ELECTROMAGNETISM — LECTURES 12-24

- Microscopic aspects of magnetism – magnetic media
  2 lectures.
- Electromagnetic Induction
  $2\frac{1}{2}$ lectures.
- Electromagnetic Waves in free space
  2 lectures.
- Electromagnetic Waves in plasmas and metals
  $1\frac{1}{2}$ lectures.
- Transmission Lines
  2 lectures.
- Waveguides and resonant cavities
  2 lectures.

SUMMARY SHEET — MAGNETOSTATICS — S12

- For magnetic media: $\boldsymbol{H} = \frac{1}{\mu_0}(\boldsymbol{B} - \mu_0 \boldsymbol{M})$
- Magnetostatics:
  $\nabla \times \boldsymbol{H} = \boldsymbol{J}_{\text{free}}$ (Maxwell 4)
  $\oint d\boldsymbol{l} \cdot \boldsymbol{H} = I$ (Ampère)
- For many media (but NOT for permanent magnets!) $\boldsymbol{M} \propto \boldsymbol{H}$

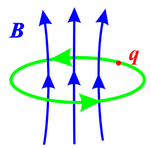
$$\left.\begin{array}{l} \boldsymbol{M} = \chi \boldsymbol{H} \\ \boldsymbol{B} = \mu\mu_0 \boldsymbol{H} \end{array}\right\} \Rightarrow \mu = 1 + \chi$$

- At boundaries:
  $\boldsymbol{B}_\perp$ is continuous
  $\boldsymbol{H}_{||}$ (or ϕ_{m}) is continuous
- $\boldsymbol{B}$ is the fundamental field: $\nabla \cdot \boldsymbol{B} = 0$
- $\boldsymbol{H}$ lines can begin and end on magnetisation poles
  (North poles are equivalent of positive magnetic charges).
- Examples: current loop: solenoid; (Use of Biot-Savart)
  magnetisable sphere (potential theory).
- $\boldsymbol{B}$ and $\boldsymbol{H}$ inside materials depend on shape of body.

SUMMARY SHEET — S13

Microscopic origin of magnetic behaviour

- **Diamagnetism**:



$$\boldsymbol{m} = g\frac{q}{2m_\mathrm{q}}\boldsymbol{J}$$

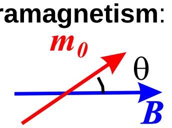
(g = gyromagnetic ratio)

All materials show magnetisation *opposed* to the applied field.

Due to perturbation of electron orbits.

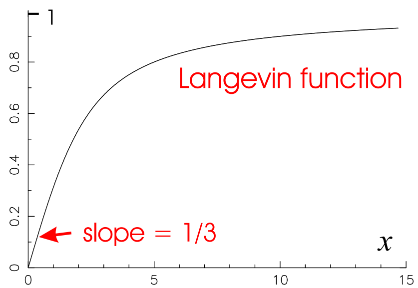
$$\frac{\Delta\omega}{\omega_0} = \tfrac{1}{2}\omega_\mathrm{L} = \frac{eB}{2m_\mathrm{e}}; \quad \frac{\Delta r}{r_0} = 0 \Rightarrow \langle\Delta m\rangle = \frac{e^2\langle r_0^2\rangle B}{6m_\mathrm{e}}$$

- **Paramagnetism**: Atoms with permanent dipole moment $\boldsymbol{m}_0$.



- Dipoles line up in the $\boldsymbol{B}$ field.

$$\Pr(\theta)d\theta \propto \tfrac{1}{2}\sin\theta d\theta \exp\left(\frac{m_0 B\cos\theta}{kT}\right)$$



$$\Rightarrow \frac{\langle m_\parallel\rangle}{m_0} = \langle\cos\theta\rangle$$

$$= L(x) = \coth(x) - \frac{1}{x}$$

$$x = \frac{m_0 B}{kT}$$

- **Curie's Law** $\quad \dfrac{\langle m_\parallel\rangle}{m_0} \approx \dfrac{m_0 B}{3kT}$

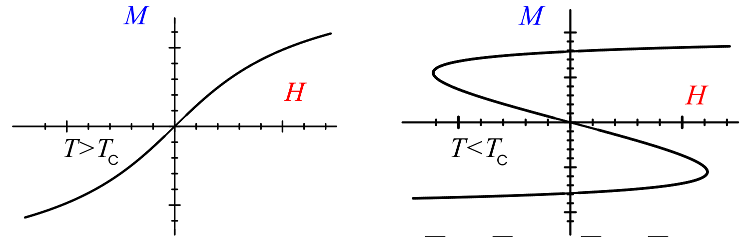
$$\chi = \frac{C}{T}$$

SUMMARY SHEET — S14

Paramagnetism and Ferromagnetism

- **Weiss theory**: cooperative quantum effects — Weiss constant λ.

  Use $\boldsymbol{B}_\mathrm{local} = \mu_0(\boldsymbol{H} + \lambda\boldsymbol{M})$ in Langevin function ($\lambda \approx 10^3$ for Fe)



- Qualitatively different behaviour for $T > T_\mathrm{c}$ and $T < T_\mathrm{c}$.
- Modified paramagnetic law for temperatures above T_c

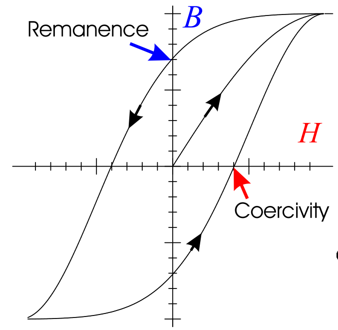
$$\chi = \frac{C}{T - \lambda C} = \frac{C}{T - T_\mathrm{c}}$$

Curie-Weiss Law

$$C = \frac{m_0^2 n\mu_0}{3k}; \qquad T_\mathrm{c} = \lambda C$$

For iron $\mu_0 n m_0 \approx$ 2 T; $C \approx$ 1 K; $T_\mathrm{c} = 1043$ K.

- Spontaneous magnetisation for temperatures below T_c.
- Domain structure of ferromagnetic materials.



- Permanent magnets.
- Magnetic hystersis curves.
- Wide range of coercivity.
- Remanence usually $\gg \mu_0 \times$ coercivity.

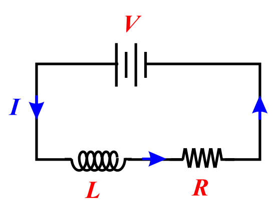
SUMMARY SHEET — S15

Faraday's Law of Electromagnetic induction

- Electromotive force $\mathcal{E} = -\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial t}$(magnetic flux).
- Fixed loop: $\oint d\boldsymbol{l}\cdot\boldsymbol{E} = -\int d\boldsymbol{S}\cdot\frac{\partial \boldsymbol{B}}{\partial t}$

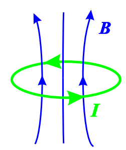
$$\Rightarrow \nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ Maxwell 2

- **Self inductance:** $\Phi = LI$

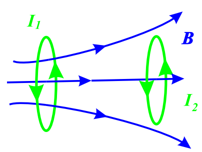


$$V = RI + L\frac{\partial I}{\partial t}$$

Magnetic energy: $\frac{1}{2}LI^2$



- **Mutual inductance:** $\Phi_2 = M_{21} I_1$



Symmetry: $M_{21} = M_{12} = M$

(Reciprocity Theorem)

Energy: $U_{\mathrm{M}} = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2$

SUMMARY SHEET — S16

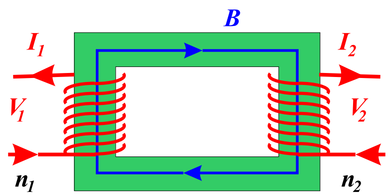
Inductance and Transformers

- Magnetic energy of coupled inductances:

$$U_{\mathrm{M}} = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 + M I_1 I_2$$

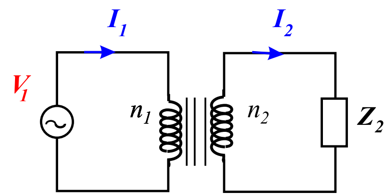
- Mutual inductance $M = k(L_1 L_2)^{1/2} \quad (0 \le k \le 1)$.
- Transformer: $\frac{V_1}{V_2} = \frac{n_1}{n_2}$

$$\frac{L_1}{L_2} = \left(\frac{n_1}{n_2}\right)^2$$



- Example: $V_1 \propto e^{j\omega t}$

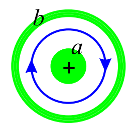
$$\frac{V_1}{I_1} = \frac{1}{\frac{1}{j\omega L_1} + \frac{1}{Z_2 (n_1/n_2)^2}}$$



- Self inductances:

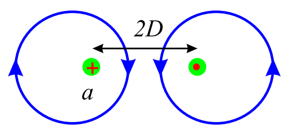
Long solenoid: $L = \mu_0 N^2 S/l$

Coaxial cylinders: $L = \frac{\mu_0}{2\pi}\log(b/a)$ per unit length



Two long wires:

$$L = \frac{\mu_0}{\pi}\log(2D/a)$$ per unit length



SUMMARY SHEET — S17

Energy in Electromagnetic fields

- Magnetic energy: $W_{\rm M} = \frac{1}{2}\int d\tau\, \boldsymbol{A}\cdot\boldsymbol{J} = \frac{1}{2}\int d\tau\, \boldsymbol{B}\cdot\boldsymbol{H}$
- Electrostatic energy: $W_{\rm E} = \frac{1}{2}\int d\tau\, \rho V = \frac{1}{2}\int d\tau\, \boldsymbol{E}\cdot\boldsymbol{D}$
- Total energy: $= \int d\tau\, \frac{1}{2}\,(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H}) \neq \int d\tau\, \frac{1}{2}(\rho V + \boldsymbol{A}\cdot\boldsymbol{J})$

Maxwell's Displacement Current

- Required to ensure charge conservation: $\nabla\cdot\boldsymbol{J} + \dfrac{\partial\rho}{\partial t} = 0$

$$\nabla\times\boldsymbol{H} = \boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial t} \qquad \text{(Maxwell 4)}$$

Learn Maxwell's Equations

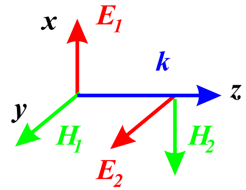
$$\nabla\cdot\boldsymbol{D} = \rho_{\rm free} \qquad \nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$
$$\nabla\cdot\boldsymbol{B} = 0 \qquad \nabla\times\boldsymbol{H} = \boldsymbol{J}_{\rm free} + \frac{\partial\boldsymbol{D}}{\partial t}$$

Electromagnetic Waves in Free Space

- Free space: $\rho = \boldsymbol{J} = 0;\ \boldsymbol{D} = \epsilon_0\boldsymbol{E};\ \boldsymbol{B} = \mu_0\boldsymbol{H}$
- Maxwell's equations ⇒ $\nabla^2\boldsymbol{E} = \epsilon_0\mu_0\dfrac{\partial^2\boldsymbol{E}}{\partial t^2}$ (same for $\boldsymbol{H}$).
- ⇒ electromagnetic waves of velocity $c = (\epsilon_0\mu_0)^{-1/2}$
  (velocity of light)

SUMMARY SHEET — S18

Electromagnetic Waves



- Electromagnetic waves are transverse.

- Plane wave travelling in z direction ⇒ Pairs of equations for transverse (E_x, H_y) and $(E_y, -H_x)$ polarisations.
- All waves travel with the same speed $c = (\epsilon_0\mu_0)^{-1/2}$
- Electric and magnetic fields are in phase.
- Impedance of free space: $\dfrac{E_x}{H_y} = \dfrac{E_y}{-H_y} = \sqrt{\dfrac{\mu_0}{\epsilon_0}} = 377\,\Omega$
- Electric energy density in wave equal to magnetic energy density:

$$\tfrac{1}{2}\epsilon_0|\boldsymbol{E}|^2 = \tfrac{1}{2}\mu_0|\boldsymbol{H}|^2$$

Flow of Electromagnetic Energy

- Work done on fields $\oint d\boldsymbol{S}\cdot\boldsymbol{E}\times\boldsymbol{H} + \dfrac{\partial}{\partial t}\displaystyle\int d\tau\tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H})$
- Poynting vector $\boldsymbol{N} = \boldsymbol{E}\times\boldsymbol{H}$ = rate of energy flow
  (power per unit area).
- For electromagnetic waves $|\boldsymbol{N}| = cU$

Summary Sheet — S19

Wave momentum and radiation pressure.

- Poynting vector $\boldsymbol{N} = \boldsymbol{E}\times\boldsymbol{H} \Rightarrow$ momentum density: $\boldsymbol{g} = \boldsymbol{N}/c^2$
- Radiation pressure: $\boldsymbol{R} = \boldsymbol{N}/c$ (for absorption; $\times 2$ if reflected)

Plane waves in insulating media

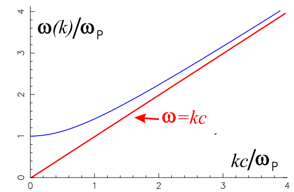
- ϵ, μ real and constant. $\epsilon_0 \to \epsilon\epsilon_0, \mu_0 \to \mu\mu_0$

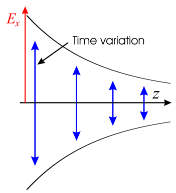
$$n = \frac{c}{c'} = \sqrt{\epsilon\mu} \qquad Z = Z_0\sqrt{\frac{\mu}{\epsilon}} \qquad (\boldsymbol{E} \text{ and } \boldsymbol{H} \text{ in phase})$$

- Equal energy density in electric and magnetic fields.

Plane waves in plasmas

- Plasma contains free electrons — oscillate in $\boldsymbol{E}$ field of wave.
- $\boldsymbol{E} = \boldsymbol{E}_0 e^{-i\omega t} \Rightarrow \boldsymbol{r} = \dfrac{e\boldsymbol{E}}{m_e\omega^2} \Rightarrow \boldsymbol{P} = -\dfrac{Ne^2}{m_e\epsilon_0\omega^2}\epsilon_0\boldsymbol{E}$
- $\Rightarrow \epsilon = 1 - \dfrac{\omega_p^2}{\omega^2} \qquad (\omega_p^2 = \dfrac{Ne^2}{m_e\epsilon_0})$





- Above ω_p waves dispersive.

$$\omega^2 = \omega_p^2 + k^2c^2$$

- $v_\phi > c\,;\ v_g < c\,;\ v_\phi v_g = c^2$
- Below ω_p waves evanescent.

  All parts oscillate in phase.
- $\boldsymbol{H}$ is $\pi/2$ behind $\boldsymbol{E}$. No net $\boldsymbol{N}$.

Summary Sheet — S20

Waves in plasmas.

- Plasma contains free electrons — oscillate in $\boldsymbol{E}$ field of wave.
- $\Rightarrow \epsilon = 1 - \dfrac{\omega_p^2}{\omega^2} \qquad (\omega_p^2 = \dfrac{Ne^2}{m_e\epsilon_0})$
- Above ω_p waves dispersive $\qquad \omega^2 = \omega_p^2 + k^2c^2$
- Below ω_p waves evanescent — $\boldsymbol{H}$ lags $\boldsymbol{E}$ by $\pi/2$.
- N.B. using $e^{-i\omega t}$ phase rotates clockwise in Argand diagram.

  Take real part of fields *before* calculating energies or Poynting vector.

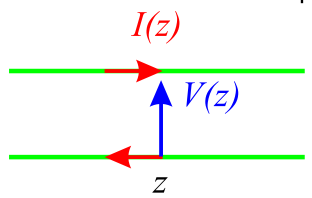
Waves in metals — skin depth.

- $\nabla\times\boldsymbol{H} = \sigma\boldsymbol{E} + \dfrac{\partial\boldsymbol{D}}{\partial t} \Rightarrow \epsilon' = \epsilon + \dfrac{i\sigma}{\omega\epsilon_0}$
- Highly conducting: $\epsilon' \approx \dfrac{i\sigma}{\omega\epsilon_0}$ imaginary. $\Rightarrow n$ complex $\propto \pm e^{i\pi/4}$.
- $E_x = E_0 \exp i(z/\delta - \omega t)e^{-z/\delta}$ (Skin depth $\delta = \sqrt{\dfrac{2}{\sigma\mu\mu_0\omega}}$)
- Severely damped wave — $\boldsymbol{H}$ lags $\boldsymbol{E}$ by $\pi/4$ — net $\boldsymbol{N}$.
- Example: alternating current in wire flows in skin around surface — resistance increases. $R = \dfrac{1}{2\pi a\delta\sigma}$ per unit length.

Summary Sheet — S21

Transmission lines.

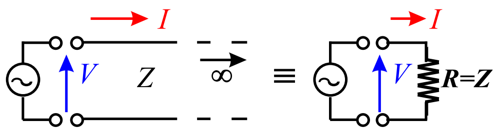
- Pairs of conductors can transport energy as electromagnetic waves.



$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

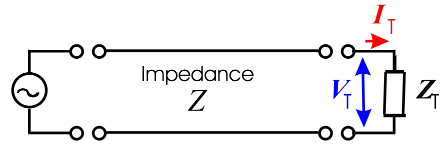
- $\Rightarrow \quad \dfrac{\partial^2 V}{\partial z^2} = LC\dfrac{\partial^2 V}{\partial t^2}$ (same for I)
- Wave equation for V, I — wave speed $v = \pm(LC)^{-1/2}$
- Impedance $\dfrac{V}{I} = \dfrac{\omega L}{k} = \sqrt{\dfrac{L}{C}} = Z$ ($-Z$ for wave $\to -z$)



- Semi-infinite line looks like real resistance $R = Z$.
- Examples:
  - Coaxial: $v = c$; $Z = Z_0\dfrac{\log(b/a)}{2\pi}$
  - Wires: $v = c$; $Z = Z_0\dfrac{\log(2D/a)}{\pi}$
  - Strips: $v = c$; $Z = Z_0\dfrac{d}{a}$
- Power flow $+VI$ travelling to $+z$.
- Transmission lines with resistive losses — damped waves.

Summary Sheet — S22

Terminated transmission lines.

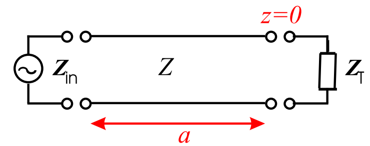


- No reflection if $Z_T = Z$
- More generally $V_i = V_1 e^{i(kz-\omega t)}$; $V_r = rV_1 e^{i(-kz-\omega t)}$; $V_T = tV_1$

Match
$$\left.\begin{array}{lcrcl} V & \Rightarrow & 1+r & = & t \\ I & \Rightarrow & (1-r)/Z & = & t/Z_T \end{array}\right\} \Rightarrow \begin{array}{rcl} r & = & \dfrac{Z_T - Z}{Z_T + Z} \\ t & = & \dfrac{2Z_T}{Z_T + Z} \end{array}$$

- Energy flow: VI conserved.

$$V^2 \times \left(\frac{1}{Z} = \frac{r^2}{Z} + \frac{t^2}{Z_T}\right)$$

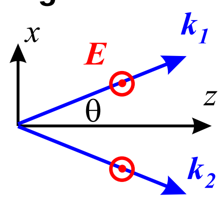


- $\dfrac{Z_{in}}{Z} = \dfrac{Z_T \cos ka - iZ \sin ka}{Z \cos ka - iZ_T \sin ka}$
- Shorted or open circuit $\to$ pure reactance.
- $a = \lambda/4 \Rightarrow Z_{in}Z_T = Z^2$

$\Rightarrow$ can match load to impedance of a transmission line.

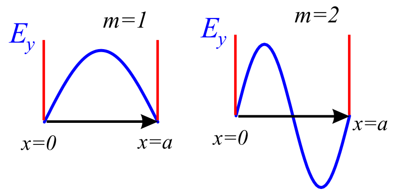
Summary Sheet — S23

Waveguides.



$$e^{i(\boldsymbol{k}_1\cdot\boldsymbol{x}-\omega t)} + e^{i(\boldsymbol{k}_2\cdot\boldsymbol{x}-\omega t)}$$
$$= e^{i(kz\cos\theta-\omega t)}\, 2\cos(kx\sin\theta)$$

- Have series of nodal planes with $E_y = 0$
  $\Rightarrow$ can propagate electromagnetic waves between conducting plates as sum of 2 plane waves.
- Magnetic field boundary conditions are OK provided currents flow in the walls.



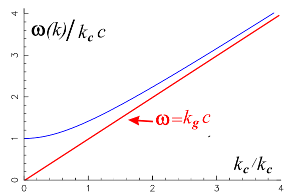
- Can get different modes
  $$E_y \propto \sin\left(\frac{m\pi x}{a}\right) = \sin(k_x x)$$

- Simplest case TE$_{10}$ in rectangular guide.
- General TE$_{mn}$ case $(E_x, E_y, 0)\ \ (H_x, H_y, H_z)$
  (only $\boldsymbol{E}$ or $\boldsymbol{H}$ can be transverse.)
  $$E_x = E_0 k_y \cos(k_x x)\sin(k_y y)\cos(k_z z-\omega t)$$
  $$E_y = -E_0 k_x \sin(k_x x)\cos(k_y y)\cos(k_z z-\omega t)$$
- $$k_x = \frac{m\pi}{a};\qquad k_y = \frac{n\pi}{b};\qquad k_z = k_{\rm g};$$
  $$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

Summary Sheet — S24

Waveguides.

- Can find $\boldsymbol{H}$ for TE$_{mn}$ modes from $\boldsymbol{E}$ using $\nabla\times\boldsymbol{E} = -\mu_0\dfrac{\partial\boldsymbol{H}}{\partial t}$
- $\Rightarrow H_y = \dfrac{E_x}{Z_{\rm g}} \qquad \left(Z_{\rm g} = Z_0\dfrac{\omega}{k_{\rm g}c}\right)$
- Same for H_y — also find $H_z \neq 0$.
- Waveguide equation $\qquad k_{\rm g}^2 = k_0^2 - k_{\rm c}^2$
- Cut-off frequency for given (a,b) and (m,n)
  $$\nu_{\rm c} = \frac{ck_{\rm c}}{2\pi} = c\left(\frac{m^2}{4a^2}+\frac{n^2}{4b^2}\right)^{1/2}$$



- Phase and group velocities
  $$\frac{\omega^2}{c^2} = k_{\rm c}^2 + k_{\rm g}^2$$
  $$v_\phi > c;\ v_{\rm g} < c;\ v_\phi v_{\rm g} = c^2$$

- Transverse magnetic modes TM$_{mn}$ exactly the same, but with $\boldsymbol{H}$ transverse. $\Rightarrow \boldsymbol{E}$ now has longitudinal component.
- Conducting guide cannot support TM$_{m0}$ or TM$_{0n}$ modes.
- Circular cross sections also used.
- Optical fibres — resonant cavities.