Part IB Advanced Physics
Dr. S.F. Gull

Michaelmas Term 1999 (last 12 lectures)
Lent Term 2000 (first 12 lectures)
M, W, F: 12.00 noon

ELECTROMAGNETISM

Synopsis

Introduction:

Maxwell's equations; Lorentz force; revision of grad, div and curl; divergence theorem, Stokes' theorem.

Electrostatics:

Coulomb's law; $\boldsymbol{E}$ and V; Gauss' theorem; Laplace and Poisson equations; electric dipoles; uniqueness theorem; conducting sphere in $\boldsymbol{E}$ field, method of images; point charge near conducting sphere, line charge near conducting cylinder, capacitance of parallel cylinders; electrostatic energy; force on charged conductor; capacitors and batteries. Isotropic dielectrics; polarisation charges; Gauss' theorem; permittivity and susceptibility; properties of $\boldsymbol{D}$ and $\boldsymbol{E}$; boundary conditions at dielectric surfaces; relationship between $\boldsymbol{E}$ and $\boldsymbol{P}$; thin slab in field, dielectric sphere in field; images in semi-infinite dielectrics; local fields inside dielectrics; Clausius–Mossotti equation.

Magnetostatics:

Forces between current elements; Gauss' theorem; dipoles; magnetic scalar potential; Ampère's theorem; magnetic vector potential. Magnetic media; magnetisation; permeability and magnetic susceptibility; properties of $\boldsymbol{B}$ and $\boldsymbol{H}$; boundary conditions at surfaces; methods of calculating $\boldsymbol{B}$ and $\boldsymbol{H}$, magnetisable sphere in a uniform field; electromagnets; magnetic circuits*; diamagnetism; paramagnetism; Langevin function; ferromagnetism, Curie–Weiss law; domains.

Changing electromagnetic fields:

Electromagnetic induction, Faraday's law; magnetic energy; self-inductance; inductance of long solenoid, coaxial cylinders, parallel cylinders; mutual inductance; transformers; displacement current; Maxwell's equations; electromagnetic waves; plane waves in isotropic media; energy flow; Poynting theorem; radiation pressure and momentum; insulating media; plasmas; conditions above and below plasma frequency; evanescent fields; conducting media; skin effect.
Transmission lines; characteristic impedance; coaxial, parallel-wire, strip transmission lines; power flow; terminated lines; matching; reflection and transmission coefficients; impedance of short terminated lines; impedance matching.
Waveguides; $\boldsymbol{E}$ and $\boldsymbol{H}$, σ and $\boldsymbol{J}$; TE and TM modes; geometrical treatment; dispersion relation; waveguide equation; cut-off frequency; characteristic impedance; phase and group speeds.

* The Christmas Vacation will occur about here.

BOOKS

Bleaney, B.I. & Bleaney, G.: Electricity and Magnetism: OUP (two volumes). This is an excellent and classic text which will see you through Part IB and Part II.

Grant, I.S. & Phillips, W.R.: Electromagnetism: Wiley. This treatment is at about the level of the course. It is easier to read than Bleaney & Bleaney, but does not go so far.

Solymar, L.: Lectures on electromagnetic theory: OUP. This is a good exposition of the development of macroscopic electromagnetism from Maxwell's equations as axioms.

Duffin, W.J.: Electricity and Magnetism: McGraw Hill. Start here if you find electromagnetism a mystery.

A Summary of Formulae

(See also the Appendix at the back)

Maxwell's equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

$$\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Lorentz force:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

Coulomb's law:

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon\epsilon_0}\frac{q_1 q_2}{r^3}\boldsymbol{r}$$

Force on current element:

$$d\boldsymbol{F} = I\, d\boldsymbol{s}\times\boldsymbol{B}$$

Force between current elements:

$$d\boldsymbol{F}_2 = \frac{\mu\mu_0 I_1 I_2}{4\pi r^3}\, \boldsymbol{s}_2\times(d\boldsymbol{s}_1\times\boldsymbol{r})$$

Biot–Savart law:

$$d\boldsymbol{B} = \frac{\mu\mu_0 I}{4\pi r^3}\, d\boldsymbol{s}\times\boldsymbol{r}$$

Electrostatic energy:

$$W = \tfrac{1}{2}\int \mathrm{d}\tau\, \rho V$$

Energy density:

$$U = \tfrac{1}{2}(\boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{B}\cdot\boldsymbol{H})$$

Poynting vector:

$$\boldsymbol{N} = \boldsymbol{E}\times\boldsymbol{H}$$

Electrostatic potential:

$$\boldsymbol{E} = -\boldsymbol{\nabla} V$$

Laplace's equation:

$$\nabla^2 V = 0$$

Poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon\epsilon_0}$$

Magnetic scalar potential:

$$\boldsymbol{H} = -\boldsymbol{\nabla}\phi_{\text{m}}$$

$$\phi_{\text{m}} = \frac{I\Omega}{4\pi}$$

Magnetic vector potential:

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$$

Polarisation charge density:

$$\rho_{\text{p}} = -\boldsymbol{\nabla}\cdot\boldsymbol{P}$$

Displacement:

$$D = \epsilon_0\boldsymbol{E} + \boldsymbol{P} = \epsilon\epsilon_0\boldsymbol{E}$$

Clausius–Mossotti: $$\frac{\epsilon - 1}{\epsilon + 2} = \frac{n\alpha}{3\epsilon_0}$$

Magnetic flux density and magnetic field strength: $$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \mu\mu_0\boldsymbol{H}$$

Magnetisation current density: $$J_{\mathrm{m}} = \boldsymbol{\nabla}\times\boldsymbol{M}$$

Boundary conditions:

$D_\perp$ and $E_{||}$ continuous

$B_\perp$ and $H_{||}$ continuous

Faraday's law: $$\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{E} = -\frac{\partial}{\partial t}\int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B}$$

boundary $\longleftrightarrow$ surface

Self inductance: $$\Phi = LI$$

Mutual inductance: $$\Phi_1 = MI_2; \quad \Phi_2 = MI_1$$

Electromagnetic waves: $$\nabla^2\boldsymbol{E} = \mu\mu_0\epsilon\epsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2}$$

$$\nabla^2\boldsymbol{B} = \mu\mu_0\epsilon\epsilon_0\frac{\partial^2\boldsymbol{B}}{\partial t^2}$$

Impedance: $$Z = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}$$

Refractive index: $$n = \sqrt{\epsilon\mu}$$

Transmission line: $$Z = \sqrt{\frac{L}{C}}$$

Waveguide: $$\omega^2/c^2 = k_x^2 + k_y^2 + k_z^2$$

Advice on Answering Physics Problems

- **Always draw a diagram.**
  A diagram helps clarify your thoughts and define the symbols you use. Make it big enough.
- **Stay in symbols until the end.**
  This enables you to check the dimensions at every stage. An exception to this rule arises occasionally where some terms may be dimensionless factors (e.g. γ in relativity) and work out as simple fractions.
- **Check the dimensions.**
  The answer is definitely wrong if the dimensions do not balance.
- **Check the arithmetic.**
- **Be tidy.**
  If no-one can decipher your scratchings then you have wasted your time.
- **Don't be satisfied with your first attempt.**
  Even if you managed to do the problem first time, you will learn a lot by thinking about it again. Your work will also be neater and more organised.

Other obvious tips include writing down the question number, stating your assumptions, and including the units of the answer at the end.

ELECTROMAGNETISM

Electrostatics

1. Charges q are placed at the four corners of a square of side $\sqrt{2}a$. A particle of charge q and mass m is placed at the centre of the square, and constrained to move only in the plane of the square. Show that, if displaced slightly from the centre and released, it performs simple harmonic motion of angular frequency

$$\omega = \sqrt{\frac{q^2}{2\pi\epsilon_0 a^3 m}} \, .$$

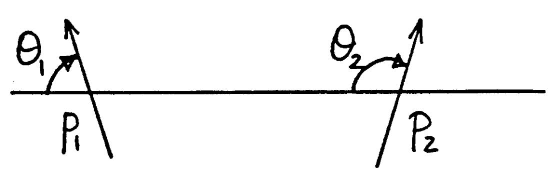
2. A small dipole centred at the origin of coordinates has Cartesian vector components $(0, 0, p)$, where $p = qa$, q is the charge, and a the separation. By differentiating the expression for the potential at point (x, y, z), find the x, y and z components of the electric field when $(x^2 + y^2 + z^2) \gg a^2$.

3. Two coplanar identical small electric dipoles of moment p are supported on pivots a large distance d apart. Each dipole can rotate only in the plane. Their angles of twist, θ_1 and θ_2, are measured clockwise from the line joining their centres as shown in the diagram. Show, by treating each dipole as the sum of two components at right angles, that the potential energy of one in the field of the other is

$$-\frac{p^2}{8\pi\epsilon_0 d^3} \left(3\cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2)\right) .$$

Hence explain how the dipoles would move if released from rest in each of the following configurations:

(a) $\theta_1 = \theta_2 = 0$;
(b) $\theta_1 = \theta_2 = \pi/2$;
(c) $\theta_1 = \pi/2$; $\theta_2 = -\pi/2$;
(d) $\theta_1 = 0$; $\theta_2 = \pi$.



4. The electrical system of a typical thundercloud can be represented by a vertical dipole with a charge of +40 C at a height of 10 km and a charge of −40 C vertically below it at a height of 6 km. What is the electric field at the ground immediately below the thundercloud and at what distance from there is the field at the ground zero? (Treat the ground as a perfect conductor.)

5. Two point charges q and $-q$ are a distance $2d$ apart. Show that, if a conducting sphere of small radius a $(a \ll 2d)$ is placed midway between the two charges, the force on each is increased by a factor of approximately $(1 + 16(a^3/d^3))$.

6. Two infinite conducting plane sheets meet at an angle of 60°. A particle of mass m, carrying a charge q, is constrained to move on the plane bisecting this angle. Obtain expressions for

   (a) the force on the particle when it is a distance x from the line on which the sheets meet,

   (b) the time taken to reach the intersection when it is released from rest at $x = a$.

7. Calculate the electrostatic energy of a charge Q uniformly distributed throughout a sphere of radius a. Hence calculate a "classical radius" of the electron on the dubious assumption that its rest mass is due to the electrostatic energy.

8. Two isolated spherical conductors of radii 30 and 90 mm are charged to 1.5 and 3 kV respectively. They are then connected by a fine resistive wire. How much heat is generated in the wire if its resistance is sufficient to overdamp the system?

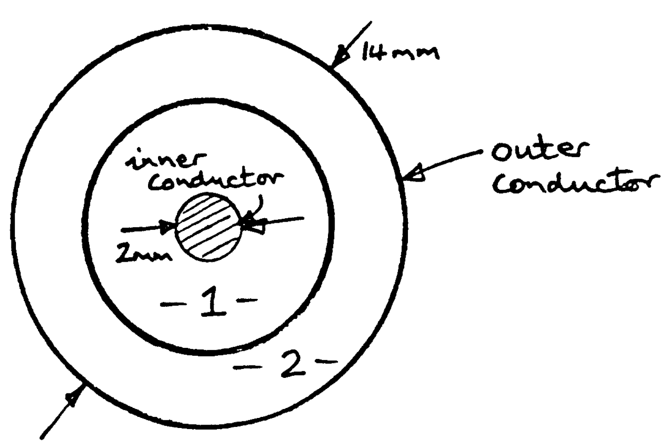
9. Derive the expression for the pressure on a charged surface of a conductor by considering the energy of a charged parallel-plate capacitor when the capacitor is

   (a) at constant charge,

   (b) at constant voltage.

10. Two conducting discs of radius r are arranged one above the other with their faces horizontal. The lower disc is fixed and the upper is suspended a distance $a \ll r$ above it by springs. When a potential difference V is applied, the equilibrium separation becomes b. Show that the system is stable in the second case only if $b > 2a/3$.

11. A coaxial cable is constructed using two dielectrics as shown in the figure. Dielectric 1 occupies one sixth of the total area between the conductors, whose radii are 1 and 7 mm. The relative permittivities of the dielectrics are $\epsilon_1 = 5$ and $\epsilon_2 = 3$. Calculate the capacitance per unit length of the cable.



12. A planar slab of a dielectric material for which $\epsilon = 2$ has a disc-shaped cavity in its interior. Both slab and cavity are much wider than they are thick, and the normal to the plane of the cavity is at 45° to that of the slab. The slab is placed in a uniform electric field E_0. If the field inside the cavity is E_1, what is the ratio E_1/E_0 when the direction of E_0 is parallel to the normal to the plane of the cavity?

13. Show that an arbitrary charge distribution in a semiconductor of conductivity σ and permittivity ϵ decays with time constant $\tau = \epsilon\epsilon_0/\sigma$.
    [τ is known as the *dielectric relaxation time*.]

14. An infinite plane boundary separates a dielectric medium of relative permittivity ϵ from vacuum. A charge q is placed in the vacuum a distance d from the boundary. Using the method of images, determine the electrostatic potential in the two regions and show that the charge q experiences a force of attraction to the boundary given by

$$F = \left(\frac{q}{2d}\right)^2 \frac{(\epsilon - 1)}{4\pi\epsilon_0(\epsilon + 1)}.$$

15. Show that a dielectric sphere placed in a uniform field E_0 acquires a dipole moment per unit volume

$$P = \frac{3(\epsilon - 1)\epsilon_0 E_0}{\epsilon + 2},$$

where ϵ is the relative permittivity of the dielectric.

An inhomogeneous dielectric medium consists of many such spheres embedded at random in a matrix of relative permittivity 1. The spheres occupy a fraction $f(\ll 1)$ of the total volume. Use the Clausius–Mossotti theory to describe how the field acting on a sphere depends on the polarization of all the others. Hence show that the bulk relative permittivity of the inhomogeneous medium is

$$\epsilon_{\mathrm{m}} = \frac{(\epsilon + 2) + 2f(\epsilon - 1)}{(\epsilon + 2) - f(\epsilon - 1)}.$$

16. Outline the derivation of the Clausius–Mossotti equation

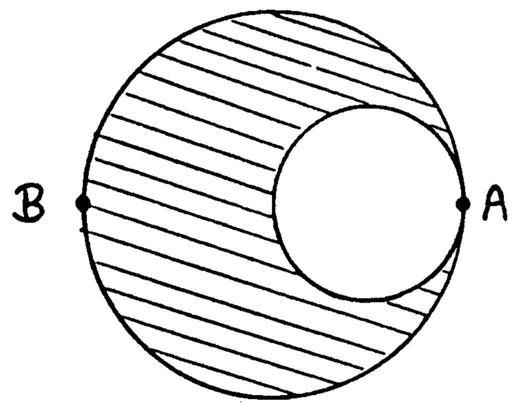
$$\alpha = \frac{3\epsilon_0(\epsilon - 1)}{N(\epsilon + 2)},$$

which relates atomic polarizability α, number density N and relative permittivity ϵ.

At 77 K the relative permittivity of liquid nitrogen is 1.43616 and the density is 808 kg m^{-3}. The corresponding values of gaseous nitrogen at the same temperature are 1.002155 and 4.50 kg m^{-3}. Comment on these results in the light of the theory and derive a value for the molecular polarizability.

Magnetostatics

17. A long wire having the cross-section shown below carries a steady current. What is the ratio of the magnetic fields at A and B?



18. A cylindrical column of mercury of radius a carries a current I, uniformly distributed over its cross-section. Derive a formula for the pressure p as a function of radius r in the column. If $a = 5$ mm and $I = 100$ A, what is the difference between the pressure at the centre and that on the edge of the column?

19. A short magnetic dipole, of moment $\boldsymbol{m}$, is placed on, and aligned with, the axis of a long solenoid of radius a and having N turns per unit length. If the current in the solenoid is I, calculate the force on the dipole when it is outside the solenoid and at a distance x_0 from the end of the solenoid.

20. Sketch on separate diagrams the field lines of $\boldsymbol{B}$ and $\boldsymbol{H}$ inside and outside a uniformly magnetised bar magnet.

21. A paramagnetic sphere of radius a and relative permeability μ is placed in a vacuum in a uniform magnetic field $\boldsymbol{B}$. Show that the total magnetic moment $\boldsymbol{M}$ induced in the sphere is given by

$$\mu_0 \boldsymbol{M} = \frac{4\pi a^3(\mu - 1)}{\mu + 2}\boldsymbol{B}.$$

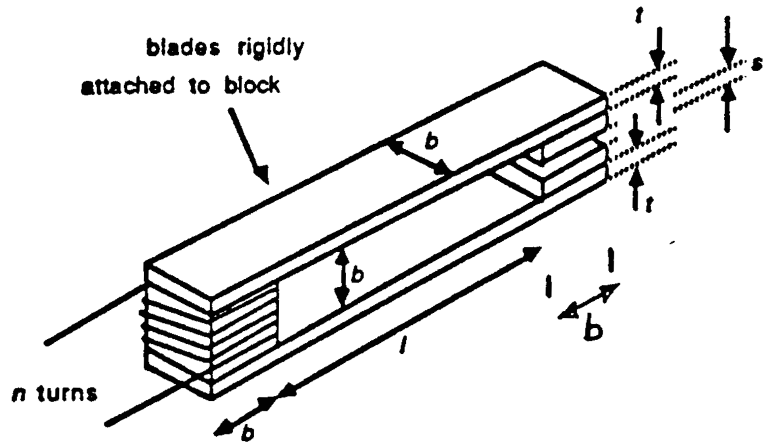
If two such spheres are a distance d apart ($d \gg a$) along the direction of $\boldsymbol{B}$, what is the force between them?

22. A solid metal sphere is placed in an alternating uniform magnetic field of amplitude H_0. Eddy currents in the surface of the sphere result in zero flux density B inside the sphere. Show that the sphere acquires an alternating magnetisation M of magnitude $3H_0/2$.

A small solid aluminium sphere is situated 0.1 m above, and on the axis of, a horizontal coil carrying a high-frequency alternating current. If the coil has 100 turns and a diameter of 0.2 m, what current amplitude is sufficient to support the sphere?

[Density of aluminium $\approx 2.7\times10^3$ kg m^{-3}.]

23. An electromagnet is constructed out of high permeability iron in the form shown in



the figure.

The force F required between the pole-pieces to bend the blades and reduce the gap from s_0 to s is given by

$$F \approx \frac{bt^3(s_0 - s)E}{8l^3},$$

where E is the Young modulus. When the magnet is energised, the attractive magnetic force between the pole-pieces is $B^2/2\mu_0$ per unit area, where B is the induction gap. Show that:

(a) when a current I flows through the coil, the magnetic force on the pole-pieces is given by

$$F' \approx \frac{\mu_0 n^2 I^2 b^2}{2s^2},$$

where n is the number of turns in the coil;

(b) the system becomes unstable so that the pole-pieces snap together when $s = 2s_0/3$.

Hence find an expression for the critical current at which this occurs.

24. The magnetic moment of an electron in the ground state of the hydrogen atom is 1 Bohr magneton. Compare this with the moment induced by a field of 1 T.

25. Show that the following data for the magnetic susceptibility of nickel are consistent with the Curie–Weiss law.

T/°C :	500	600	700	800	900
$\chi/10^{-5}$:	38.4	19.5	15.0	10.6	9.73

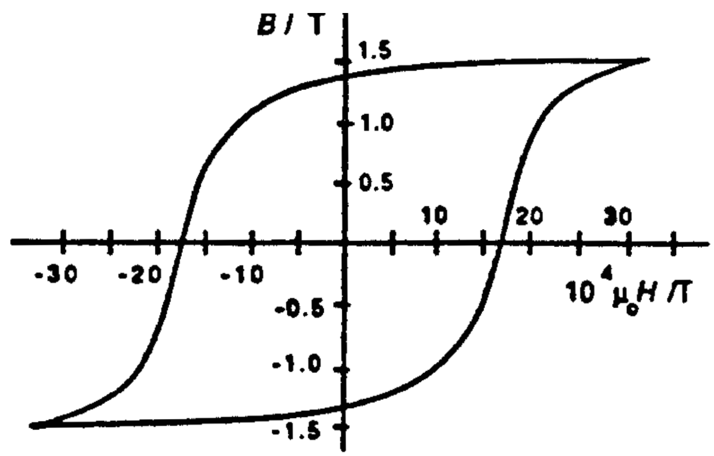
Deduce the Curie constant, the Curie temperature and the effective number of Bohr magnetons per atom.

[Relative atomic mass = 58.7; density = 8.85×10^3 kg m^{-3}.]

26. Tungsten steel with the hysteresis loop shown in the figure is to be used to make a permanent magnet. The magnet is formed by bending a rod 60 cm long into a ring with a small air gap. The shape of the hysteresis loop for $\boldsymbol{B} > 0, \boldsymbol{H} < 0$ may be assumed to be an ellipse, with semi-major axes along the $\boldsymbol{B}$ and $\boldsymbol{H}$ axes. Neglecting flux leakage, show that the $\boldsymbol{B}$ field in the air gap after magnetisation is approximately given by

$$B/T = \frac{1.4}{\sqrt{1 + Aa^2}} \, ,$$

where a is the width of the air gap and A is approximately 2 mm^{-2}.



If a coil of 100 turns is wound on the ring, what current would be required to reduce the field in the gap to zero?

Varying electromagnetic fields, inductance and energy

27. A pendulum consists of a conducting sphere of radius r and density ρ suspended by a fine (conducting) wire of length $l(\gg r)$ from an earthed pivot. The pendulum swings with a small amplitude in a vertical plane. Show that a uniform horizontal magnetic field B applied at right angles to this plane changes the period of the pendulum by one part in $8\rho r^2/3\epsilon_0 l^2 B^2$. Does the period increase or decrease?

28. A circular loop of wire of density ρ and conductivity σ rotates freely about a diameter which is perpendicular to a magnetic field B. Show that the speed of rotation decays with time constant $\tau = 4\rho/B^2\sigma$.

    A spherical aluminium satellite in a geostationary orbit spins about an east-west axis in the earth's field of 120 nT. Estimate the fraction by which the speed of rotation decreases in a year because of the field.
    $[\rho = 2.7\times10^3 \text{ kg m}^{-3}; \ \sigma = 3.8\times10^7 \ \Omega^{-1} \text{ m}^{-1}.]$

29. An alternating potential difference $V_0 \exp(-i\omega t)$ is applied to a parallel-plate capacitor of capacitance C whose electrodes are two circular discs of radius a. Find an expression for the amplitude of the magnetic field between the plates. Compare the field to that of the earth (50 μT) when $V_0 = 300$ V, $\omega = 10^4$ rad s^{-1}, $a = 300$ mm and the plates are 1 mm apart.

30. Calculate the mutual inductance of two coaxial single-turn circular loops of wire of radii r and $r'(\ll r)$ when their centres are a distance x apart. Hence find the flux linking a circular loop of radius r when a small magnetic dipole of moment m is aligned with and on its axis at a distance x from the centre.

If the dipole approaches from a large distance and passes through the loop at a constant speed u, show that the instantaneous voltage V induced in the loop is given by $V = 3\mu_0 m u r^2 x / 2(x^2 + r^2)^{5/2}$. Sketch the form of this function of x.

It is suggested that individual neutrons might be detected by observing the voltage induced in a small loop of wire by their magnetic fields as they pass through. Discuss the feasibility of this technique.

[The magnetic moment of the neutron is 9.7×10^{-27} J T^{-1}.]

31. An audio-frequency transformer, designed to operate at frequencies as low as 20 Hz, is used to transform the 5 Ω resistance of a secondary load into a reflected resistance that equals the 75 Ω output resistance of the voltage generator attached to the primary coil. The magnetic core of the transformer is a toroid of average radius 5 cm, cross-sectional area 30 cm^2, and relative permeability $\mu = 500$. Calculate the minimum number of primary and secondary turns that must be wound on the core so as to achieve impedance matching, whilst having a primary inductance that presents an impedance of at least 150 Ω.

[You may assume perfect coupling and zero coil resistance.]

32. A long paramagnetic circular rod of diameter 5 mm is suspended from a balance so that its lower end is between the poles of an electromagnet that produces a horizontal field. When a field of 1 T is switched on, the apparent mass of the rod increases by 1.5 g. What is the magnetic susceptibility of the material?

33. Two long solenoids of radius 20 mm wound with 2 turns mm^{-1} are placed end to end, nearly touching. When 1 A passes through both solenoids in the same sense, what is the force between them? Is it attractive or repulsive?

Electromagnetic waves

34. Estimate the electric field strength 100 km from a 100 kW television transmitter.

35. Calculate the energy flux, energy density and r.m.s. electric field of sunlight

    (a) at the surface of the Earth,

    (b) just above the surface of the Sun.

[Assume that the Sun behaves as a black body with surface temperature 6000 K. Its angular diameter as seen from the Earth is half a degree. The Stefan–Boltzmann constant is 5.67×10^{-8} W m^{-2} K^{-4}.]

36. Plane monochromatic waves are propagating in free space parallel to the x-axis in both positive and negative directions. At the origin the field strengths are given by

$$\begin{aligned} E_z &= E_0 \cos \omega t, & E_y &= 3E_0 \cos \omega t, \\ H_z &= -(E_0/Z_0) \cos \omega t, & H_y &= -(E_0/Z_0) \cos \omega t. \end{aligned}$$

Find the amplitudes and polarisations of the forward and backward travelling waves, and the net energy flux.

37. Show that spherical dust grains within the Solar System which have a radius less than a critical radius r will be repelled by the Sun, whilst those with a radius greater than r will be attracted towards it. Calculate r given that the mass of the Sun is 2×10^{30} kg, its luminosity is 4×10^{26} W and the density of the dust grains is of order 10^3 kg m^{-3}. (Neglect the pressure exerted by the Solar Wind.)

38. The electron density in the ionosphere reaches a maximum value of 1.5×10^{12} m^{-3}. What range of frequencies could be used for radio control of a satellite?

39. Show that the refractive index n of a plasma is given by

$$n^2 = 1 - \frac{Ne^2}{\epsilon_0 m_e \omega^2},$$

where N is the number density of free electrons, m_e the electron mass and ω the angular frequency of the electromagnetic radiation. Use this result to derive the dispersion relation for the radiation and show that the product of the phase and group velocities is equal to c^2.

A pulsar emits pulses of radio waves containing a wide range of frequencies. The interval between the times of arrival at the Earth of pulses at 400 MHz and 200 MHz is found to be 4 s. The mean electron density in interstellar space is known to be about 3×10^4 m^{-3}. What is the distance to the pulsar? What is the lowest frequency at which such observations could be made from the ground?

40. Show that the wave impedance Z of a good conductor at low frequencies to an incident wave $E_0 \exp i(kx - \omega t)$ can be expressed as $Z = (1-i)a$ with $a^2 = \omega \mu \mu_0 / 2\sigma$ where σ is the conductivity. Hence show that the power reflection coefficient is given approximately by

$$R \approx 1 - 2\sqrt{\frac{2\epsilon_0 \omega}{\sigma}}\ .$$

Calculate the electrical conductivity of aluminium given that the power reflection coefficient for 5 μm radiation incident normally is 97%.

[Assume $\mu = 1$.]

41. Radiation is incident normally at the plane surface of a good conductor. Derive an equation for the variation of the electric field with time and depth below the surface. Comment on the flow of energy in this situation.

It is possible to communicate with submerged submarines using very low frequency radio waves. If the frequency used is 16 kHz and the receiver has sufficient sensitivity to operate satisfactorily with electric fields whose amplitude is 100 times less than that just under the surface, to what depth can a submarine remain in radio contact?

[Sea water has $\sigma \approx 4\ \Omega^{-1}\ \mathrm{m}^{-1}$, $\epsilon = 80$ and $\mu = 1$. Assume the radiation is incident vertically.]

42. (a) A radio-frequency current flows in a wire of conductivity σ and radius $a \gg \delta$, the skin depth. Calculate the power dissipated in the wire and show that the effective resistance is as if all the current were flowing uniformly in a layer of thickness δ.

(b) Show also that when there is an oscillating tangential magnetic field of strength H_0 at a plane surface of a conductor, the energy dissipation per unit area is $H_0^2/2\sigma\delta$.

(c) Show that $H_0^2/2\sigma\delta$ is also the mean value of the Poynting vector at the surface.

Transmission lines and waveguides

43. A coaxial transmission line consists of a cylindrical conductor of radius 1 mm and an outer conductor chosen to make the characteristic impedance 75 Ω. The space between the conductors is filled with a gas which can stand a maximum field of 10^5 V m^{-1} without dielectric breakdown. Estimate the maximum mean radio frequency power that can be transmitted along this line into a matching load.

44. An annular sheet 1 mm thick of a material of resistivity 0.5 Ω m connects the inner and outer conductors of an air-spaced coaxial transmission line at a point on the line. A low frequency signal is fed into one end of the line and the other is terminated by its characteristic impedance. Calculate the relative amplitude of the wave reflected from the resistive sheet.

45. A length of lossless transmission line is first short-circuited at one end then open-circuited; the impedance measured at the other end is Z_1 in the first case and Z_2 in the second. Show that $Z_1 Z_2 = Z_0^2$, where Z_0 is the characteristic impedance of the line. (This is a convenient way of measuring Z_0 for a cable of unknown electrical length.)

46. An open-ended quarter-wavelength, air-spaced, parallel-wire transmission line is found to be in resonance with an oscillator when its length is 0.25 m. When a capacitance of 1 pF is connected across the open end, it is found that the length of the line must be reduced to 0.125 m to obtain resonance. Show that the characteristic impedance of the line is approximately 530 Ω.

47. A film of magnesium fluoride (MgF_2) (refractive index = 1.38) one-quarter wavelength thick is deposited on crown glass (refractive index = 1.52). Show that the reflected intensity of light is reduced to about 1 per cent.

Alternate layers, each $\lambda/4$ thick, of transparent media with refractive indices n_1, n_2, are deposited onto a substrate of refractive index n_0. With x layers of each material, show that for incident light the surface appears to have a refractive index $n_0(n_2/n_1)^{2x}$. If $n_0 = 1.52, n_1 = 1.38$, and $n_2 = 2.35$ (ZnS), how many layers are needed to produce reflection of at least 99.99 per cent of the incident energy?

48. An electromagnetic wave is propagated in the x direction in a simple rectangular waveguide of width a and height b. The waveguide is operating in the TM_{11} mode and the y component of the magnetic field $\boldsymbol{B}$ is given by

$$B_y = B_0 \frac{\pi}{b} \cos\left(\frac{z}{b}\right) \sin\left(\frac{y}{a}\right) \exp i(k_x x - \omega t)$$

where

$$k_x = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2 - \left(\frac{\pi}{b}\right)^2}$$

B_0 is a constant and the origin of the coordinate system is taken at an inside corner.

(a) Calculate all the components of $\boldsymbol{B}$ and $\boldsymbol{E}$, and demonstrate that they satisfy the proper boundary conditions.

(b) Sketch the instantaneous configurations of $\boldsymbol{B}$ and E_x in a length of the waveguide.

(c) Show that the TM_{11} mode will not propagate if $\omega < c\sqrt{(\pi/a)^2 + (\pi/b)^2}$.

(d) Show that the absolute cutoff frequency for TE modes in this waveguide is lower than for the TM mode.

49. A square waveguide of side 110 mm is used to transmit waves whose free space wavelength is 100 mm. Enumerate the different modes of propagation and calculate the wavelength in the guide for each. Draw sketches to show the configurations of the fields in the modes which have the shortest and longest λ_g. Sketch also the lines of current flow in the waveguide.

[Remember to distinguish TE and TM modes.]

50. It is proposed to transmit television signals requiring a bandwidth of 8 MHz by means of a waveguide operating at 15 GHz. If the cut-off frequency of the guide is 12 GHz, what range of group velocities will be present in the signal?

Answers to problems

(Worked solutions will be issued at lectures 12 and 24)

2. $\{x, y, z(1 - r^2/3z^2)\}(3pz/4\pi\epsilon_0 r^5)$, where $r^2 = x^2 + y^2 + z^2$.

4. 12.8 kV m^{-1} upwards; 11.0 km.

6. (a) A/x^2; (b) $\pi(ma^3/8A)^{1/2}$, where $A = (5 - 4/\sqrt{3})q^2/16\pi\epsilon_0$.

7. $3Q^2/20\pi\epsilon_0 a$; 1.7×10^{-15} m.

8. 2.8×10^{-6} J.

9. $\sigma^2/2\epsilon_0$, where σ is the surface charge density.

11. 111 pF m^{-1}.

12. 1.52.

16. $(\epsilon - 1)/\rho(\epsilon + 2)$ is constant within about 2%, despite the large change in ρ; $\alpha = 1.95\times10^{-40}$ F m^2.

17. 3/5.

18. $P_0 + \mu_0 I^2(a^2 - r^2)/4\pi^2 a^4$; 12.7 Pa.

19. $\mu_0 NmIa^2/2(x_0^2 + a^2)^{3/2}$.

21. $24a^6\pi B^2(\mu - 1)^2/\mu_0(\mu + 2)^2 d^4$.

22. 173 A r.m.s.

23. $I_c = \sqrt{s_0^3 t^3 E/27 l^3 \mu_0 n^2 b}$.

24. The ratio is about 2×10^{-6}.

25. 4.5×10^{-2} K; 660 K; 0.44.

26. About 8 A.

28. About 1 part in 10^3.

29. $V_0\omega C/2\pi\epsilon_0 c^2 a$; ratio is 10^{-4}.

30. Flux linked is $\mu_0 r^2 m/2(x^2 + r^2)^{3/2}$.

31. $n_1 = 446$; $n_2 = 115$.

32. 1.9×10^{-3}.

33. 3.2×10^{-3}N.

34. About 50 mV m^{-1}, depending on your assumption about the radiation pattern of the antenna.

35. (a) 1.4 kW m^{-2}; 730 V m^{-1} r.m.s.; 4.7×10^{-6} J m^{-3}.

(b) 74 MW m^{-2}; 235 kV m^{-1} r.m.s.; 0.49 J m^{-3}.

36. Forward wave $\sqrt{2}E_0$ at 45° to y axis; backward wave $2E_0$ parallel to y axis; mean flux density E/Z_0 along $-x$.

37. 600 nm.

38. > 11 MHz.

39. 5.3×10^{19} m; lowest frequency $\approx$ 11 MHz (see Q. 38) if pulsar is in the zenith, otherwise higher.

40. 2.9×10^{7} Ω^{-1} m^{-1}.

41. 9.2 m.

43. Maximum mean power is 104 W.

44. 0.27.

47. 10 layers are needed.

48.

$$\{B_x, B_y, B_z\}/B_0 = \left\{0, \frac{\pi}{b}\cos\frac{\pi z}{b}\sin\frac{\pi y}{a}, -\frac{\pi}{a}\sin\frac{\pi z}{b}\cos\frac{\pi y}{a}\right\} \exp i(k_x x - \omega t);$$

$$E_x/B_0 = i\frac{c^2}{\omega}\left(\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right)\sin\frac{\pi y}{a}\sin\frac{\pi z}{b} \exp i(k_x x - \omega t),$$

$$E_y/B_0 = -\frac{\pi k_x c^2}{a\omega}\cos\frac{\pi y}{a}\sin\frac{\pi z}{b}\frac{c^2}{\omega} \exp i(k_x x - \omega t),$$

$$E_z/B_0 = -\frac{\pi k_x c^2}{b\omega}\sin\frac{\pi y}{a}\cos\frac{\pi z}{b} \exp i(k_x x - \omega t).$$

49. TE_{01}, TE_{10} : $\lambda_g = 0.11$ m; TE_{02}, TE_{20} : $\lambda_g = 0.24$ m; TM_{11}, TE_{11} : $\lambda_g = 0.13$ m.

50. 1.7×10^{5} m s^{-1}

[Please advise me of any errors, misprints or omissions in these answers — Steve Gull (steve@mrao.cam.ac.uk)].

APPENDIX

Electromagnetic Formulae

The following pages and index have been kindly supplied by Dr. Graham Woan (now at the University of Glasgow) as an appendix to the course. They have been extracted from his forthcoming book entitled "A Cambridge book of physics formulae". Some of the formulae go beyond the scope of Part IB electromagnetism, but you may find this appendix useful in the future.