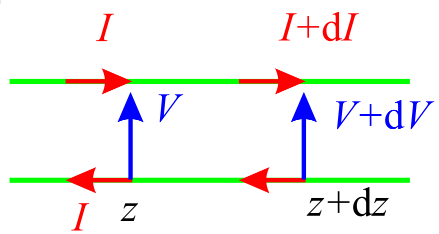
Transmission Lines and Waveguides

- Pairs of conductors can be used to carry electromagnetic waves.
- Can transport energy from one point to another efficiently.
- Examples: Pairs of wires (separation $\ll \lambda$);
  Coaxial cables;
  Parallel flat strips.
- If the conductors are identical (e.g. wires or flat strips) they carry equal and opposite currents.
  Transmission line is *balanced*.
- If conductors are not identical (e.g. coaxial cable)
  Transmission line is *unbalanced*.

Transmission Lines

- Consider two parallel wires.



- If there are waves on the line, V and I will be functions of position z.
- If $V(z)$ and $V(z+\mathrm{d}z)$ are different it must be due to the inductance of the line.
- Let inductance of line per unit length be L.

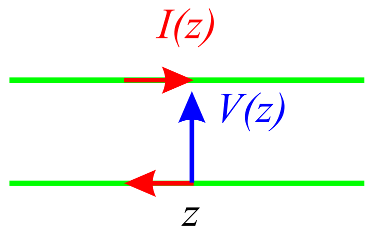
$$\mathrm{d}V = -(L\mathrm{d}z)\frac{\partial I}{\partial t}$$

If $I(z)$ and $I(z+\mathrm{d}z)$ are different it must be due to the capacitance of the line: C per unit length.

- Charge flowing out : $\mathrm{d}q = (C\mathrm{d}z)\mathrm{d}V$

$$\Rightarrow \mathrm{d}I = -\frac{\partial \mathrm{d}q}{\partial t} = -C\mathrm{d}z\frac{\partial V}{\partial t}$$

Transmission Lines II



- Transmission line equations (no losses):

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

- Equations valid for any geometry of transmission line.

- $\Rightarrow$ wave equation for V, I

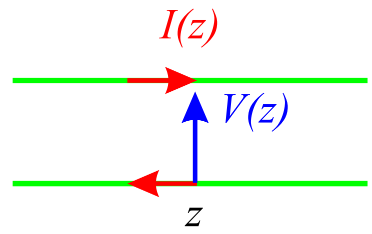
$$\frac{\partial^2 V}{\partial z^2} = -L\frac{\partial^2 I}{\partial z \partial t} = LC\frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2}$$

- Waves travel on transmission line at speed

$$v = \pm(LC)^{-1/2}$$

Waves on Transmission Lines



- Look for wavelike solutions:

$$V = V_0 \exp i(kz - \omega t)$$

$$I = I_0 \exp i(kz - \omega t)$$

- Speed of waves $v = \omega/k = (LC)^{-1/2}$

$$\Rightarrow \quad ikV = i\omega LI$$

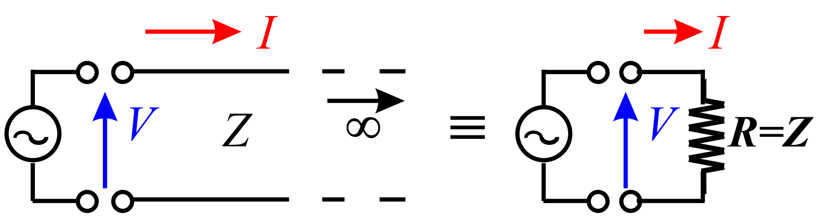
- Ratio $V/I \equiv Z$ is the *impedance* of the line

$$\frac{V}{I} = \frac{\omega L}{k} = \sqrt{\frac{L}{C}} = Z$$

- N.B. Impedance for wave $\exp i(-kz - \omega t)$ travelling to $-z$ is $-Z$.

Impedance of Transmission Line

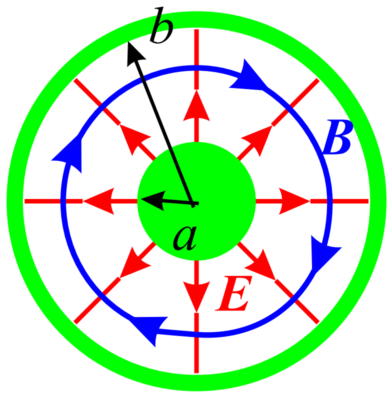
- Impedance: $Z = \pm\sqrt{\frac{L}{C}}$
- Semi-infinite transmission line attached to output of voltage generator.



- Voltage generator supplies I for given V just as if it were connected to a resistance of value Z.
- $\Rightarrow$ A semi-infinite line can be replaced by a real resistance Z without affecting the conditions on the line up to that point.

Examples: Coaxial Transmission Lines

- Coaxial cylinders — radii a and b.



- Electric field E radial.
- Magnetic field H azimuthal.

$\Rightarrow$ Poynting vector $E \times H$ longitudinal.

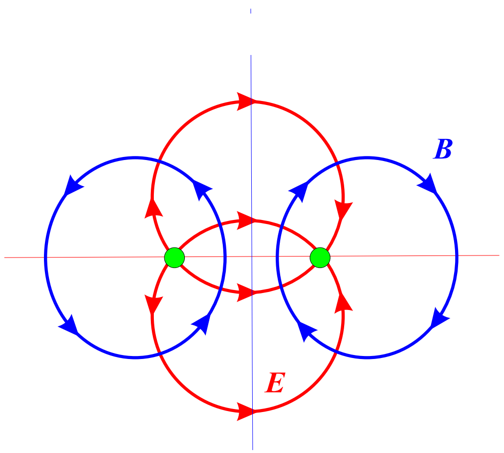
$$C = \frac{2\pi\epsilon_0}{\log(b/a)}; \qquad L = \frac{\mu_0}{2\pi}\log(b/a)$$

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\epsilon_0\mu_0}} = c$$

$$Z = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{\log(b/a)}{2\pi} = Z_0\frac{\log(b/a)}{2\pi}$$

Examples of Transmission Lines II

- Parallel wires — radii a, separation $2D$ ($2D \gg a$):



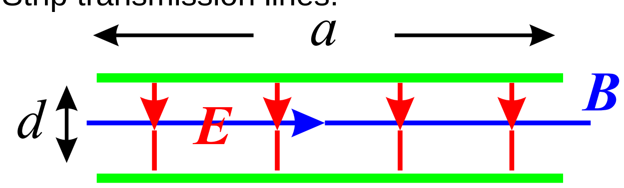
- Electric and magnetic fields tranverse ⇒ Poynting vector $\boldsymbol{N} = \boldsymbol{E} \times \boldsymbol{H}$ longitudinal.
- Electric and magnetic fields are orthogonal.

$$C = \frac{\pi \epsilon_0}{\log(2D/a)} ; \qquad L = \frac{\mu_0}{\pi} \log(2D/a)$$

$$v = c ; \qquad Z = Z_0 \frac{\log(2D/a)}{\pi}$$

Examples of Transmission Lines III

- Strip transmission lines:

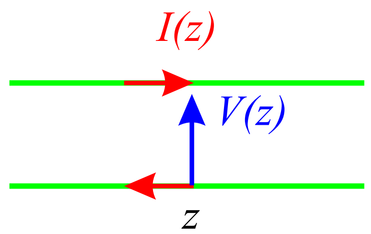


$$C = \frac{\epsilon_0 a}{d} ; \qquad L = \frac{\mu_0 d}{a}$$

$$v = c ; \qquad Z = Z_0 \frac{d}{a}$$

- All three examples have electric and magnetic field patterns that are transverse and orthogonal.
- All three examples have $v = c$ for air spacing.
- Impedance $Z = Z_0 \times$ (geometric factor).
- For dielectric spacing (coaxial cable) replace $\epsilon_0 \to \epsilon \epsilon_0$ in above formulae.

Power Flow in Transmission Lines



- Energy is stored in $\boldsymbol{E}$ and $\boldsymbol{H}$ fields as waves travel along line at velocities $\pm v$.
- Energy stored between $z = a$ and $z = b$ is

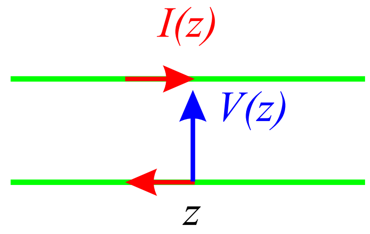
$$U = \int_a^b \mathrm{d}z \, \left(\tfrac{1}{2}LI^2 + \tfrac{1}{2}CV^2\right)$$

$$\frac{\partial U}{\partial t} = \int_a^b \mathrm{d}z \left(LI\frac{\partial I}{\partial t} + CV\frac{\partial V}{\partial t}\right)$$

$$= -\int_a^b \mathrm{d}z \left(I\frac{\partial V}{\partial z} + V\frac{\partial I}{\partial z}\right) = -[IV]_a^b$$

- Power flow to $+z$ is $+IV = V^2/Z$.
- Power flow negative for wave travelling to $-z$.

Resistive Losses in Transmission Lines



- If transmission line has resistance R per unit length
  Transmission line equations become:

$$\frac{\partial V}{\partial z} = -RI - L\frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

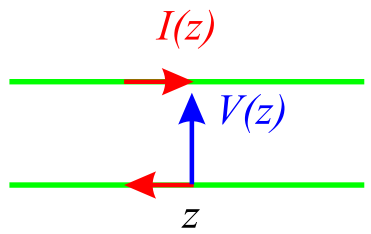
- $\Rightarrow$ modified wave equation for V, I

$$\frac{\partial^2 V}{\partial z^2} = -L\frac{\partial^2 I}{\partial z \partial t} - R\frac{\partial I}{\partial z}$$

$$= LC\frac{\partial^2 V}{\partial t^2} + RC\frac{\partial V}{\partial t}$$

$$\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2} + RC\frac{\partial I}{\partial t}$$

Resistive Losses in Transmission Lines II



- Look for waves $\propto \exp i(kz - \omega t)$
- $\Rightarrow k^2 = LC\omega^2 + iRC\omega$
- $\Rightarrow$ Damping of waves of frequency ω in direction of propagation.

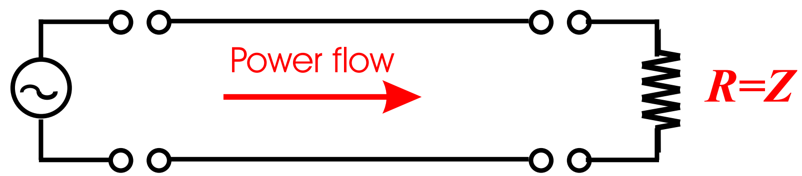
Further topics for discussion

- Try to draw the field lines of $\boldsymbol{E}$, $\boldsymbol{H}$ and $\boldsymbol{N}$ for transmission line with resistive losses.
- We have analysed case of transmission line with resistance along the wires — more general case also has shunt conductance G per unit length *between* the wires.

  See if you can analyse this case.

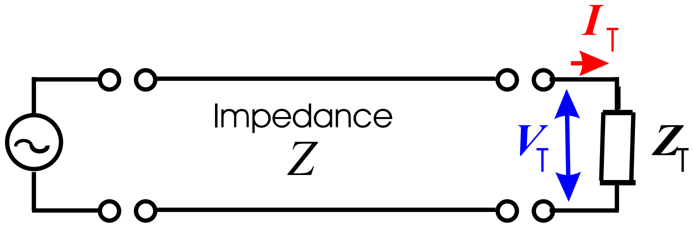
Terminated Lines

- Energy is conveyed by waves away from the voltage generator.



- Semi-infinite lines behave the same way as a resistance of value Z as far as the sources is concerned.
- If we terminate the line with a resistance $R = Z$, the remaining line is unaffected.
- If the terminating "load resistor" $R = Z$, the load is "matched" to the line and no reflection of waves occurs.
- If the line is terminated by a resistance $Z_T \neq Z$ waves will be reflected.

TERMINATED LINES II



- Transmission line terminated by resistance $Z \neq Z_T$.
- Must have $V_T/I_T = Z_T$
- Waves will now be reflected.

  Incident wave: $V_i = V_1 \exp i(kz - \omega t)$

  $I_i = I_1 \exp i(kz - \omega t)$

  Reflected wave: $V_r = V_2 \exp i(-kz - \omega t)$

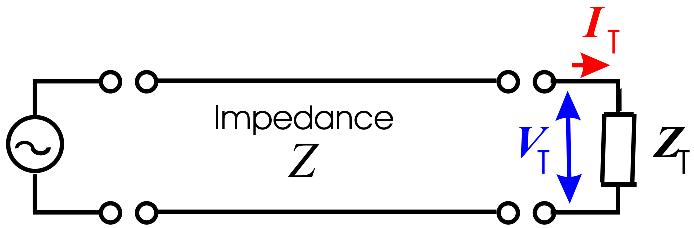
  $I_r = I_2 \exp i(-kz - \omega t)$

- N.B. $V_1/I_1 = Z$; $V_2/I_2 = -Z$.

$$V_T = V_i + V_r = e^{-i\omega t}\left[V_1 e^{ikz} + V_2 e^{-ikz}\right]$$

$$I_T = I_i + I_r = e^{-i\omega t}\left[\frac{V_1}{Z}e^{ikz} - \frac{V_2}{Z}e^{-ikz}\right]$$

REFLECTION OF WAVES



- Take origin $z = 0$ at load

$$\frac{V_T}{I_T} = \frac{V_1 + V_2}{V_1/Z - V_2/Z} = Z_T$$

- Reflection coefficient (amplitude) $r \equiv V_2/V_1$

$$r = \frac{Z_T - Z}{Z_T + Z}$$

- Define “transmitted” voltage $V_T = V_1 + V_2 = tV_1$

$$t = 1 + r = \frac{2Z_T}{Z_T + Z}$$

- No reflection ($r = 0$) only if $Z_T = Z$.

Reflection of Waves

- Phase changes can occur:

  (1) for complex Z_T;

  (2) for different values of real Z_T.

Z_T/Z	r	t
< 1	negative	positive
1	0	1
> 1	positive	positve

- Power transfer into load resistor $Z_T = R$.

$$P_L = \frac{\langle V_T^2 \rangle}{R} = \frac{4R^2}{(Z+R)^2} \frac{\langle V_1^2 \rangle}{R}$$

- Incident power $P_i = \frac{\langle V_i^2 \rangle}{Z} \Rightarrow \frac{P_L}{P_i} = \frac{4RZ}{(R+Z)^2}$

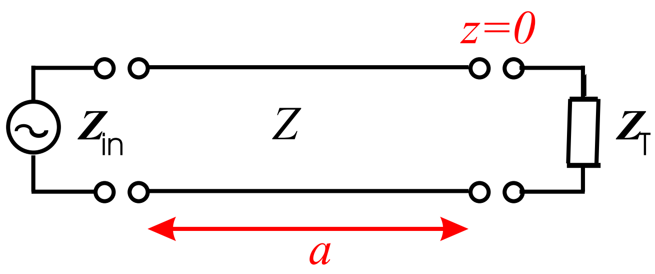
- $\Rightarrow P_L \le P_i$ equality only if $Z = R$

- Power budget (VI): Incident = Reflected + Transmitted

$$V_1^2 \left(\frac{1}{Z} = \frac{r^2}{Z} + \frac{t^2}{Z_T} \right)$$

Waves on Short Terminated Lines

- Line of length a, impedance Z. Terminated by load Z_T.



- Standing waves on line if $Z_T \ne Z$

  $\Rightarrow V/I$ varies along the line

  $\Rightarrow$ input impedance Z_{in} depends on distance a.

$$V_i = V_1 e^{i(kz-\omega t)} \qquad V_r = rV_1 e^{i(-kz-\omega t)}$$

$$I_i = V_i/Z \qquad I_r = -V_r/Z$$

$$\Rightarrow Z_{in} = \left. \frac{V_i + V_r}{I_i + I_r} \right|_{z=-a} = \frac{e^{-ika} + re^{ika}}{e^{-ika} - re^{ika}} Z$$

- But $r = \frac{Z_T - Z}{Z_T + Z} \Rightarrow \frac{Z_{in}}{Z} = \frac{Z_T \cos ka - iZ \sin ka}{Z \cos ka - iZ_T \sin ka}$

Waves on Short Terminated Lines II

$$\frac{Z_{in}}{Z} = \frac{Z_T \cos ka - iZ \sin ka}{Z \cos ka - iZ_T \sin ka}$$

- Generally complex, but special cases.
- Shorted: $Z_T = 0$. $Z_{in}/Z = -i \tan ka$

  Pure reactance, no dissipation.

  Inductance $(0 < ka < \pi/2)$,

  Capacitance $(\pi/2 < ka < \pi)$.

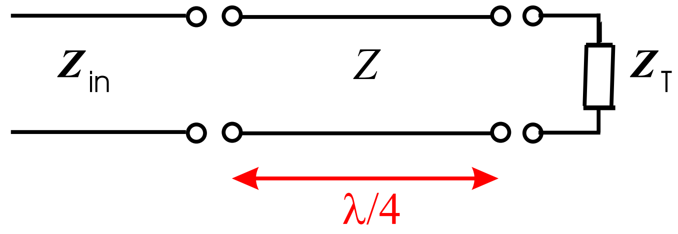
  (N.B. We are now using $V \propto e^{-i\omega t} \Rightarrow$ impedance of inductor is $Z_L = -i\omega L$, capacitor: $Z_C = 1/(-i\omega C)$.)
- Open circuit: $Z_T = \infty$. $Z_{in}/Z = i \cot ka$

  Capacitance $(0 < ka < \pi/2)$,

  Inductance $(\pi/2 < ka < \pi)$.

Input Impedance of Terminated Lines

- Special case: $a = \lambda/4$ $ka = \pi/2$



$$\frac{Z_{in}}{Z} = \frac{-iZ}{-iZ_T}$$

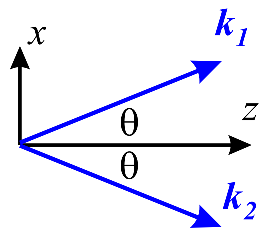
$$\boxed{Z_{in} Z_T = Z^2}$$

- Important case:

  can use $\lambda/4$ lines to match two impedances or to match line with impedance Z_{in} to load Z_T.
- Only works for particular frequency (for which $a = \lambda/4$).
- Exactly like antireflection coating on a lens.

WAVEGUIDES

- Electromagnetic waves can propagate down hollow metal tubes, of various cross sections, called *waveguides*.
- Invetigate the modes of propagation.
- Preliminaries: consider 2 plane waves, travelling at angle $\pm\theta$ to the z-axis



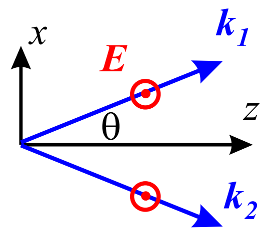
$$\mathbf{k}_1 = (k\sin\theta, 0, k\cos\theta)$$
$$\mathbf{k}_2 = (-k\sin\theta, 0, k\cos\theta)$$

$$\begin{aligned} \mathbf{k}_1\cdot\mathbf{x} &= kx\sin\theta + kz\cos\theta \\ \mathbf{k}_2\cdot\mathbf{x} &= -kx\sin\theta + kz\cos\theta \end{aligned}$$

- Consider waves polarised in y-direction, with equal amplitude

$$E_y = E_0\left(e^{i(\mathbf{k}_1\cdot\mathbf{x}-\omega t)} + e^{i(\mathbf{k}_2\cdot\mathbf{x}-\omega t)}\right)$$

WAVEGUIDES — PRELIMINARIES



- Two waves, polarised in y-direction

$$e^{i(\mathbf{k}_1\cdot\mathbf{x}-\omega t)} + e^{i(\mathbf{k}_2\cdot\mathbf{x}-\omega t)}$$

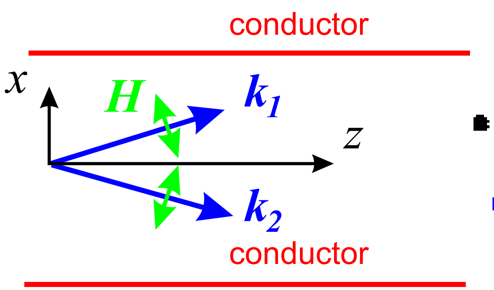
$$\begin{aligned} E_y &= E_0\, e^{i(kz\cos\theta-\omega t)}\left[e^{ikx\sin\theta} + e^{-ikx\sin\theta}\right] \\ &= E_0\, e^{i(kz\cos\theta-\omega t)}2\cos(kx\sin\theta) \end{aligned}$$

- There are a series of nodal planes where the E_y component is zero.
- First ones occur at $kx\sin\theta = \pm\pi/2$
- $\Rightarrow$ Standing wave in the x-direction.
- Propagating wave in the z-direction.
- Velocity $\dfrac{\omega}{k_g}$, where $k_g = k\cos\theta$.

$$\Rightarrow \frac{\omega}{k'} = \frac{c}{\cos\theta} > c \qquad \text{(c.f. plasma waves)}$$

WAVEGUIDES III

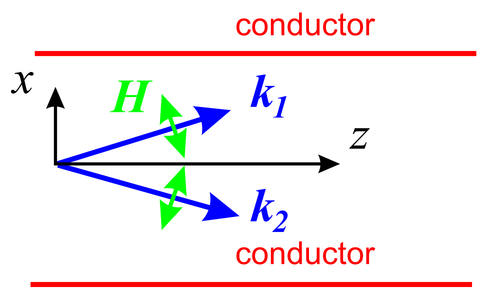
- Nodal planes have $E_y = 0$. $\Rightarrow$ Could put conducting plane in the nodes, with no effect on the electric field.
- Turn this around; if we have parallel metal planes, electromagnetic waves can propagate between them by modes in which the fields are the sum of pairs of plane waves travelling at angles $\pm\theta$ to the propagation axis.



- $E_y = 0$ in planes $kx \sin\theta = \pm\pi/2$

- But we have to consider the boundary conditions for the magnetic field as well.
- $\boldsymbol{H}$ is perpendicular to $\boldsymbol{k}$ and $\boldsymbol{E}$.
  $\Rightarrow \boldsymbol{H}$ has both H_z and H_x components.
- H_x components *add*, just like $E_y \Rightarrow$ zero at conductors.

WAVEGUIDES IV



- Boundary condition for $\boldsymbol{H}$:

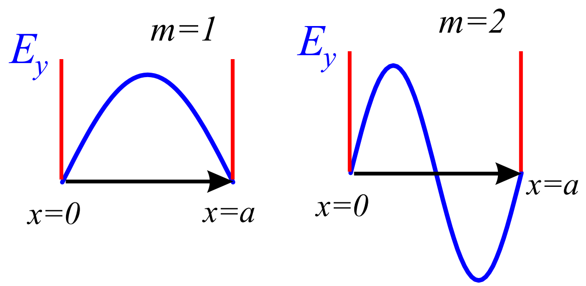
$$\nabla\cdot\boldsymbol{B} = 0 \Rightarrow B_\perp = 0 \Rightarrow H_\perp = H_x = 0.$$

  So boundary conditions on $\boldsymbol{H}$ satisfied.

- H_z components are in different directions for the two waves $\Rightarrow$ conductors are *antinodes* for H_z.
  This is OK, because *currents can flow in the walls*.
  (But can't have $E_{\|}$.)

Waveguides V

- Nodal planes occur every $\Delta x = \dfrac{\pi}{k \sin\theta}$ in x.
- Suppose planes are separated by distance a in x and θ is such that m half-periods fit across the waveguide.

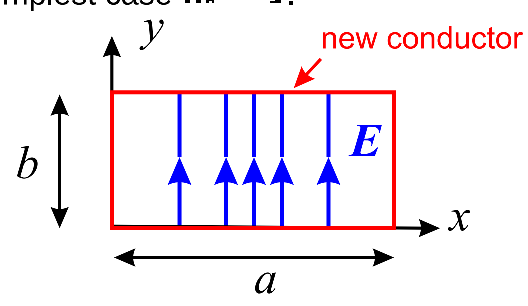


- $\Rightarrow k \sin\theta = m\pi/a$, so that $k_g \equiv k \cos\theta$ satisfies

$$k_g^2 = k^2 - \frac{m^2\pi^2}{a^2}$$

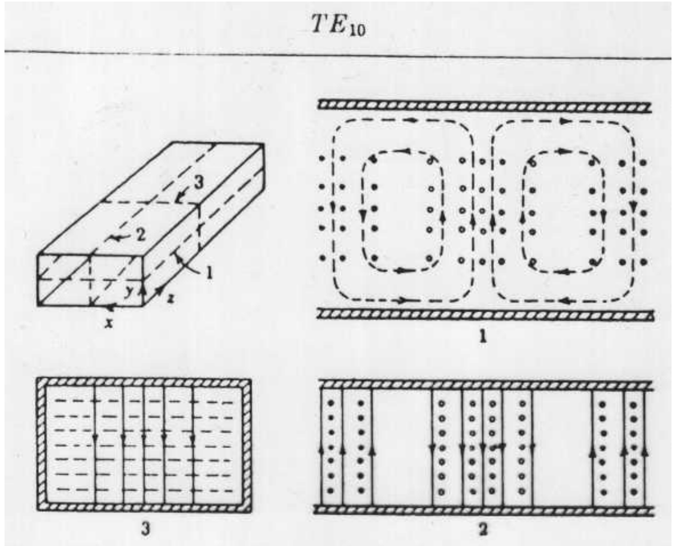
Waveguides VI

- Take simplest case $m = 1$.



- E in y-direction $\Rightarrow$ perpendicuar to any xz plane.
- Therefore can place conductor at any height y in the xz plane and boundary condition $E_{||} = 0$ will be satisfied.
- H field in the xz plane, so $H_y = 0$. This satisfies the boundary condition $B_\perp = 0$ on the xz plane.
- $\Rightarrow$ can make rectangular waveguide of any height b.
- Example of one mode of propagation, called the TE_{10} mode (Transverse Electric).

Transverse Electric Modes



- This diagram shows the electric and magnetic fields of the TE_{10} mode.
- The field lines of $\boldsymbol{E}$ are the solid lines – the $\boldsymbol{H}$ lines are dotted.
- Sketch #3 is a section across the guide and shows that charges build up on the $y = 0$ and $y = b$ planes.
- Section #1 shows that there are longitudinal magnetic fields so that currents flow in the $x = 0$ and $x = a$ planes.

Waveguides – Transverse Electric Modes

- More generally, can propagate modes in which the electric field is transverse to the waveguide. Called "Transverse Electric" modes.
- If $\boldsymbol{E}$ is transverse everywhere, $\boldsymbol{H}$ cannot also be transverse $\Rightarrow$ $\boldsymbol{H}$ has longitudinal component.
- Can make $\boldsymbol{E}$ vary with both x and y across the waveguide.

$$E_x = \cos(k_g z - \omega t)\ \sin\frac{n\pi y}{b}\ E_1(x)$$

$\uparrow$ travelling wave in z — $\uparrow$ standing wave in y — $\uparrow$ x-variation

- Similarly $E_y = \cos(k_g z - \omega t)\ \sin\dfrac{m\pi x}{a}\ E_2(y)$
- N.B. n, m must be integers: $k_x = \dfrac{m\pi}{a}$, $k_y = \dfrac{n\pi}{b}$

  Called the TE_{mn} mode.
- How are $E_1(x)$ and $E_2(y)$ related?

WAVEGUIDES – TE$_{mn}$ MODE

- $\rho_{\text{free}} = 0$ inside guide, so $\nabla \cdot \boldsymbol{E} = 0$

$$\Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

$$\Rightarrow \quad \sin(k_y y)\frac{\partial E_1}{\partial x} + \sin(k_x x)\frac{\partial E_2}{\partial y} = 0$$

- Satisfied by:

$$E_1(x) = E_0 k_y \cos(k_x x); \quad E_2(y) = -E_0 k_x \cos(k_y y)$$

- TE$_{mn}$ mode:

$$\begin{aligned} E_x &= E_0 k_y \cos(k_x x)\sin(k_y y)\cos(k_z z - \omega t) \\ E_y &= -E_0 k_x \sin(k_x x)\cos(k_y y)\cos(k_z z - \omega t) \\ E_z &= 0 \end{aligned}$$

where $(k_x, k_y, k_z) \equiv \left(\frac{m\pi}{a}, \frac{n\pi}{b}, k_g\right)$

- N.B. k_x, k_y quantised, but k^2 must still satisify

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2$$

WAVEGUIDES – TE$_{mn}$ MODE II

- Lowest TE$_{mn}$ mode is TE$_{10}$ (or TE$_{01}$).
  (TE$_{00}$ vanishes identically.)
- T$_{10}$ mode: $\quad E_y = E_0 \sin\left(\frac{\pi x}{a}\right)\cos(k_g z - \omega t)$
- Can find form of $\boldsymbol{H}$ from Maxwell equation:

$$\nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}$$

$$\Rightarrow \frac{\partial E_y}{\partial z} = \mu_0 \frac{\partial H_x}{\partial t}; \qquad \frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t}$$

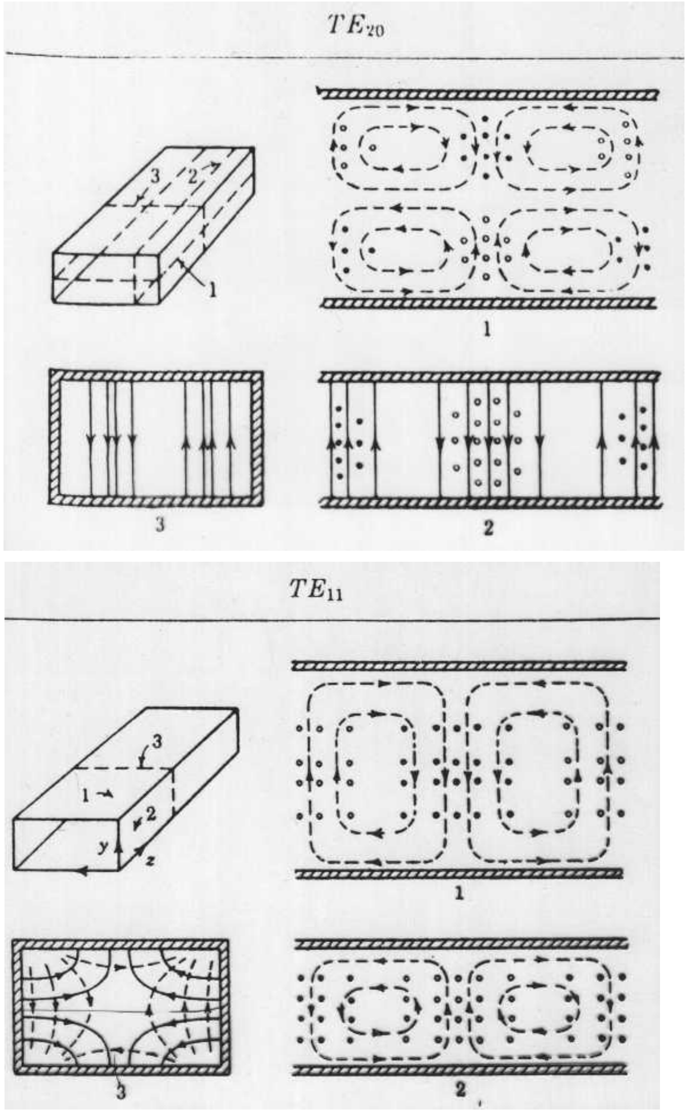
- $\Rightarrow \quad H_x = -\frac{E_0}{\mu_0}\frac{k_g}{\omega}\sin\left(\frac{\pi x}{a}\right)\cos(k_g z - \omega t)$

$$Z_g \equiv \frac{E_y}{-H_x} = \frac{\omega \mu_0}{k_g} = Z_0 \frac{k_0}{k_g}$$

- Can also find $H_z \neq 0$

$$H_z = \frac{E_0}{\mu_0}\frac{\pi}{\omega a}\cos\left(\frac{\pi x}{a}\right)\sin(k_g z - \omega t)$$

More Transverse Electric Modes



- These diagrams show the electric and magnetic fields of the TE_{20} and the TE_{11} modes.

Waveguide Equation

- To satisfy wave equation we must have

$$k_z^2 = k_g^2 = \frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}$$

- For given guide and mode (m, n) $k_x^2 + k_y^2 \equiv k_c^2$ is fixed.
- Defining $k_0^2 \equiv \omega/c$, we have the waveguide equation

$$k_g^2 = k_0^2 - k_c^2$$

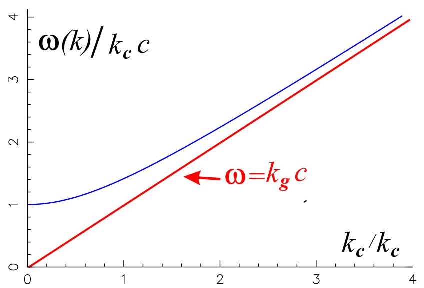
- For propagation k_g^2 must be positive.
- Cut-off frequency:

$$\nu_c = \frac{ck_c}{2\pi} = c\left(\frac{m^2}{4a^2} + \frac{n^2}{4b^2}\right)^{1/2}$$

- Below cut-off frequency for given guide and given mode waves will not propagate and we have evanescent waves.

Phase and Group Velocities

- Dispersion relation: $\frac{\omega^2}{c^2} = k_c^2 + k_g^2$
- k_c is constant (function of (m, n)).
- Phase velocity: $v_\phi = \frac{\omega}{k_g}$
- Group velocity: $v_g = \frac{d\omega}{dk_g}$



- $\frac{\omega d\omega}{c^2} = k_g dk_g$
  $\Rightarrow \frac{\omega}{k_g}\frac{d\omega}{dk_g} = c^2$
- $v_\phi > c; \quad v_g < c; \quad v_\phi v_g = c^2$
  (just like plasma waves)

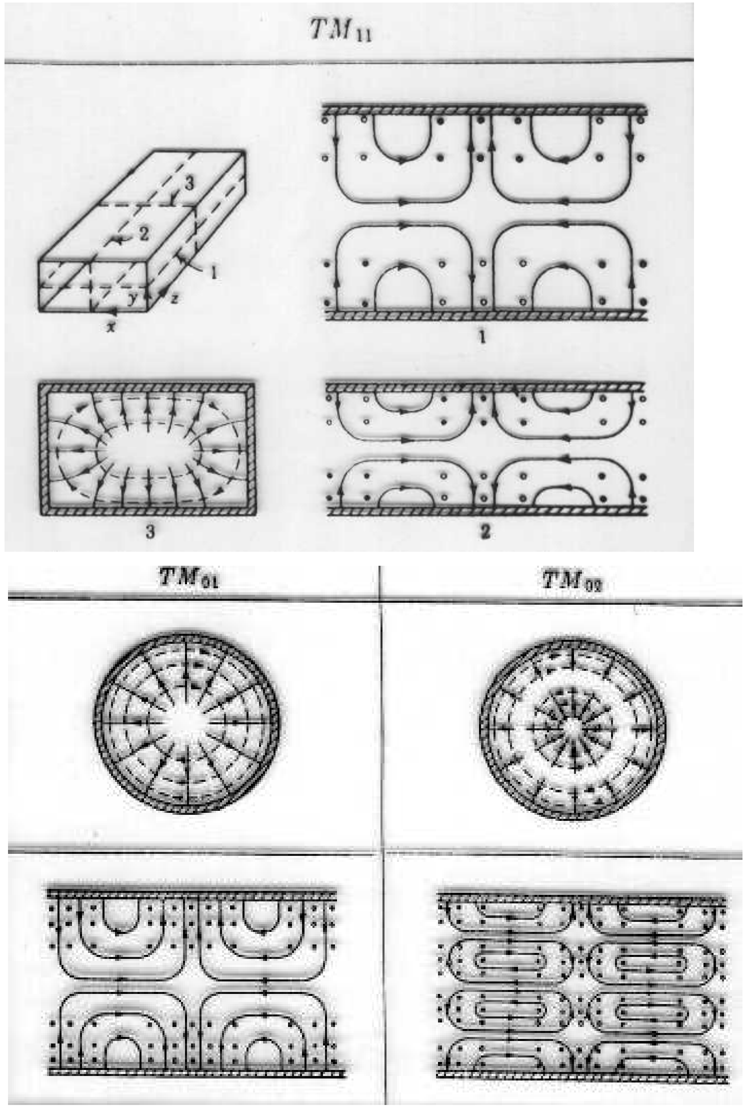
Transverse Magnetic Modes

- Identical waveguide equation and cut-off frequency for case where $\boldsymbol{H}$ is transverse. These are called "transverse magnetic" TM_{mn} modes.
- $\boldsymbol{E}$ must now have a longitudinal component.
- Conducting guide cannot support any TM_{0n} or TM_{m0} modes, because no magnetic charge can build up on the walls.
- But could have TM_{10} mode in suitable *magnetisable* waveguide. ($\boldsymbol{H}$ lines can end on magnetisaion poles.)

Other geometries.

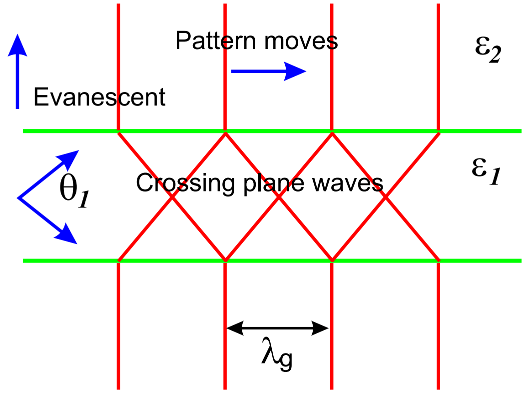
- Circular waveguides are also useful. Easily handled using Bessel functions.
- Transmission lines — can be considered as transverse electromagnetic waveguide modes (TEM_{00}).

Other Waveguide Modes

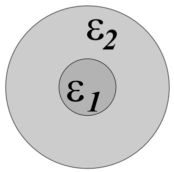


- Here are some other modes for you to study — apologies for the scanning quality.

Dielectric Waveguides — Optical Fibres



- At dielectric interface, can have total internal reflection if $k_1 \sin\theta_1 > k_2$
- If $\epsilon_1 > \epsilon_2$ we can contain the waves in the dielectric. But $E_\parallel \neq 0$ at the boundary $\Rightarrow$ wave evanescent in medium #2.
- Optical fibres: make sure outer layer is large enough so that the evanescent wave has decayed sufficiently.



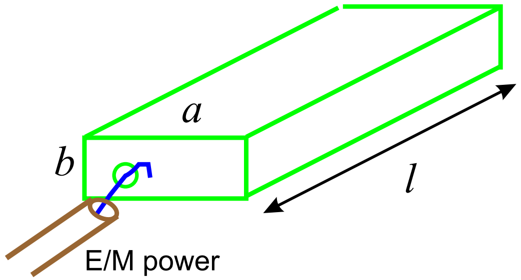
- Typical parameters: Outer 125 μ

Inner: 7 μ (single mode); 50 μ (multi-mode)

- Attenuation ≈ 0.1 db km^{-1}.

Resonant Cavities

- Take a length l of waveguide and close it off with conductor.



- Introduce electromagnetic energy by probe.
- Boundary condition $E_{\parallel} = 0$ at ends $\Rightarrow$ $k_z l = p\pi$
- Only discrete frequencies are now possible for introduction of electromagnetic waves.
- Resonant cavity:

$$\frac{\omega^2}{c^2} = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} + \frac{p^2\pi^2}{l^2}$$

- Well-known condition for discrete spectrum of waves in a box.
- $Q = \dfrac{2\pi \times \text{ stored energy}}{\text{energy loss per cycle}} \approx 10^4$ at cm λ for T_{10}.