Electromagnetism — Lectures 1-12

- Introduction 2 lectures.
  Vector calculus, revision.
- Electrostatics 6 lectures.
  Potential theory;
  dielectrics;
  microscopic origin of polarisation.
- Magnetostatics 4 lectures.
  magnetic fields from currents;
  magnetic media.

Summary Sheet — Introduction — S1

Electromagnetism:
Summarised by
Maxwell's equations.

(M1) $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}}$

(M2) $\boldsymbol{\nabla}\times\boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$

(M3) $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$

(M4) $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \dfrac{\partial \boldsymbol{D}}{\partial t}$

and Lorentz force $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$

Revise vectors: $\boldsymbol{a}\cdot\boldsymbol{b} \quad \boldsymbol{a}\times\boldsymbol{b}$

Vector indentities: $\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c} = \boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{a} = -\boldsymbol{c}\times\boldsymbol{b}\cdot\boldsymbol{a}$

Important one: $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{c}\,\boldsymbol{b} - \boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{c}$

$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{a}\cdot\boldsymbol{c}\,\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{c}\,\boldsymbol{a}$

Vector calculus: $\boldsymbol{\nabla}\Phi \quad \boldsymbol{\nabla}\cdot\boldsymbol{E} \quad \boldsymbol{\nabla}\times\boldsymbol{E} \quad \boldsymbol{\nabla}^2\Phi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\Phi$

Divergence theorem

$$\oint \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{E} = \int \mathrm{d}\tau\,\boldsymbol{\nabla}\cdot\boldsymbol{E}$$

surface ↔ volume

Stokes' theorem

$$\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{E} = \int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{\nabla}\times\boldsymbol{E}$$

perimeter ↔ surface

Conservation of charge $\dfrac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$

Summary Sheet — Electrostatics — S2

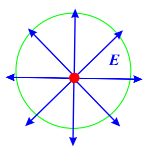
Electrostatics: $\boldsymbol{B}, \boldsymbol{H}, \boldsymbol{J}$ all zero; $\frac{\partial}{\partial t} = 0$

Maxwell's equations:

$$\nabla \cdot \boldsymbol{D} = \rho_{\text{free}}$$
$$\nabla \times \boldsymbol{E} = 0$$

Electric field: $\boldsymbol{E}$ = 'force per unit charge'.

Electric potential: $V(\boldsymbol{r})$ $\boldsymbol{E} = -\nabla V$.



Field of a point charge: $\boldsymbol{E} = \frac{q}{4\pi\epsilon_0 r^3}\boldsymbol{r}$

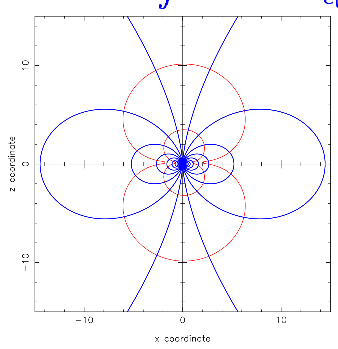
Potential of a point charge: $V = \frac{q}{4\pi\epsilon_0 r}$

$\boldsymbol{E}$ is conservative $\Rightarrow \nabla \times E = 0$.

Electric field lines perpendicular to **Equipotential surfaces**

(lines of force)

Gauss' theorem $\oint \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{E} = \frac{1}{\epsilon_0}\int \mathrm{d}\tau\,\rho \Rightarrow \nabla\cdot\boldsymbol{E} = \rho/\epsilon_0$

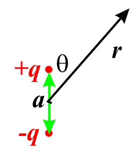


Couple on dipole: $\boldsymbol{G} = \boldsymbol{p}\times\boldsymbol{E}$

Energy of dipole: $U = -\boldsymbol{p}\cdot\boldsymbol{E}$

Summary Sheet — Electrostatics — S3

Dipoles: Dipole moment: $\boldsymbol{p} = q\boldsymbol{a}$.



Potential: $V = \frac{|\boldsymbol{p}|\cos\theta}{4\pi\epsilon_0 r^2}$

In constant $\boldsymbol{E}$ field:

no force; couple: $\boldsymbol{G} = \boldsymbol{p}\times\boldsymbol{E}$; energy: $U = -\boldsymbol{p}\cdot\boldsymbol{E}$.

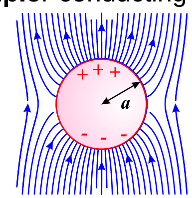
In non-uniform $\boldsymbol{E}$ field: force $\boldsymbol{F} = \boldsymbol{p}\cdot\nabla\,\boldsymbol{E} = \overset{*}{\nabla}(\boldsymbol{p}\cdot\overset{*}{\boldsymbol{E}})$ $(= -\nabla U)$ (care!)

Potential theory: $\nabla^2 V = -\rho/\epsilon_0$ Poisson's equation $\nabla^2 V = 0$ Laplace's equation

Uniqueness Theorem:

Justifies trickery to find solution of potential problem.

Example: conducting sphere in uniform field.



Potential outside: $\left(-E_0 r + \frac{p}{4\pi\epsilon_0 r^2}\right)\cos\theta$

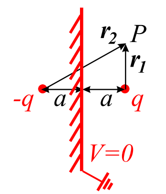
(uniform + dipole)

Satsifies B.C. $V = 0$ at $r = a$,

provided $p = 4\pi\epsilon_0 a^3 E_0$. $\Rightarrow (E_\theta = 0, E_r = 3E_0\cos\theta)$ at $r = a$

$\Rightarrow$ surface charge $3\epsilon_0 E_0 \cos\theta$.

Method of images



- Easy way of solving problems involving plane conductors — use symmetry.

Potential: $V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

Summary Sheet — Images — S4

Method of images Conducting planes are easy[a]: just like mirrors.

Conducting spheres and cylinders Learn these basic facts.

(It's much more difficult to show $V = 0$ at $r = a$ if you don't remember where the image charge is.)

Sphere

Radius of sphere a, point charge at b.

Uncharged sphere: add charge qa/b at centre.

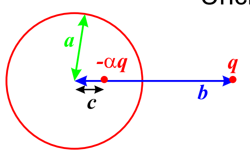


Image at $c = a^2/b$

Charge: $-qa/b$

Cylinder

Radius of cylinder a, line charge at b.

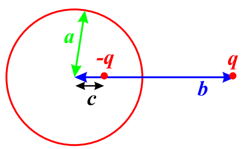


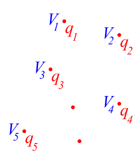
Image at $c = a^2/b$

Charge: $-q$

It works if the charge is **inside** the conductor, too (image is outside).

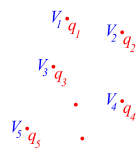
Electrostatic energy

Energy of a set of charges q_i at potentials V_i is $\frac{1}{2}\sum_i q_i V_i$.



[a] Or impossible. . .

Summary Sheet — Electrostatic Energy — S5



Electrostatic energy of a set of charges q_i at potentials V_i

$$U = \tfrac{1}{2}\sum_i q_i V_i \to \tfrac{1}{2}\int \mathrm{d}\tau\, \rho V .$$

Depends on state of system — not how it was set up.

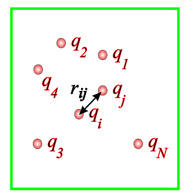
Alternative View: energy is localised in the $\boldsymbol{E}$ field.

Energy density: $U_E = \frac{1}{2}\epsilon_0|\boldsymbol{E}|^2$. (In general $U = \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D}$.)

These forms for $U = \frac{1}{2}\int \mathrm{d}\tau\, \rho V = \frac{1}{2}\int \mathrm{d}\tau\, \epsilon_0|\boldsymbol{E}|^2$ look very different, but give the same answer in electrostatics when integrated over all space.

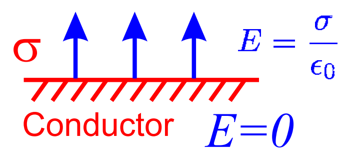
Energy of assembly of charges

Explains factor of 2 another way: $U = \displaystyle\sum_{i,j<i} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$



No self-energy of point charge.

Consistency: $V_i = \displaystyle\sum_{i \neq j}\left(\frac{q_j}{4\pi\epsilon_0 r_{ij}}\right)$



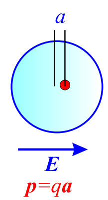
- Charges at surface of conductor feel a force $\boldsymbol{F} = \dfrac{\sigma^2}{2\epsilon_0}$

Work is done by: (1) forces moving conductors $\mathrm{d}W = F\mathrm{d}x$;

(2) by batteries maintaining fixed potentials $\mathrm{d}W = V\mathrm{d}Q$.

Dielectrics: insulators become **polarised** in electric field.

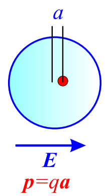
Induced dipole: $\boldsymbol{p}$ per atom; $\boldsymbol{P} = N\boldsymbol{p}$ per unit volume.

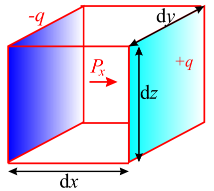


Summary Sheet — Dielectrics — S6

Dielectrics: insulators become **polarised** in electric field.

Induced dipole: $\boldsymbol{p}$ per atom; $\boldsymbol{P} = N\boldsymbol{p}$ per unit volume.

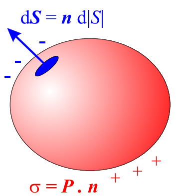




- Element of polarisation $\boldsymbol{P}\,\mathrm{d}\tau$ has surface charge density on faces

  $\sigma = \boldsymbol{P}\cdot\boldsymbol{n}$

Polarisation charge density ρ_p due to variations in $\boldsymbol{P}$.



- No net charge (no free charges)

$$\Rightarrow \oint |\mathrm{d}\boldsymbol{S}|\,\sigma + \int \mathrm{d}\tau\,\rho_p = 0$$

$$\Rightarrow \boxed{\rho_p = -\boldsymbol{\nabla}\cdot\boldsymbol{P}}$$

Gauss in dielectrics: $\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{1}{\epsilon_0}(\rho_{free} + \rho_p) \Rightarrow \boldsymbol{\nabla}\cdot(\epsilon_0\boldsymbol{E} + \boldsymbol{P}) = \rho_{free}$

Define Electric Displacement:

$\boxed{\boldsymbol{D} \equiv \epsilon_0\boldsymbol{E} + \boldsymbol{P}}$ $\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{free}}$ (M1)

Isotropic dielectric medium (linear): $\boldsymbol{D} = \epsilon_r\epsilon_0\boldsymbol{E}$ ϵ_r = permittivity

$(\epsilon_r = 1 + \chi)$ $\boldsymbol{P} = \chi\epsilon_0\boldsymbol{E}$ χ= susceptibility.

- Field lines of $\boldsymbol{D}$ begin and end on free charges; $\boldsymbol{E}$ determines the force on charge and is the fundamental field.

At boundaries: (1) $D_\perp$ is continuous; (2) $E_{||}$ and V are continuous.

[Learn these boundary conditions.]

Summary Sheet — Dielectric Media— S7

Define Electric Displacement for medium with polarisation density $\boldsymbol{P}$:

$\boxed{\boldsymbol{D} \equiv \epsilon_0\boldsymbol{E} + \boldsymbol{P}}$ $\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{free}}$ (M1)

Isotropic dielectric medium (linear): $\boldsymbol{D} = \epsilon_r\epsilon_0\boldsymbol{E}$ ϵ_r = permittivity

$(\epsilon_r = 1 + \chi)$ $\boldsymbol{P} = \chi\epsilon_0\boldsymbol{E}$ χ= susceptibility.

At boundaries: (1) $D_\perp$ is continuous; (2) $E_{||}$ and V are continuous.

[Learn these boundary conditions.]

Dielectric sphere in uniform field: use potential theory.

Step 1: propose trial solution (ansätz): $V_{in} = -E_{in} r\cos\theta$ $(r < a)$

$$V_{out} = -E_0 r\cos\theta + \frac{A\cos\theta}{r^2} \quad (r > a)$$

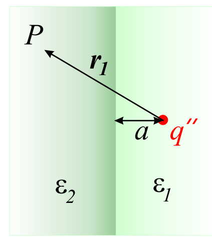
Step 2: match V (or $E_{||}$) at $r = a \Rightarrow E_{in} = E_0 - A/a^3$

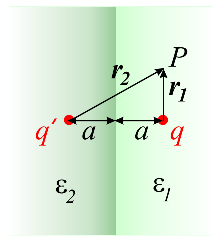
match $D_\perp$ at $r = a \Rightarrow \epsilon_r E_{in} = E_0 + 2A/a^3$

Step 3: solve for A, E_{in}.

Similar methods work for cylinders and some other geometries.

Images in dielectrics





Step 1: propose trial solution:

for $x < a$ use q'' at original position;

for $x > a$ use image q'.

Step 2: match V and $D_\perp$ at $x = 0$

Step 3: solve for $q'' = \frac{2\epsilon_1}{\epsilon_1 + \epsilon_2}q$; $q' = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}q$

Method of images works for line charge outside (or inside) cylinder, the case of a point charge outside a sphere needs more general approach.

SUMMARY SHEET — POLARISATION IN DIELECTRICS— S8

Energy in linear dielectrics ($\boldsymbol{D} \propto \boldsymbol{E}$)

Two different forms: $U = \frac{1}{2}\int \mathrm{d}\tau\, \rho_{\text{free}} V = \frac{1}{2}\int \mathrm{d}\tau\, \boldsymbol{E}\cdot\boldsymbol{D}$.

(For non-linear dielectric media, need to integrate; dissipative media show hysteresis.)

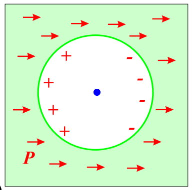
Local fields in dielectrics.

For molecule: $\boldsymbol{p} = \alpha \boldsymbol{E}_{\text{local}}$. Bulk polarisation: $\boldsymbol{P} = N\boldsymbol{p}$.

Macroscopic field: $\boldsymbol{E}(\epsilon_r - 1)\epsilon_0 = \boldsymbol{P}$.

- Macroscopic $\boldsymbol{E}$ contains contribution from every molecule,
  so $\boldsymbol{E}_{\text{local}} \neq \boldsymbol{E}$ since each molecule cannot polarise itself.
- Lorentz showed that the contributions from molecules in sphere surrounding molecule of interest cancel exactly.
- Polarisation of rest of medium outside sphere

(of any radius a) increases field: $\boldsymbol{E}_{\text{local}} = \boldsymbol{E} + \dfrac{\boldsymbol{P}}{3\epsilon_0}$



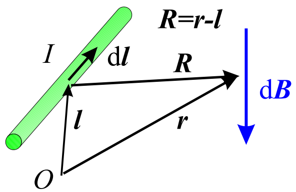
$\Rightarrow \boxed{\dfrac{\epsilon_r - 1}{\epsilon_r + 2} = \dfrac{N\alpha}{3\epsilon_0}}$ (Clausius-Mossotti relation)

Magnetostatics: no charges; no electric fields. Steady currents; time-independent magnetic field.

Force on current element: $\mathrm{d}\boldsymbol{F} = I\mathrm{d}\boldsymbol{l}\times\boldsymbol{B}$ ($\boldsymbol{B}$ is magnetic flux density).

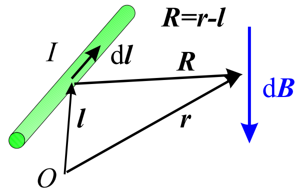
Magnetic field generated by currents.

Biot-Savart Law: $\mathrm{d}\boldsymbol{B} = \dfrac{\mu_0}{4\pi}\dfrac{I\mathrm{d}\boldsymbol{l}\times\boldsymbol{R}}{R^3}$



SUMMARY SHEET — MAGNETOSTATICS— S9

Magnetostatics: no charges; no electric fields. Steady currents; time-independent magnetic field.



Force on current element: $\mathrm{d}\boldsymbol{F} = I\mathrm{d}\boldsymbol{l}\times\boldsymbol{B}$.

Magnetic field generated by currents.

Biot-Savart Law: $\mathrm{d}\boldsymbol{B} = \dfrac{\mu_0}{4\pi}\dfrac{I\mathrm{d}\boldsymbol{l}\times\boldsymbol{R}}{R^3}$

No sources of $\boldsymbol{B}$ (free magnetic monopoles) $\Rightarrow \boxed{\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0}$ (M3)

Elementary dipoles: current loops have dipole field: $\boxed{\mathrm{d}\boldsymbol{m} = I\mathrm{d}\boldsymbol{S}}$

In constant $\boldsymbol{B}$ field:

no force; couple: $\boldsymbol{G} = \boldsymbol{m}\times\boldsymbol{B}$; energy: $U = -\boldsymbol{m}\cdot\boldsymbol{B}$.

Non-uniform $\boldsymbol{B}$ field: force $\boldsymbol{F} = \boldsymbol{m}\cdot\boldsymbol{\nabla}\,\boldsymbol{B} = \overset{*}{\boldsymbol{\nabla}}\,(\boldsymbol{m}\cdot\overset{*}{\boldsymbol{B}})$ $(= -\boldsymbol{\nabla}U)$ (care!)

Generalisation of Gauss' and Stokes' Theorems (Non-examinable)

Gauss: $\oint \mathrm{d}\boldsymbol{S}\,(\bullet) = \int \mathrm{d}\tau\,\boldsymbol{\nabla}\,(\bullet) \Rightarrow \begin{array}{l}\oint \mathrm{d}\boldsymbol{S}\Phi = \int \mathrm{d}\tau\,\boldsymbol{\nabla}\Phi\\ \oint \mathrm{d}\boldsymbol{S}\times\boldsymbol{A} = \int \mathrm{d}\tau\,\boldsymbol{\nabla}\times\boldsymbol{A}\end{array}$

Stokes: $\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{A} = \int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} = \int (\mathrm{d}\boldsymbol{S}\times\boldsymbol{\nabla})\cdot\boldsymbol{A}$

$\Rightarrow$ Generalised Stokes:

$\oint \mathrm{d}\boldsymbol{l}\,(\bullet) = \int (\mathrm{d}\boldsymbol{S}\times\boldsymbol{\nabla})\,(\bullet) \Rightarrow \begin{array}{l}\oint \mathrm{d}\boldsymbol{l}\Phi = \int \mathrm{d}\boldsymbol{S}\times\boldsymbol{\nabla}\Phi\\ \oint \mathrm{d}\boldsymbol{l}\times\boldsymbol{A} = \int (\mathrm{d}\boldsymbol{S}\times\boldsymbol{\nabla})\times\boldsymbol{A}\end{array}$

Various higher-tech proofs for current loops.

Magnetostatics: $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$; $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J} \Rightarrow \boldsymbol{\nabla}\times\boldsymbol{B} = 0$ except at currents $\Rightarrow$ can sometimes define

Magnetic scalar potential ϕ_m. $\boxed{\boldsymbol{B} = -\mu_0\boldsymbol{\nabla}\phi_m}$ (better $\boldsymbol{H} = -\boldsymbol{\nabla}\phi_m$)

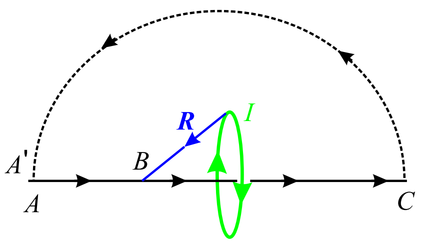
SUMMARY SHEET — MAGNETOSTATICS — S10

Magnetostatics: $\nabla\cdot\boldsymbol{B}=0$; $\nabla\times\boldsymbol{H}=\boldsymbol{J} \Rightarrow \nabla\times\boldsymbol{B}=0$ except at currents ⇒ can sometimes define

Magnetic scalar potential ϕ_m. $\boldsymbol{B}=-\mu_0\nabla\phi_m$ (better $\boldsymbol{H}=-\nabla\phi_m$)

Magnetic vector potential of current loop: $\phi_m = \dfrac{I\Omega}{4\pi}$

[Ω is the solid angle of the loop = $\displaystyle\int \frac{d\boldsymbol{S}\cdot\boldsymbol{R}}{R^3}$.]



$$\Omega(A') = \Omega(A) - 4\pi$$

$$\Rightarrow \int_A^{A'} d\phi_m = -I = -\frac{1}{\mu_0}\oint d\boldsymbol{l}\cdot\boldsymbol{B}$$

$$\Rightarrow \oint d\boldsymbol{l}\cdot\boldsymbol{B} = \mu_0 I \quad \text{(Ampère)} \qquad \nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} \quad \text{(M4)}$$

Ampère's theorem ⇒ $\boldsymbol{B} = \dfrac{\mu_0 I}{2\pi r}\hat{\boldsymbol{u}}_\phi$ for current-carrying wire.

Scalar potential is mathematical fiction, but is sometimes useful for magnetic media or problems where currents are confined to wires: e.g. wire: $\phi_m = -\dfrac{I\phi}{2\pi}$. [$\phi$ is azimuthal angle.]

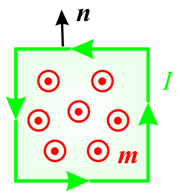
Magnetic vector potential $\boldsymbol{A}$ is the real physical field. $\boldsymbol{A}$ forms relativistic 4-vector with V and generates $\boldsymbol{B}$ via $\nabla\times\boldsymbol{A}=\boldsymbol{B}$.

Magnetic media have magnetisation density $\boldsymbol{M}$ per unit volume.

Magnetisation currents due to variation of $\boldsymbol{M}$: $\boldsymbol{J}_m = \nabla\times\boldsymbol{M}$.

SUMMARY SHEET — MAGNETOSTATICS — S11

Magnetic media have magnetisation density $\boldsymbol{M}$ per unit volume.



Magnetised block has surface current density

$$\boldsymbol{J}_s = \boldsymbol{M}\times\boldsymbol{n}.$$

Magnetisation current $\boldsymbol{J}_m$ due to variation of $\boldsymbol{M}$.

No free currents ⇒ $\oint d|\boldsymbol{S}|\,\boldsymbol{J}_s + \int d\tau\,\boldsymbol{J}_m = 0 \Rightarrow \boldsymbol{J}_m = \nabla\times\boldsymbol{M}$.

Define magnetic field strength: $\boldsymbol{H} \equiv \dfrac{1}{\mu_0}(\boldsymbol{B}-\mu_0\boldsymbol{M})$

Magnetostatics:

$$\nabla\times\boldsymbol{H} = \boldsymbol{J}_{free} \quad \text{(Maxwell 4)}$$
$$\oint d\boldsymbol{l}\cdot\boldsymbol{H} = I \quad \text{(Ampère)}$$

For many media (but NOT for permanent magnets) $\boldsymbol{M} \propto \boldsymbol{H}$
$\boldsymbol{M} = \chi\boldsymbol{H}$; $\boldsymbol{B} = \mu_r\mu_0\boldsymbol{H}$ $\Rightarrow \mu_r = 1+\chi$

At boundaries:

$\boldsymbol{B}_\perp$ is continuous
$\boldsymbol{H}_\parallel$ (or ϕ_m) is continuous

Properties of $\boldsymbol{B}$ and $\boldsymbol{H}$

$\boldsymbol{B}$ is the fundamental field (force on moving charges): $\nabla\cdot\boldsymbol{B}=0$

$\boldsymbol{H}$ lines can begin and end on magnetisation poles (North poles are equivalent of positive magnetic charges).

Examples: current loop: solenoid; (Use of Biot-Savart)
magnetisable sphere (potential theory).

$\boldsymbol{B}$ and $\boldsymbol{H}$ inside materials depend on shape of body.