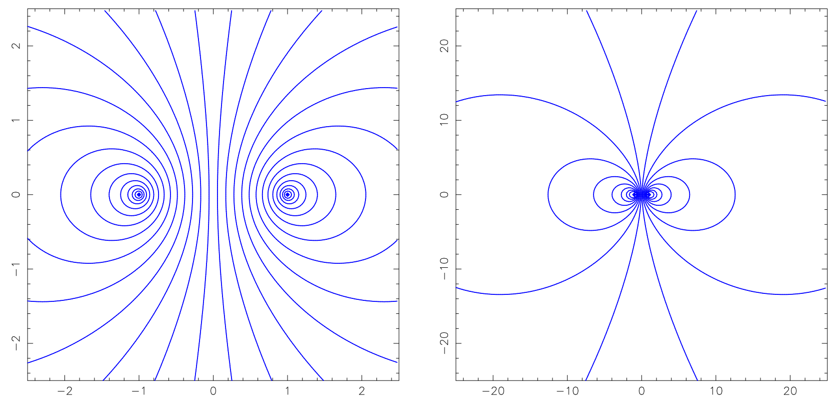
But $\boldsymbol{\nabla}(1/R) = -\boldsymbol{R}/R^3$, (Can you prove this?)

$$\Rightarrow \mathrm{d}\boldsymbol{B} = -\frac{\mu_0 I}{4\pi}\mathrm{d}\boldsymbol{l}\times\boldsymbol{\nabla}\left(\frac{1}{R}\right) = \frac{\mu_0 I}{4\pi}\boldsymbol{\nabla}\times\left(\frac{\mathrm{d}\boldsymbol{l}}{R}\right)$$

- Hence $\boldsymbol{\nabla}\cdot\mathrm{d}\boldsymbol{B} = 0$ (divergence of a curl).

Current Loops and Magnetic Dipoles

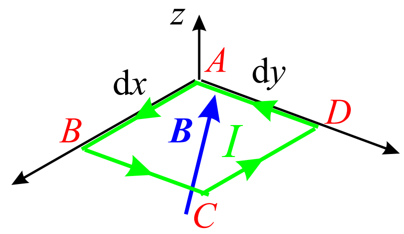
- There are no magnetic charges (magnetic monopoles). The simplest magnetic sources are dipoles.



- Figure shows field lines of $\boldsymbol{B}$ due to a circular current loop. (The solution is expressible in terms of the complete elliptic integral.)
- At large distance the field lines are the same as those of a dipole.
- $\Rightarrow$ Magnetic dipoles correspond to small current loops.

Elementary Dipoles

- There are no magnetic charges (magnetic monopoles). The simplest magnetic sources are dipoles.
- Dipoles correspond to small current loops.

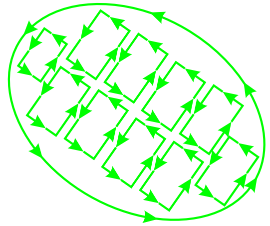


- Elementary current loop in uniform field $\boldsymbol{B}$.
- $\Rightarrow$ No net force on a small loop (cancels).
- But there is a couple (just like electric dipole).
- Consider force in z-direction on line CD = $-I\mathrm{d}x B_y$; it has a couple about the x-axis: $G_x = -B_y I \mathrm{d}x\mathrm{d}y$.
- Similarly,the couple about the y-axis is $G_y = B_x I \mathrm{d}x\mathrm{d}y$. [Take origin at A; G_y due to BC.]
- $G_z = 0$, so defining $\mathrm{d}\boldsymbol{m} \equiv I\mathrm{d}\boldsymbol{S} = (0, 0, I\mathrm{d}x\mathrm{d}y)$ for this loop, we have $\mathrm{d}\boldsymbol{G} = (-\mathrm{d}m_z B_y, \mathrm{d}m_z B_x, 0)$

$$\Rightarrow \mathrm{d}\boldsymbol{G} = \mathrm{d}\boldsymbol{m}\times\boldsymbol{B}$$

Elementary Dipoles II

- Small current loop acts like a small magnetic dipole $\mathrm{d}\boldsymbol{m}$
- Note **sign convention**: direction of current I defines orientation of $(\mathrm{d}\boldsymbol{m} \equiv I\mathrm{d}\boldsymbol{S})$ by right-hand rule.
- Couple on loop $\mathrm{d}\boldsymbol{G} = \mathrm{d}\boldsymbol{m}\times\boldsymbol{B}$



- Consider finite current loop:

$\boldsymbol{m} = I\int \mathrm{d}\boldsymbol{S}$ due to cancellation of currents except at edge.

- Compare with electrostatics:

$$\boldsymbol{G} = \boldsymbol{p}\times\boldsymbol{E};\quad U = -\boldsymbol{p}\cdot\boldsymbol{E}$$

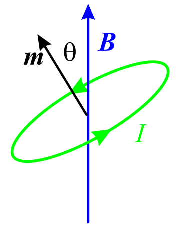
- Potential energy of a magnetic dipole:

$$U = -\boldsymbol{m}\cdot\boldsymbol{B} = -I\int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B} = -I\Phi$$

- $\Phi = \int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B}$ = magnetic flux through S.

Potential Energy — Force on Dipole

- Potential energy of a magnetic dipole: $U = -\boldsymbol{m}\cdot\boldsymbol{B}$.
- Derived by analogy with electrostatics, but what is the physical picture for the current loop?

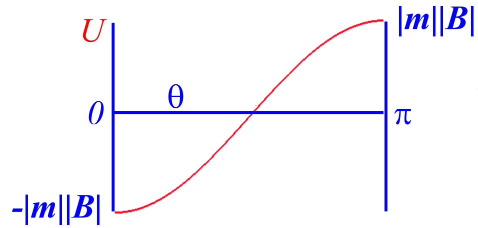


- Couple $\boldsymbol{G} = \boldsymbol{m}\times\boldsymbol{B}$

$|\boldsymbol{G}| = |\boldsymbol{m}||\boldsymbol{B}|\sin\theta$

Couple does work $\mathrm{d}W = |\boldsymbol{m}||\boldsymbol{B}|\sin\theta\,\mathrm{d}\theta$ as θ is varied

$\Rightarrow W = -|\boldsymbol{m}||\boldsymbol{B}|\cos\theta = -\boldsymbol{m}\cdot\boldsymbol{B}$.



Take $U = 0$ at $\theta = \pi/2$.

- Force on dipole in non-uniform field:

$$\boldsymbol{F} = \boldsymbol{m}\cdot\boldsymbol{\nabla}\,\boldsymbol{B} = \overset{*}{\boldsymbol{\nabla}}\,(\boldsymbol{m}\cdot\overset{*}{\boldsymbol{B}})\qquad \overset{\text{(Care!)}}{(=-\boldsymbol{\nabla}U)}$$

[Proof: $\boldsymbol{m}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) = \overset{*}{\boldsymbol{\nabla}}\,(\boldsymbol{m}\cdot\overset{*}{\boldsymbol{B}}) - \boldsymbol{m}\cdot\boldsymbol{\nabla}\,\boldsymbol{B} = 0$

$(\boldsymbol{\nabla}\times\boldsymbol{B} = 0$ for interior of loop).]

Non-Examinable — Mathematical Interlude

- The divergence theorem $\oint \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{A} = \int \mathrm{d}\tau\,\boldsymbol{\nabla}\cdot\boldsymbol{A}$
  is much more general than you think!
- Generalised form: $\oint \mathrm{d}\boldsymbol{S}(\bullet) = \int \mathrm{d}\tau\,\boldsymbol{\nabla}(\bullet)$
  (operator identity) the argument $\bullet$ can be anything...
- Two new useful theorems: for scalar field Φ and vector $\boldsymbol{A}$

$$\oint \mathrm{d}\boldsymbol{S}\,\Phi = \int \mathrm{d}\tau\,\boldsymbol{\nabla}\Phi\,; \qquad \oint \mathrm{d}\boldsymbol{S}\times\boldsymbol{A} = \int \mathrm{d}\tau\,\boldsymbol{\nabla}\times\boldsymbol{A}$$

- Stokes' theorem $\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{A} = \int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{\nabla}\times\boldsymbol{A}$
  is better written $\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{A} = \int (\mathrm{d}\boldsymbol{S}\times\boldsymbol{\nabla})\cdot\boldsymbol{A}$
- Generalised form: $\oint \mathrm{d}\boldsymbol{l}(\bullet) = \int (\mathrm{d}\boldsymbol{S}\times\boldsymbol{\nabla})(\bullet)$
- Two new useful theorems:

$$\oint \mathrm{d}\boldsymbol{l}\,\Phi = \int \mathrm{d}\boldsymbol{S}\times\boldsymbol{\nabla}\Phi\,; \quad \oint \mathrm{d}\boldsymbol{l}\times\boldsymbol{A} = \int (\mathrm{d}\boldsymbol{S}\times\boldsymbol{\nabla})\times\boldsymbol{A}$$

- (1) Remember the brackets!
  (2) The $\boldsymbol{\nabla}$ operates on **everything**.

Force on Current Loop — Higher-Technology

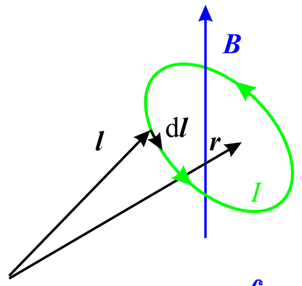
- Don't use electrostatic analogy — calculate the forces directly.
- Force on current loop: $\boldsymbol{F} = I \oint \mathrm{d}\boldsymbol{l}\times\boldsymbol{B}$
- Generalised Stokes: $\boldsymbol{F} = I \int (\mathrm{d}\boldsymbol{S}\times\boldsymbol{\nabla})\times\boldsymbol{B}$

$$\boldsymbol{F} = I \int (\boldsymbol{\nabla}(\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B}) - \mathrm{d}\boldsymbol{S}\,\boldsymbol{\nabla}\cdot\boldsymbol{B}) = I \int \boldsymbol{\nabla}(\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B})$$

$$\boldsymbol{F} = I \int (\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{\nabla}\boldsymbol{B} + \mathrm{d}\boldsymbol{S}\times(\boldsymbol{\nabla}\times\boldsymbol{B}))$$

- $\boldsymbol{\nabla}\times\boldsymbol{B} = 0$ for interior of the loop $\Rightarrow$ $\boldsymbol{F} = \int I\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{\nabla}\boldsymbol{B}$
- For small loop: $\boldsymbol{m} \equiv I \int \mathrm{d}\boldsymbol{S}$.
- If loop smaller than the scale of variation of $\boldsymbol{B}$:
  $\boldsymbol{F} = \boldsymbol{m}\cdot\boldsymbol{\nabla}\,\boldsymbol{B}$ as required.

Couple on Current Loop — Non-Examinable



- Uniform magnetic field.

  $\boldsymbol{l}$ on loop; $\boldsymbol{r}$ is general point.

- Force: $\boldsymbol{F} = I \oint \mathrm{d}\boldsymbol{l}\times\boldsymbol{B}$.
- Couple: $\boldsymbol{G} = I \oint \boldsymbol{l}\times(\mathrm{d}\boldsymbol{l}\times\boldsymbol{B})$.
- Generalised Stokes: $\boldsymbol{G} = I \int \boldsymbol{r}\times((\mathrm{d}\boldsymbol{S}\times \overset{\leftrightarrow}{\boldsymbol{\nabla}})\times\boldsymbol{B})$
- Not quite as bad as it looks...

  Notes: $\overset{\leftrightarrow}{\boldsymbol{\nabla}}$ acts on everything; but $\boldsymbol{B}$ is constant.

  Move $\boldsymbol{r}$ to end and expand inner triple product:

$$\boldsymbol{G} = I \int (\boldsymbol{B}\cdot\boldsymbol{\nabla}(\mathrm{d}\boldsymbol{S}\times\boldsymbol{r}) - \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}\boldsymbol{\nabla}\times\boldsymbol{r})$$

  Now use $\boldsymbol{\nabla}\times\boldsymbol{r} = 0$ and $\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{r} = \boldsymbol{B}$. (prove this!)

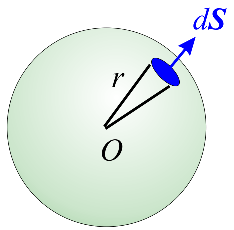
- $\Rightarrow \boldsymbol{G} = I \int \mathrm{d}\boldsymbol{S}\times\boldsymbol{B} = \boldsymbol{m}\times\boldsymbol{B}$
- That's enough vector calculus for now...

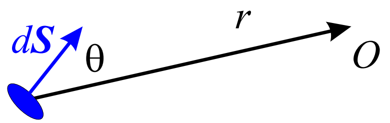
Magnetic Scalar Potential

- In electrostatics we define $\boldsymbol{E} = -\boldsymbol{\nabla}V$, where V is the electrostatic potential.
- We are justified in doing this because $\boldsymbol{\nabla}\times\boldsymbol{E} = 0$ in electrostatics.
- In magnetostatics $\boldsymbol{\nabla}\times\boldsymbol{B} = 0$ except where currents are flowing.
- We can **sometimes** define a **magnetic scalar potential** ϕ_{m} to solve problems, analogous to V in electrostatics.
- Definition: $\boldsymbol{B} = -\mu_0\boldsymbol{\nabla}\phi_{\mathrm{m}}$.
- Warning: for magnetic media, a better definition is $\boldsymbol{H} = -\boldsymbol{\nabla}\phi_{\mathrm{m}}$. ($\boldsymbol{B} = \mu_0\boldsymbol{H}$ in free space)
- Unlike V, the magnetic scalar potential ϕ_{m} is **not** a physical field, but only a mathematical convenience.

The Solid Angle of a Loop

- The **solid angle** of an area $\mathrm{d}\boldsymbol{S}$ on a sphere of radius r is defined to be $\mathrm{d}\Omega \equiv \frac{|\mathrm{d}\boldsymbol{S}|}{r^2}$



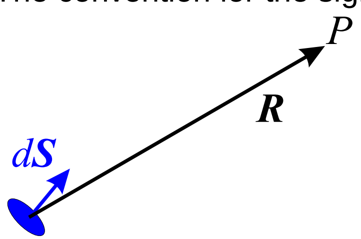


- Solid angle is measured in *steradians*; a whole sphere subtends 4π steradians.

- More generally, account for $\cos\theta$ factor:

$$\mathrm{d}\Omega = \frac{|\mathrm{d}\boldsymbol{S}|}{r^2}\cos\theta = \frac{\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{r}}{r^3}$$

- The convention for the sign is (as in Bleaney&Bleaney):



$$\mathrm{d}\Omega = \frac{\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{R}}{R^3}$$

Magnetic Scalar Potential of a Current Loop

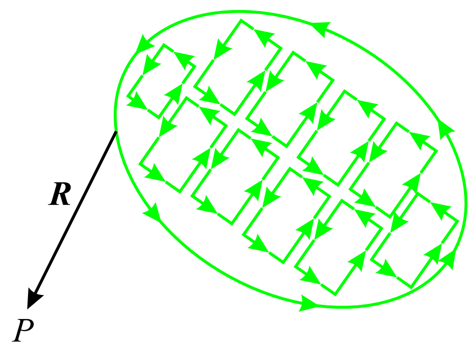
- Small dipole: $\mathrm{d}\boldsymbol{m} = I\mathrm{d}\boldsymbol{S}$

$$\Rightarrow \phi_{\mathrm{m}} = \frac{|\mathrm{d}\boldsymbol{m}|\cos\theta}{4\pi R^2}$$

[c.f. electrostatic dipole]

- $\phi_{\mathrm{m}} = \frac{\mathrm{d}\boldsymbol{m}\cdot\boldsymbol{R}}{4\pi R^3} = \frac{I\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{R}}{4\pi R^3} = \frac{I\mathrm{d}\Omega}{4\pi}$

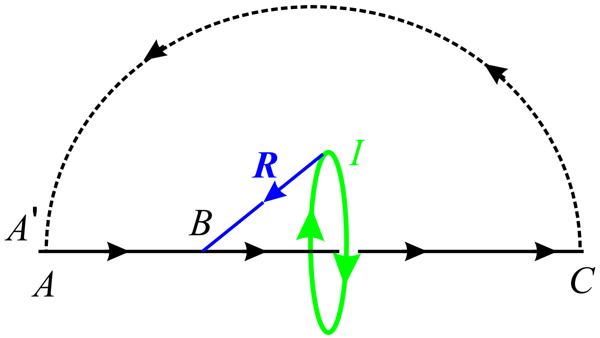
[$\mathrm{d}\Omega$ is the solid angle of current loop at $\boldsymbol{R}$.]



- Must be true for larger loop as well. $\phi_{\mathrm{m}} = \frac{I\Omega}{4\pi}$

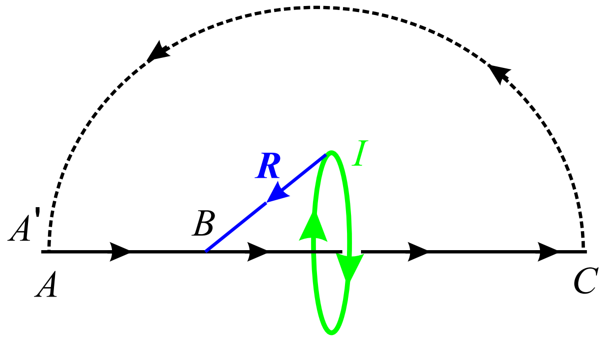
- Magnetic scalar potential is proportional to the solid angle subtended by loop at P.

Ampère's Theorem



- N.B. Direction of loop is $A \to C$.
- At A (at large distance) $\Omega \approx 0$.
- At B (intermediate distance) $\Omega < 0$.
  Going through loop $\Omega = -2\pi$.
- Continuing, Ω **continues to decrease** and at large distance $\Omega \approx -4\pi$.
  $\phi_{\rm m}(A) = 0\,; \qquad \phi_{\rm m}(C) = -I$.
- On return path $\Delta\Omega = 0$, since it does not go through the loop.
- $\Rightarrow \Omega_{A'} \neq \Omega_A$ (by -4π).
- Magnetic scalar potential is not single-valued: if a path encircles a current $\phi_{\rm m} \to \phi_{\rm m} - I$.

Ampère's Theorem II

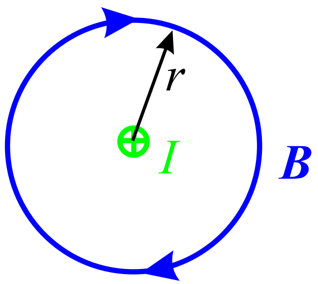


- $\displaystyle\int_A^{A'} {\rm d}\phi_{\rm m} = \int_A^{A'} {\rm d}\boldsymbol{l}\cdot\boldsymbol{\nabla}\phi_{\rm m} = -\frac{1}{\mu_0}\oint {\rm d}\boldsymbol{l}\cdot\boldsymbol{B}$
- $\phi_{\rm m}(A') = \phi_{\rm m}(A) - I \Rightarrow \boxed{\oint {\rm d}\boldsymbol{l}\cdot\boldsymbol{B} = \mu_0 I}$ (Ampère)
- $\displaystyle\oint {\rm d}\boldsymbol{l}\cdot\boldsymbol{B} = \mu_0 I = \mu_0 \int {\rm d}\boldsymbol{S}\cdot\boldsymbol{J}$
  [$\boldsymbol{J}$ = current density (A m^{-2})]
- Apply Stokes' theorem:

$$\oint {\rm d}\boldsymbol{l}\cdot\boldsymbol{B} = \int {\rm d}\boldsymbol{S}\cdot\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0 \int {\rm d}\boldsymbol{S}\cdot\boldsymbol{J}$$

$$\Rightarrow \boxed{\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0 \boldsymbol{J}}$$

Ampère's Theorem III

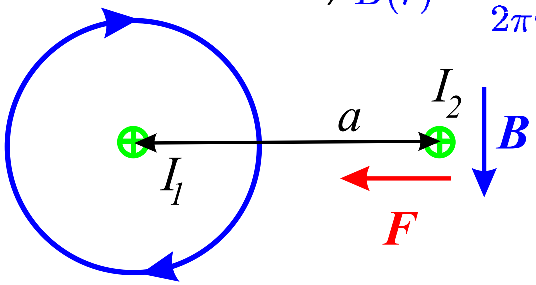


Field at distance r from long wire carrying current I (into plane)

$$\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{B} = 2\pi r B(r)$$

$$\int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{J} = I$$

$$\Rightarrow B(r) = \frac{\mu_0 I}{2\pi r}$$



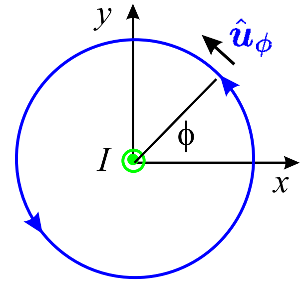
Force: $\dfrac{\mu_0 I_1 I_2}{2\pi a}$

- Force between two long wires.
- Definition of the Amp: put $I_1 = I_2$ = 1 A; a=1 m.

$$\Rightarrow F = \frac{\mu_0}{2\pi} = 2\times 10^{-7} \text{ N m}^{-1}$$

Magnetic Scalar Potential Revisited

- $\phi_{\rm m}$ is useful for some problems involving magnetic media.
- $\phi_{\rm m} = \dfrac{I\Omega}{4\pi}$ is a useful insight.
- Works when $\boldsymbol{J} = 0$ and sometimes useful when currents are confined to wires.
- Long thin wire:



I out of the plane.

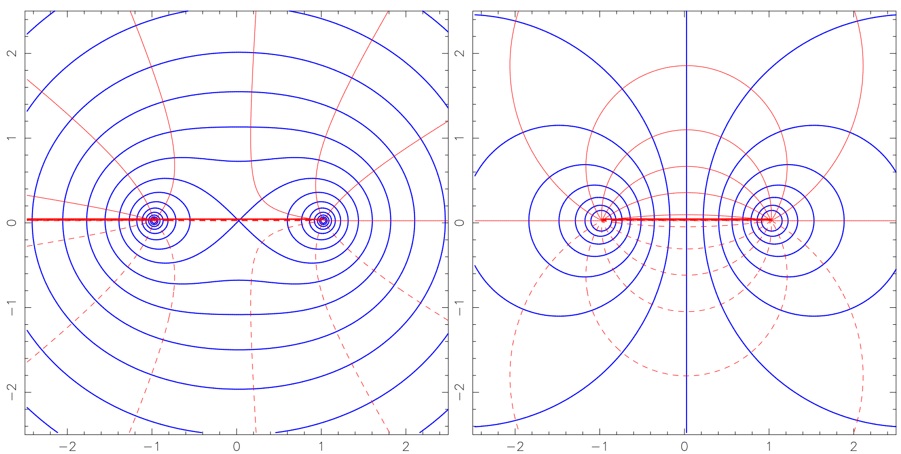
$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{u}}_\phi.$$

We can use $\phi_{\rm m} = -\dfrac{I\phi}{2\pi}$

[$\hat{\boldsymbol{u}}_\phi$ is the unit vector in direction of cylindrical angle ϕ.]

- This $\phi_{\rm m}$ is multi-valued, but doesn't matter since $\boldsymbol{B}$ is well-defined.
- But $\nabla\times\boldsymbol{B} \neq 0$, so the scalar potential must ultimately fail.

Current-Carrying Wires



- Figure shows field lines of $\boldsymbol{B}$ (thick, blue) and equipotentials of ϕ_{m} (thin, red) for two wires carrying equal parallel currents (left) and antiparallel currents (right).
- Note that ϕ_{m} is multivalued:

  (1) for currents in same direction, there is a discontinuity along the negative x-axis;

  (2) for opposite currents the discontinuity is in the plane joining the wires — ϕ_{m} is continuous for paths enclosing **both** wires.

Magnetic Vector Potential

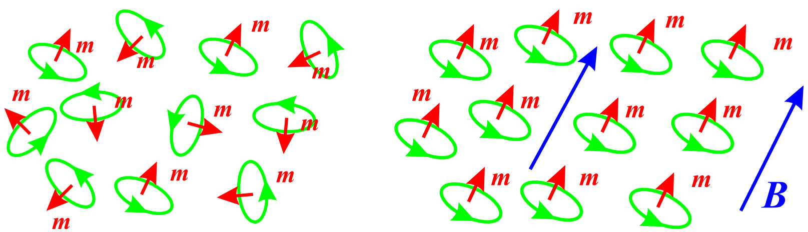
- V is well-defined in electrostatics — potential energy per unit charge.
- V is generated by ρ.
- When ρ changes, we get a current $\boldsymbol{J}$.
- $(\rho, \boldsymbol{J})$ form a relativistic 4-vector.
- There is another **vector** potential $\boldsymbol{A}$ that forms a relativistic 4-vector with the electrostatic potential V. (next year. . .)
- $\boldsymbol{A}$ generates $\boldsymbol{B}$ via $\boxed{\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}}$ $\Rightarrow \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$.
- $\boldsymbol{\nabla}\cdot\boldsymbol{A}$ is not determined (gauge choice);

  c.f. zero-point of V.
- We have already seen that $\boldsymbol{B}$ can be written:

  $$\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\boldsymbol{\nabla}\times\oint\frac{\mathrm{d}\boldsymbol{l}}{R}.$$
- $\Rightarrow$ Formula for $\boldsymbol{A}$ is easy. (next year). . .

Magnetic Media

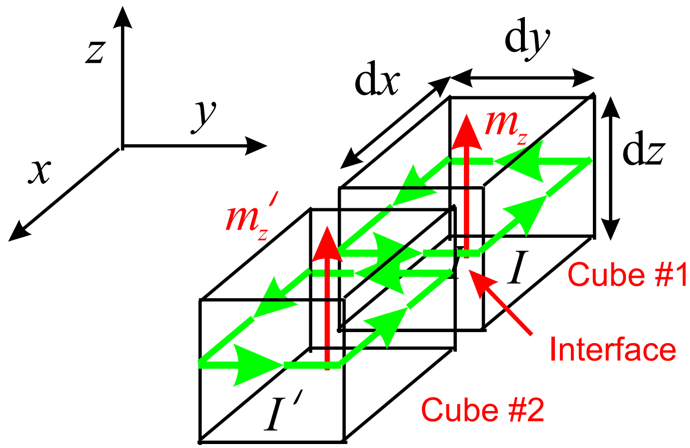
- Magnetic substances acquire dipole moments in magnetic fields.
- Dipole moment $\boldsymbol{M}$ per unit volume.
  [Analogous to $\boldsymbol{P}$ for dielectrics.]
- $\boldsymbol{M}$ originates in microscopic circulating currents.



$\boldsymbol{B} = 0$; no net circulation. With $\boldsymbol{B}$ field; currents combine $\Rightarrow$ net circulation.

- Circulating currents $\Rightarrow$ magnetisation current density J_{m}
- Need to relate J_{m} and $\boldsymbol{M}$.

Magnetisation Currents



- Consider two small cubes, carrying currents I, I' around faces shown.
- Magnetic dipole moment in z-direction in cell 1 is $m_z = I\mathrm{d}x\mathrm{d}y$.
  By definition: $m_z = M_z \mathrm{d}x\mathrm{d}y\mathrm{d}z \Rightarrow I = M_z \mathrm{d}z$.
- For cell 2 $m_z' = I'\mathrm{d}x\mathrm{d}y = M_z'\mathrm{d}x\mathrm{d}y\mathrm{d}z$.

  But $M_z' \approx M_z + \frac{\partial M_z}{\partial x}\mathrm{d}x;\;\; I' \approx I + \frac{\partial I}{\partial x}\mathrm{d}x$
- Net current flow (J_y) on interface in y, z plane:

$$I - I' = -\frac{\partial I}{\partial x}\mathrm{d}x = -\frac{\partial M_z}{\partial x}\mathrm{d}x\mathrm{d}z = J_y \mathrm{d}x\mathrm{d}z$$

Magnetisation currents II

- There is also a contribution to J_y from currents in the x, y-plane $= \dfrac{\partial M_x}{\partial z}$.

$$\Rightarrow J_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x} = [\nabla\times\boldsymbol{M}]_y$$

$$\Rightarrow \boxed{\boldsymbol{J}_{\rm m} = \nabla\times\boldsymbol{M}}$$

- We must add the magnetisation currents to Ampère's theorem

$$\nabla\times\boldsymbol{B} = \mu_0(\boldsymbol{J}_{\rm free} + \boldsymbol{J}_{\rm m})$$

$$\Rightarrow \nabla\times(\boldsymbol{B} - \mu_0\boldsymbol{M}) = \mu_0\boldsymbol{J}_{\rm free}$$

- Define **magnetic field strength**:

$$\boxed{\boldsymbol{H} \equiv \frac{1}{\mu_0}(\boldsymbol{B} - \mu_0\boldsymbol{M})}$$

- In magnetostatics: $\boxed{\nabla\times\boldsymbol{H} = \boldsymbol{J}_{\rm free}}$ (M4)

Ampère's Theorem in Magnetic Media

- Integral form: $\boxed{\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{H} = I}$

- For many media (but **not** permanent magnets) $\boldsymbol{M}$ is proportional to $\boldsymbol{H}$.

- We define: $\boldsymbol{M} = \chi_{\rm m}\boldsymbol{H}$ (magnetic susceptibility)

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \mu_0(1 + \chi_{\rm m})\boldsymbol{H}$$

- Define also: $\boldsymbol{B} = \mu_{\rm r}\mu_0\boldsymbol{H}$ relative permeability

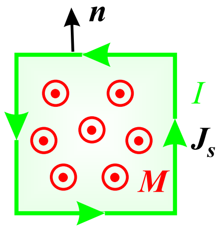
- Related via $\mu_{\rm r} = 1 + \chi$.

- We use $\mu_{\rm r}$ to describe the magnetic field, but use χ when we are interested in the properties of the medium.

- For most insulators we can assume that $\mu_{\rm r} \approx 1$.

Magnetisation Currents — Non-Examinable

- Alternative proof provides insight into the nature of magnetisation currents.



$$M_z \mathrm{d}x\mathrm{d}y\mathrm{d}z = I\mathrm{d}x\mathrm{d}y$$

$$I = J_\mathrm{s}\mathrm{d}z$$

- Magnetised block has currents circulating around its faces ($\boldsymbol{n}$ is unit normal to surface).
- Surface current density is $\boldsymbol{J}_\mathrm{s} = \boldsymbol{M}\times\boldsymbol{n}$.
- Suppose there are no free currents.

No overall current flow $\Rightarrow \oint |\mathrm{d}\boldsymbol{S}|\boldsymbol{J}_\mathrm{s} + \int \mathrm{d}\tau\, \boldsymbol{J}_\mathrm{m} = 0$

But $\oint |\mathrm{d}\boldsymbol{S}|\boldsymbol{J}_\mathrm{s} = \oint |\mathrm{d}\boldsymbol{S}|\boldsymbol{M}\times\boldsymbol{n} = -\oint \mathrm{d}\boldsymbol{S}\times\boldsymbol{M}$

$$-\oint \mathrm{d}\boldsymbol{S}\times\boldsymbol{M} + \int \mathrm{d}\tau\, \boldsymbol{J}_\mathrm{m} = 0$$

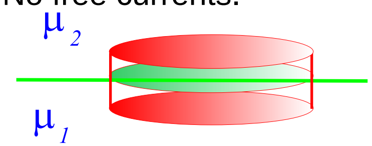
↓ Generalised divergence ↓

$$-\int \mathrm{d}\tau\, \nabla\times\boldsymbol{M} + \int \mathrm{d}\tau\, \boldsymbol{J}_\mathrm{m} = 0$$

$$\Rightarrow \boxed{\boldsymbol{J}_\mathrm{m} = \nabla\times\boldsymbol{M}}$$

Boundary Conditions for $\boldsymbol{B}$ and $\boldsymbol{H}$

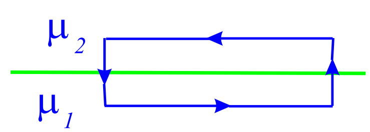
- No free currents.



- For $\boldsymbol{B}$ consider Gaussian 'pill-box' in plane of the boundary.

$$\nabla\cdot\boldsymbol{B} = 0 \overset{\text{Gauss}}{\Rightarrow} \oint \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B} = 0$$

$$\Rightarrow B_{1\perp} = B_{2\perp} \Rightarrow B_\perp \text{ is continuous}$$



- For $\boldsymbol{H}$ consider loop shown

$$\int \mathrm{d}\boldsymbol{S}\cdot\nabla\times\boldsymbol{H} = 0 \overset{\text{Stokes}}{\Rightarrow} \oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{H} = 0$$

$$\Rightarrow H_{1\parallel} = H_{2\parallel} \Rightarrow H_\parallel \text{ is continuous}$$

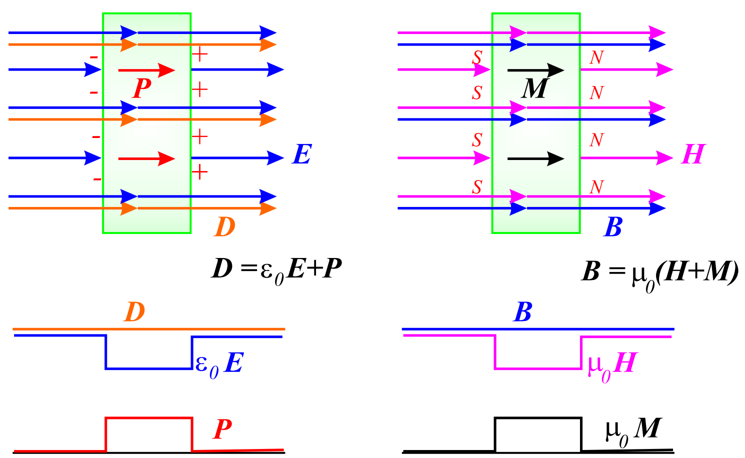
- For magnetic media, we now must define

$$\boxed{\boldsymbol{H} = -\nabla\phi_\mathrm{m}}$$

The potential ϕ_m is single-valued, so must be continuous at the boundary. This is the same condition as that on $H_\parallel$.

PROPERTIES OF $\boldsymbol{B}$ AND $\boldsymbol{H}$ — $\boldsymbol{E}$ AND $\boldsymbol{D}$ ANALOGY

- $\boldsymbol{B}$ is the fundamental field in magnetostatics.
- $\boldsymbol{B}$ gives the force on moving charges/currents.
- $\boldsymbol{H}$ is defined for mathematical convenience when considering media.
- Field lines of $\boldsymbol{B}$ are closed loops and never end.
- Lines of $\boldsymbol{H}$ can begin and end on “magnetic poles” — the North and South poles of magnetisation dipoles.
- Boundary conditions:
  $\boldsymbol{B}$ behaves like $\boldsymbol{D}$ since $B_\perp$ continuous.
  $\boldsymbol{H}$ behaves like $\boldsymbol{E}$ since $H_\parallel$ continuous.



METHODS OF CALCULATING $\boldsymbol{B}$ AND $\boldsymbol{H}$

There are four main tools.

(1) Simple circuits with high symmetry: apply Ampère’s theorem. $\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{H} = I$

(2) Along paths not intersecting loops, we can use the scalar potential. $\phi_\mathrm{m} = \dfrac{I\Omega}{4\pi}$

(3) We can always use the Biot-Savart law $\mathrm{d}\boldsymbol{B} = \dfrac{\mu_0 I \mathrm{d}\boldsymbol{l}\times\boldsymbol{R}}{4\pi R^3}$ and integrate around the current loops.

(4) Use the vector potential $\mathrm{d}\boldsymbol{A} = \dfrac{\mu_0 I \mathrm{d}\boldsymbol{l}}{4\pi R}$ and take the curl afterwards. This is often the best approach, since the integral is easier, and taking the curl is straightforward (!).

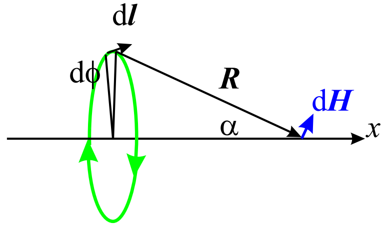
METHODS OF CALCULATING $\boldsymbol{B}$ AND $\boldsymbol{H}$ — EXAMPLE

(1) Long straight wire — revise.

(2) Long solenoid — revise: $B_z = nI$.

[n turns per unit length]

(3) Field on axis of a current loop — Biot-Savart method:



Current element gives:

$$\mathrm{d}\boldsymbol{H} = \frac{I\,\mathrm{d}\boldsymbol{l}\times\boldsymbol{R}}{4\pi R^3}.$$

After integration only the H_x component remains:

$$H_x = \frac{I}{4\pi R^2}\int_0^{2\pi} \mathrm{d}\phi\, a\sin\alpha = \frac{Ia\sin\alpha}{2R^2}$$

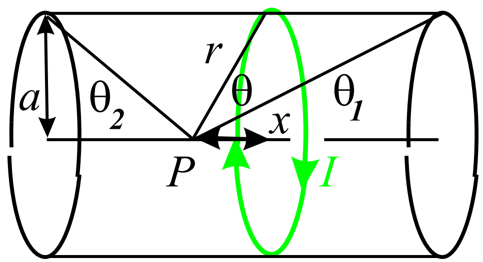
$$= \frac{Ia^2}{2(a^2+x^2)^{3/2}}$$

Alternative: evaluate ϕ_{m} on the axis = $\dfrac{I\Omega}{4\pi}$

$$\Omega = 2\pi\int_0^{\alpha}\mathrm{d}\theta\,\sin\theta = 2\pi(1-\cos\alpha)$$

and find $H_x = -\dfrac{\partial\phi_{\mathrm{m}}}{\partial x}$.

EXAMPLE — SHORT SOLENOID



Consider field at P due to loop at x:

$x = a\cot\theta$

$\mathrm{d}x = -a\mathrm{d}\theta/\sin^2\theta$

$r = a/\sin\theta$

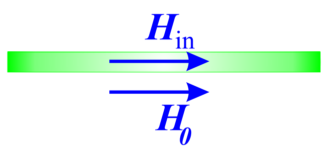
- n turns per unit length.
- Note P on negative side of loop as shown.

$$\mathrm{d}H_P = -\frac{I(n\mathrm{d}x)a\sin\theta}{2r^2} = \frac{nI}{2}\sin\theta\mathrm{d}\theta$$

$$H_P = \frac{nI}{2}\int_{\theta_1}^{\theta_2}\mathrm{d}\theta\,\sin\theta = \frac{nI}{2}(\cos\theta_1 - \cos\theta_2)$$

$\boldsymbol{B}$ and $\boldsymbol{H}$ Inside Magnetic Materials

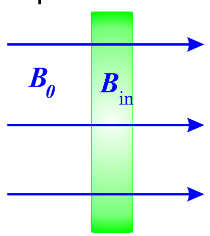
- As in electrostatics, field inside magnetic media placed in external field $\boldsymbol{H}_0$ depends on **shape** of body.
- Example: long thin rod



- Continuity of $H_{\parallel}$

$$\Rightarrow H_{\text{in}} = H_0$$
$$\Rightarrow B_{\text{in}} = \mu_r B_0$$

- Example: thin slab



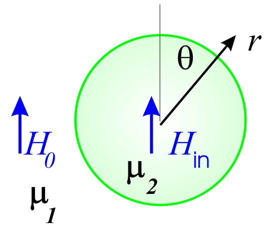
- Continuity of $B_{\perp}$

$$\Rightarrow B_{\text{in}} = B_0$$
$$\Rightarrow \mu_r H_{\text{in}} = H_0$$

- Other shapes are between these extremes: e.g. sphere, cylinder.
- In general, the field $\boldsymbol{H}_{\text{in}}$ is not parallel to $\boldsymbol{H}_0$ (tensor!).

Magnetisable Sphere in Uniform Field

- Use scalar potential ϕ_m in analogy with dielectric sphere.



Step 1: for $r < a$

$$\phi_m = -H_{\text{in}} r\cos\theta\ ;$$

for $r > a$

$$\phi_m = -H_0 r\cos\theta + \frac{A\cos\theta}{r^2}\ .$$

- Step 2: match boundary conditions.

(1) ϕ_m (or $H_{\parallel}$) $\quad \Rightarrow H_{\text{in}} = H_0 - \dfrac{A}{a^3}$

(2) $B_{\perp}$: use $B_{\perp} = -\mu_r\mu_0 \dfrac{\partial\phi_m}{\partial r}$ at $r = a$.

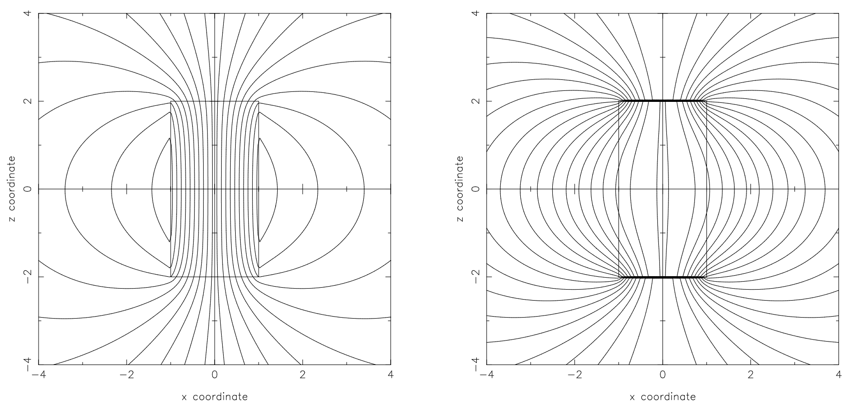
$$\mu_2 H_{\text{in}} = \mu_1\left(H_0 + \frac{2A}{a^3}\right)$$

- Step 3: solve for A, H_{in}:

$$A = \frac{a^3(\mu_2 - \mu_1)}{2\mu_1 + \mu_2} H_0\ ; \qquad H_{\text{in}} = \frac{3\mu_1}{2\mu_1 + \mu_2} H_0$$

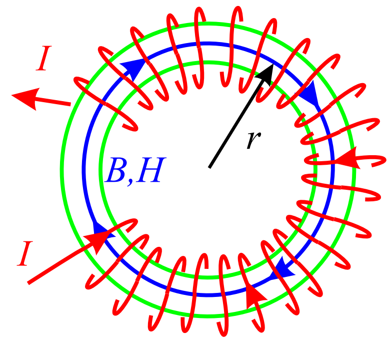
EXAMPLE — UNIFORMLY MAGNETISED CYLINDER

- Surface currents $\boldsymbol{J}_s = \boldsymbol{M}\times\boldsymbol{n}$, so the cylindrical bar magnet has the same $\boldsymbol{B}$ field as the short solenoid.
- To find the magnetic field everywhere (rather than just on the axis), it is better to calculate the $\boldsymbol{A}$ field first.



- Lines of $\boldsymbol{B}$ — "leak out" like short solenoid — continuous at the ends, but are kinked at the sides where they cross the surface currents.
- Lines of $\boldsymbol{H}$ originate at the North and end at the South magnetisation poles — antiparallel to $\boldsymbol{M}$, $\boldsymbol{B}$ in the centre of the magnet.

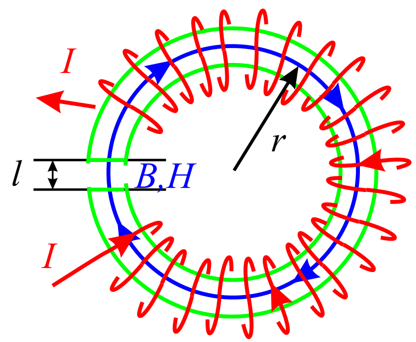
ELECTROMAGNETS



- Ring of high permeability μ_r: N turns in total; current I.
- Apply Ampère's theorem to path at radius r inside the material.

$$\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{H} = 2\pi r H_{in} = NI \Rightarrow H_{in} = \frac{NI}{2\pi r};\ B_{in} = \frac{\mu_r\mu_0 NI}{2\pi r}$$

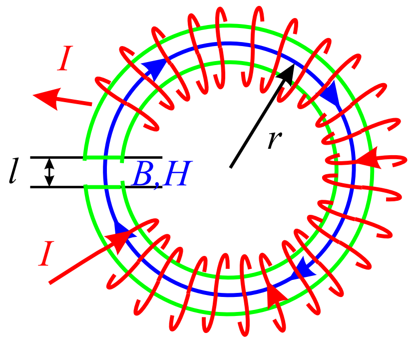
- Electromagnet will be used to apply $\boldsymbol{B}$ field to a sample of material $\Rightarrow$ need to make a small gap length l in the ring.



- $B_{gap} = B_{in}$ ($B_\perp$ continuous) $\Rightarrow \mu_r\mu_0 H_{in} = \mu_0 H_{gap}$
- Ampère's theorem:

$$\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{H} = (2\pi r - l)H_{in} + lH_{gap} = NI$$

Electromagnets II



- $B_{\text{gap}} = B_{\text{in}}$ ($B_\perp$ continuous)

  $\Rightarrow \mu_r \mu_0 H_{\text{in}} = \mu_0 H_{\text{gap}}$
- Ampère's theorem:

  $NI = (2\pi r - l)H_{\text{in}} + lH_{\text{gap}}$

$$\Rightarrow H_{\text{gap}} = \frac{\mu_r NI}{2\pi r + (\mu_r - 1)l}$$

$$\Rightarrow B_{\text{gap}} = \mu_0 H_{\text{gap}} = \frac{\mu_r \mu_0 NI}{2\pi r + (\mu_r - 1)l}$$

- For practical electromagnets $\mu_r \gg 1$ and $\mu_r l \gg r$

$$\Rightarrow B_{\text{gap}} \approx \frac{\mu_0 NI}{l}$$

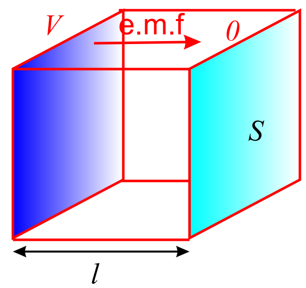
- In this limit (highly magnetisable material) it is as if *all* the path is in the gap — the rest is 'shorted out'.

Resistance and Conductivity — Reminder

- Resistor: Ohm's law $\quad V = IR$.

  V is the 'electromotive force' (e.m.f.)

  R is the 'resistance'.



- Volume of material faces of area S distance l apart.

- Apply e.m.f. across faces of material:

(1) current $\propto V$;

(2) current $\propto S$;

(3) current $\propto 1/l$;

$$I = \frac{VS\sigma}{l}; \qquad R = \frac{l}{S\sigma}$$

- σ is the *conductivity* of the material.
- $V/l = |\boldsymbol{E}|$, so in general we have the constitutive relation

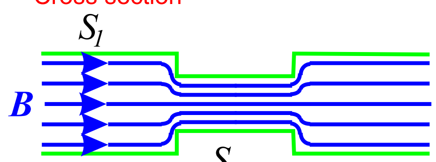
$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

Magnetic Circuits

- We can draw an analogy between electric circuits and "magnetic circuits".

Electric circuit	Magnetic circuit
ΔV = electromotive force = $\int d\boldsymbol{l}\cdot\boldsymbol{E}$	NI = "magnetomotive force" = $\int d\boldsymbol{l}\cdot\boldsymbol{H}$

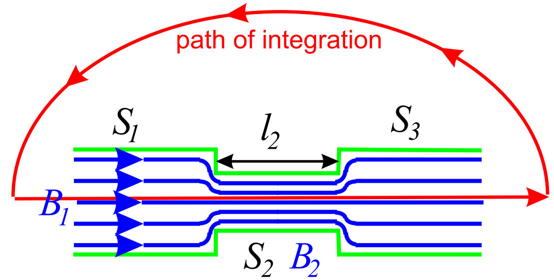
- Suppose that μ_r is so high that no field lines of $\boldsymbol{B}$ can leak out.



- Flux $\Phi = SB$ = constant = $S\mu_r\mu_0 H$

$$\Rightarrow H = \frac{\Phi}{\mu_r\mu_0 S}$$

Magnetic Circuits II



- Ampère: $NI = \oint d\boldsymbol{l}\cdot\boldsymbol{H} = \frac{\Phi}{\mu_0}\sum_{\text{loop}}\frac{l_i}{\mu_i S_i}$

$$\Phi = NI/\sum\frac{l_i}{\mu_0\mu_i S_i}$$

- Compare with D.C. circuit with resistances R_i in series:

$$I = V/\sum R_i$$

- Analogy: $\Phi \leftrightarrow I$; $NI \leftrightarrow V$; $\sum R_i \leftrightarrow \sum\frac{l_i}{\mu_0\mu_i S_i}$.
- Quantity analogous to resistance R_i is

magnetic reluctance: $\frac{l_i}{\mu_i\mu_0 S_i}$

- Note: $R_i = \frac{l_i}{\sigma_i S_i}$ (σ_i = conductivity).

Magnetic Circuits — D.C. Analogy

Electric circuit	Magnetic circuit
$\int_{\text{closed surface}} \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{J} = 0$	$\int_{\text{closed surface}} \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B} = 0$
ΔV = electromotive force = $\int \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{E}$	NI = magnetomotive force = $\int \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{H}$
$\boldsymbol{J} = \sigma\boldsymbol{E}$	$\boldsymbol{B} = \mu_{\mathrm{r}}\mu_0\boldsymbol{H}$
Resistance = $\dfrac{l}{\sigma S}$	Reluctance = $\dfrac{l}{\mu_{\mathrm{r}}\mu_0 S}$