TIME-VARYING ELECTRIC FIELDS

- Electromagnetic equations so far are NOT CONSISTENT with *charge conservation*

$$\oint d\boldsymbol{S}\cdot\boldsymbol{J} + \int d\tau \frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \nabla\cdot\boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

- We have

$$\nabla\times\boldsymbol{H} = \boldsymbol{J} \quad \Rightarrow \nabla\cdot\boldsymbol{J} = 0 \quad (!)$$

- So we must add *Maxwell's Displacement Current*

(Maxwell 4) $$\nabla\times\boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

- Take divergence:

$$0 = \nabla\cdot\boldsymbol{J} + \frac{\partial}{\partial t}\nabla\cdot\boldsymbol{D}$$

- But $\nabla\cdot\boldsymbol{D} = \rho$ (Maxwell 1), so now OK.

RECAP: MAXWELL'S EQUATIONS

$$\begin{aligned}
\nabla\cdot\boldsymbol{D} &= \rho_{\text{free}} \\
\nabla\times\boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\
\nabla\cdot\boldsymbol{B} &= 0 \\
\nabla\times\boldsymbol{H} &= \boldsymbol{J}_{\text{free}} + \frac{\partial \boldsymbol{D}}{\partial t}
\end{aligned}$$

- Constitutive relationships for media:

$(\boldsymbol{D}, \boldsymbol{E})$ permittivity

$(\boldsymbol{B}, \boldsymbol{H})$ permeability

$(\boldsymbol{J}, \boldsymbol{E})$ conductivity

- Other useful information:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \quad \text{Lorentz force}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}; \quad \boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} \quad \text{(next year)}$$

ELECTROMAGNETIC WAVES

- For *linear media*, constitutive relationships:

$$\boldsymbol{D} \propto \boldsymbol{E}; \quad \boldsymbol{B} \propto \boldsymbol{H}; \quad \boldsymbol{J} \propto \boldsymbol{E}$$

- $\Rightarrow$ Can apply the *principle of superposition*
  i.e. can make a Fourier synthesis.

- Special case — free space:

$$\rho = 0; \quad \boldsymbol{J} = 0;$$
$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E};$$
$$\boldsymbol{B} = \mu_0 \boldsymbol{H};$$

- Maxwell's equations (favouring $\boldsymbol{E}$, $\boldsymbol{H}$)

$$\nabla\cdot\boldsymbol{E}=0 \qquad \nabla\cdot\boldsymbol{H}=0$$
$$\nabla\times\boldsymbol{E}=-\mu_0\frac{\partial \boldsymbol{H}}{\partial t} \qquad \nabla\times\boldsymbol{H}=\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t}$$

ELECTROMAGNETIC WAVES

$$\nabla\cdot\boldsymbol{E}=0 \qquad \nabla\cdot\boldsymbol{H}=0$$
$$\nabla\times\boldsymbol{E}=-\mu_0\frac{\partial \boldsymbol{H}}{\partial t} \qquad \nabla\times\boldsymbol{H}=\epsilon_0\frac{\partial \boldsymbol{E}}{\partial t}$$

- Take curl of $\nabla\times\boldsymbol{E}$ equation

$$\nabla\times(\nabla\times\boldsymbol{E}) = -\mu_0\frac{\partial}{\partial t}(\nabla\times\boldsymbol{H}) = -\mu_0\epsilon_0\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

- Expand vector triple product and use $\nabla\cdot\boldsymbol{E} = 0$

$$\nabla\times(\nabla\times\boldsymbol{E}) = \nabla(\nabla\cdot\boldsymbol{E}) - \nabla^2\boldsymbol{E} = -\nabla^2\boldsymbol{E}$$

- $\Rightarrow$ Wave equation for $\boldsymbol{E}$

$$\nabla^2\boldsymbol{E} = \mu_0\epsilon_0\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

- By an exactly similar argument, we find that $\boldsymbol{H}$ satisfies a wave equation also

$$\nabla^2\boldsymbol{H} = \mu_0\epsilon_0\frac{\partial^2 \boldsymbol{H}}{\partial t^2}$$

THE MEANING OF $\nabla^2 \boldsymbol{E}$

- ∇^2 is a *scalar differential operator*.
- $\boldsymbol{E}(\boldsymbol{x})$ is a *vector field*
- In Cartesian coordinates $\nabla^2 \boldsymbol{E}$ is easy to compute, since

$$\boldsymbol{E} = E_x \boldsymbol{i} + E_y \boldsymbol{j} + E_z \boldsymbol{k}$$

  and THE BASIS VECTORS DON'T CHANGE WITH POSITION $\boldsymbol{x}$.

- So in Cartesian coordinates, $\nabla^2 \boldsymbol{E}$ just has components

$$(\nabla^2 E_x,\ \nabla^2 E_y,\ \nabla^2 E_z)$$

- Of course, $\nabla^2 \boldsymbol{E}$ can be evaluated in *any* coordinate system...
  But it's tricky! (It's easier to evaluate $\nabla \times (\nabla \times \boldsymbol{E})$)

ELECTROMAGNETIC WAVES IN FREE SPACE

- Wave equation of components of $\boldsymbol{E}, \boldsymbol{H}$. Scalar wave equation is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0$$

  where c is the wave speed.

- Velocity of electromagnetic waves is $c = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$
- c involves constants of vacuum, and is independent of motion.
  $\Rightarrow$ Same for all observers.
- Maxwell noticed this in about 1865.
  $\Rightarrow$ Maxwell's equations are consistent with Einstein's Special Relativity (1905).
- c is defined to be 299,792,458 m s^{-1}
- metre is defined in terms of c and the second.
- $\epsilon_0 \equiv (c^2 \mu_0)^{-1} = 8.85 \times 10^{-12}$ F m^{-1}

Plane waves in isotropic media

- Look for plane wave solution propagating along the z axis.
- No variation with $x, y \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$
- Look at $\nabla \times \boldsymbol{H} = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$ equation.

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = -\frac{\partial H_y}{\partial z}\boldsymbol{i} + \frac{\partial H_x}{\partial z}\boldsymbol{j}$$

- Written out in components:

$$\epsilon_0 \frac{\partial E_x}{\partial t} = -\frac{\partial H_y}{\partial z} \qquad \text{(Transverse — } P_1\text{)}$$

$$\epsilon_0 \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} \qquad \text{(Transverse — } P_2\text{)}$$

$$\epsilon_0 \frac{\partial E_z}{\partial t} = 0 \qquad \text{(Longitudinal)}$$

Plane waves II

- The other curl equation:

$$\nabla \times \boldsymbol{E} = -\frac{\partial E_y}{\partial z}\boldsymbol{i} + \frac{\partial E_x}{\partial z}\boldsymbol{j} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}$$

- Written out in components:

$$-\mu_0 \frac{\partial H_x}{\partial t} = -\frac{\partial E_y}{\partial z} \qquad \text{(Transverse — } P_2'\text{)}$$

$$-\mu_0 \frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z} \qquad \text{(Transverse — } P_1'\text{)}$$

$$-\mu_0 \frac{\partial H_z}{\partial t} = 0 \qquad \text{(Longitudinal)}$$

- Electromagnetic waves in free space are TRANSVERSE.
- Longitudinal fields are static.
- 4 transverse equations:

$$\left.\begin{array}{l} \text{2 for } E_x, H_y \\ \text{2 for } E_y, H_x \end{array}\right\} \text{ 2 independent polarisations}$$

Plane waves in free space III

- Look at E_x, H_y equations

$$\epsilon_0 \frac{\partial E_x}{\partial t} = -\frac{\partial H_y}{\partial z}$$

$$-\mu_0 \frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z}$$

- Eliminating, we recover the wave equation (same for H_y)

$$\frac{\partial^2 E_x}{\partial z^2} - \epsilon_0 \mu_0 \frac{\partial^2 E_x}{\partial t^2} = 0$$

- Solution is $E_x = E_0 \exp i(kz - \omega t)$, $\omega/k = c = (\epsilon_0 \mu_0)^{-1/2}$.

- This represents a wave of a single angular frequency ω, propagating in the positive z-direction.

$$\frac{\partial E_x}{\partial t} = -i\omega E_x \; ; \quad \frac{\partial E_x}{\partial z} = ikE_x$$

- Same for H_y.

Plane waves in free space IV

- $$\frac{\partial E_x}{\partial t} = -i\omega E_x = -\frac{1}{\epsilon_0}\frac{\partial H_y}{\partial z} = -\frac{ik}{\epsilon_0} H_y$$

$$\Rightarrow E_x = \frac{k}{\omega \epsilon_0} H_y = \sqrt{\frac{\mu_0}{\epsilon_0}} H_y$$

- Ratio E_x/H_y is *real* and *constant*.

- Units: E_x— V m^{-1}; H_y— A m^{-1}.

- Impedance of free space:

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \text{ Ohm}$$

- $E_x = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{B_y}{\mu_0} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} B_y = cB_y$.

- Relationships for both polarisations:

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{E_y}{-H_x}$$

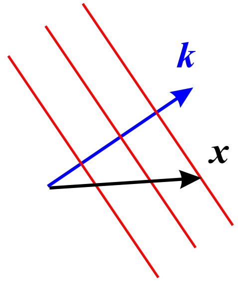
Electromagnetic Waves — Summary

- Electromagnetic waves in free space are non-dispersive—all frequencies travel with the same speed
  $c = (\epsilon_0 \mu_0)^{-1/2}$
- $\boldsymbol{H}$ and $\boldsymbol{E}$ are transverse and perpendicular to each other.
- Two independent polarisations are possible: (E_x, H_y) and (E_y, H_x).
- Ratio $E_x/H_y = \sqrt{\mu_0/\epsilon_0}$ is constant and REAL
  $\Rightarrow$ $\boldsymbol{E}$ and $\boldsymbol{H}$ are in phase with each other.
- Energy: $U_{\mathsf{E}} = \frac{1}{2}\epsilon_0 E_x^2$

$$U_{\mathsf{M}} = \tfrac{1}{2}\mu_0 H_y^2 = \tfrac{1}{2}\mu_0 \frac{\epsilon_0}{\mu_0} E_x^2 = U_{\mathsf{E}}$$

- Energy density in electric and magnetic fields is the same.

Plane Waves — Method Using Vectors



- More general case: wave number $\boldsymbol{k}$.

  Phase is $(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)$.

- Look for solution of form

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x},t) &= \boldsymbol{E}_0 \exp i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) \\ \boldsymbol{H}(\boldsymbol{x},t) &= \boldsymbol{H}_0 \exp i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) \end{aligned}$$

- $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are constant vectors, so we have

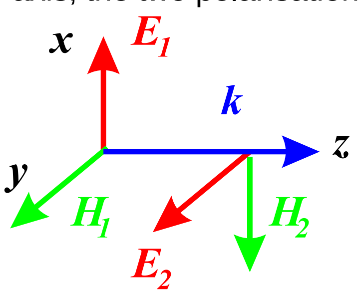
$$\begin{aligned} \nabla\times\boldsymbol{E} &= i\boldsymbol{k}\times\boldsymbol{E} \\ \nabla\cdot\boldsymbol{E} &= i\boldsymbol{k}\cdot\boldsymbol{E} \end{aligned}$$

- Maxwell's equations in free space become

$$\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0 \qquad \boldsymbol{k}\cdot\boldsymbol{H}_0 = 0$$
$$\boldsymbol{k}\times\boldsymbol{E}_0 = \omega\mu_0\boldsymbol{H}_0 \qquad \boldsymbol{k}\times\boldsymbol{H}_0 = -\omega\epsilon_0\boldsymbol{E}_0$$

More Electromagnetic Waves

- $$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_0) = \boldsymbol{k}\cdot\boldsymbol{E}_0\,\boldsymbol{k} - k^2\boldsymbol{E}_0 = -k^2\boldsymbol{E}_0$$
  $$= \omega\mu_0\boldsymbol{k}\times\boldsymbol{H}_0 = -\omega^2\epsilon_0\mu_0\boldsymbol{E}_0$$
  $$\Rightarrow \quad \boxed{\frac{\omega^2}{k^2} = c^2 = \frac{1}{\epsilon_0\mu_0}}$$
- $\boldsymbol{H}_0 = \left(\frac{1}{\mu_0 c}\right)\boldsymbol{n}\times\boldsymbol{E}_0$
  where $\boldsymbol{n} \equiv \boldsymbol{k}/|\boldsymbol{k}|$ is the unit vector along $\boldsymbol{k}$.
- $\boldsymbol{E}_0$, $\boldsymbol{H}_0$ and $\boldsymbol{n}$ are all at right angles.
- $\{\boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{n}\}$ are a right-handed set.
- If $\boldsymbol{k}$ is along the z axis, the two polarisations are as shown:



Energy Flow

- Work is done by forces when the point of application moves in the direction of the force.
- In electromagnetism $F = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$.
- The $\boldsymbol{v}\times\boldsymbol{B}$ term is perpendicular to $\boldsymbol{v}$, so cannot do any work.
- Only the electric field does work. Work done by us on the field is
  $$\mathrm{d}W = -q\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l}$$
- The rate of doing work is $-q\boldsymbol{E}\cdot\boldsymbol{v}$.
- For distributed charges the rate of work is
  $$-\int \mathrm{d}\tau\;\boldsymbol{E}\cdot\rho\boldsymbol{v} \rightarrow -\int \mathrm{d}\tau\;\boldsymbol{E}\cdot\boldsymbol{J}$$
- From Maxwell 4:
  $$\boldsymbol{J} = \nabla\times\boldsymbol{H} - \frac{\partial\boldsymbol{D}}{\partial t}$$

Energy Flow II

- The rate of work done on fields in a volume is

$$-\int d\tau\, \boldsymbol{E}\cdot\boldsymbol{J} = -\int d\tau\, \boldsymbol{E}\cdot(\nabla\times\boldsymbol{H}) + \int d\tau\, \boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t}$$

- Manipulate using handy identity:

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = \boldsymbol{H}\cdot\nabla\times\boldsymbol{E} - \boldsymbol{E}\cdot\nabla\times\boldsymbol{H}$$

- Rate of work

$$= \int d\tau \left[\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) - \boldsymbol{H}\cdot\nabla\times\boldsymbol{E} + \boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t}\right]$$

Divergence ↓ Maxwell 2 ↓

$$= \oint d\boldsymbol{S}\cdot(\boldsymbol{E}\times\boldsymbol{H}) + \int d\tau \left[\boldsymbol{H}\cdot\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t}\right]$$

- If medium is linear: $\boldsymbol{B} \propto \boldsymbol{H}$; $\boldsymbol{D} \propto \boldsymbol{E}$

$$\Rightarrow \frac{\partial}{\partial t}(\boldsymbol{B}\cdot\boldsymbol{H}) = 2\boldsymbol{H}\cdot\frac{\partial \boldsymbol{B}}{\partial t};\quad \frac{\partial}{\partial t}(\boldsymbol{E}\cdot\boldsymbol{D}) = 2\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t}$$

Poynting's Theorem

- Rate of doing work in volume is

$$= \oint d\boldsymbol{S}\cdot(\boldsymbol{E}\times\boldsymbol{H}) + \frac{\partial}{\partial t}\int d\tau \left[\tfrac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D} + \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H}\right]$$

- Second term is the rate of change of stored electromagnetic energy, as expected.
- First term is the rate of energy flow across the surface containing the volume (outwards through surface).
- $\Rightarrow$ Energy flow per unit surface area is

$$\boldsymbol{N} = \boldsymbol{E}\times\boldsymbol{H}$$

- $\boldsymbol{N}$ is the *Poynting vector* and describes the energy flux (power per unit area) in an electromagnetic field.

Poynting Vector

- Poynting vector $\boldsymbol{N} = \boldsymbol{E}\times\boldsymbol{H}$.
- Poynting vector is a non-linear function of the $\boldsymbol{E}$, $\boldsymbol{H}$ fields. Can't superpose Poynting vector patterns.
- Apply to electromagnetic wave travelling in z direction:

$$\boldsymbol{E} = (E_x, 0, 0); \quad \boldsymbol{H} = (0, H_y, 0);$$

$$\Rightarrow \boldsymbol{N} = (0, 0, E_x H_y)$$

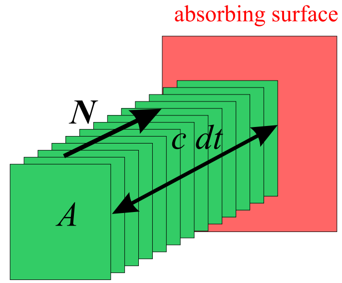
- $\boldsymbol{N}$ for wave is in positive z direction and

$$|\boldsymbol{N}| = E_x H_y = E_x^2 \sqrt{\frac{\epsilon_0}{\mu_0}} = \epsilon_0 E_x^2 \frac{1}{\sqrt{\epsilon_0 \mu_0}} = Uc$$

where $U = \epsilon_0 E_x^2$ is the energy density.

- Electromagnetic waves carry energy and *momentum*.
- Electromagnetic waves exert a force when reflected or absorbed.

Radiation Pressure and Momentum



- Radiation momentum density $\boldsymbol{g}$
- Total momentum in volume element $\mathrm{d}\boldsymbol{p} = Ac\boldsymbol{g}\mathrm{d}t$

- All momentum is absorbed in time t.
  Force: $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = Ac\boldsymbol{g}$
- $\Rightarrow$ Radiation pressure: $\boldsymbol{R} = c\boldsymbol{g}$
- If the surface reflects the radiation then $\boldsymbol{R} = 2c\boldsymbol{g}$
- What is the relationship between $\boldsymbol{N}$ and $\boldsymbol{g}$?
- Use special relativity: energy-momentum invariant

$$E^2 - p^2c^2 = m^2c^4 = 0 \text{ (for photons)} \Rightarrow E = pc$$

- Energy in volume $\mathrm{d}E = |\boldsymbol{N}|A\mathrm{d}t \Rightarrow \mathrm{d}\boldsymbol{p} = \frac{\boldsymbol{N}A\mathrm{d}t}{c}$

$$\Rightarrow \quad \boldsymbol{g} = \frac{\boldsymbol{N}}{c^2}; \quad \boldsymbol{R} = \frac{\boldsymbol{N}}{c}$$

Plane Waves in Media

We will deal with three cases

- Insulating medium ϵ, μ are constant.
- Medium with free electrons — plasma.
- Medium with conduction electrons — metals.

Insulating Media

- Polarisation $\boldsymbol{P} \propto \boldsymbol{E}$ (in phase).
- Magnetisation $\boldsymbol{M} \propto \boldsymbol{H}$ (in phase).

$$\Rightarrow \boldsymbol{D} = \epsilon\epsilon_0 \boldsymbol{E} \; ; \quad \boldsymbol{B} = \mu\mu_0 \boldsymbol{H} \; ; \quad \rho = \boldsymbol{J} = 0$$

- $\Rightarrow$ Everything like free space case with

$$\epsilon_0 \to \epsilon\epsilon_0 \; ; \quad \mu_0 \to \mu\mu_0$$

Waves in Insulating Media

- $\epsilon_0 \to \epsilon\epsilon_0 \; ; \quad \mu_0 \to \mu\mu_0$
- Velocity of waves: $c' = \dfrac{1}{\sqrt{(\epsilon\epsilon_0\mu\mu_0)}} = \dfrac{c}{\sqrt{\epsilon\mu}}$
- Refractive index: $n \equiv \dfrac{c}{c'} = \sqrt{\epsilon\mu}$

  (Often $\mu \approx 1$ in optics, so $n = \sqrt{\epsilon}$)
- Impedance of medium

$$Z_{\mathrm{m}} = \frac{E_x}{H_y} = \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}} = 377\sqrt{\frac{\mu}{\epsilon}}\ \Omega$$

- Energy densities of electric and magnetic fields are equal

$$U_{\mathrm{E}} = \tfrac{1}{2}\epsilon\epsilon_0 E_x^2 = U_{\mathrm{M}} = \tfrac{1}{2}\mu\mu_0 H_y^2$$

- Energy flux $|N| = |\boldsymbol{E}\times\boldsymbol{H}| = E_x H_y = E_x^2/Z_{\mathrm{m}}$.

Waves in Plasma

- Plasma has free electrons and ions. Electrons have lower mass so are more mobile.
  $\Rightarrow$ electrons dominate electromagnetic properties.
- Thermal motions ignored — also collisions.
- Equation of motion of an electron in a plasma

$$m_e \ddot{\boldsymbol{r}} = -e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

(N.B. charge $q = -e$ in this section.)

- In free space $\dfrac{E_x}{H_y} = \sqrt{\dfrac{\mu_0}{\epsilon_0}} \Rightarrow \dfrac{E_x}{B_y} = c$
- $\Rightarrow$ If $|\boldsymbol{v}| \ll c$ the $\boldsymbol{v} \times \boldsymbol{B}$ term is negligible.
- Consider electron in plane wave $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(kz-\omega t)}$
- At fixed $z = 0$ equation becomes $m_e \ddot{\boldsymbol{r}} = -e\boldsymbol{E}_0 e^{-i\omega t}$
- Steady-state solution: $\boldsymbol{r} = \dfrac{e}{m_e \omega^2} \boldsymbol{E}_0 e^{-i\omega t}$

Waves in Plasma II

- Electric field of wave: $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(kz-\omega t)}$
- Steady-state displacement: $\boldsymbol{r} = \dfrac{e}{m_e \omega^2} \boldsymbol{E}_0 e^{i(kz-\omega t)}$
- $\Rightarrow$ dipole moment: $\boldsymbol{p} = -e\boldsymbol{r} = -\dfrac{e^2}{m_e \omega^2} \boldsymbol{E}$.
- N electrons per unit volume

$$\Rightarrow \boldsymbol{P} = N\boldsymbol{p} = -\frac{Ne^2}{m_e \omega^2} \boldsymbol{E}.$$

- $$\epsilon = 1 + \chi = 1 + \frac{|\boldsymbol{P}|}{\epsilon_0 |\boldsymbol{E}|}$$

$$\boxed{\epsilon = 1 - \frac{Ne^2}{m_e \epsilon_0 \omega^2} = 1 - \frac{\omega_p^2}{\omega^2}}$$

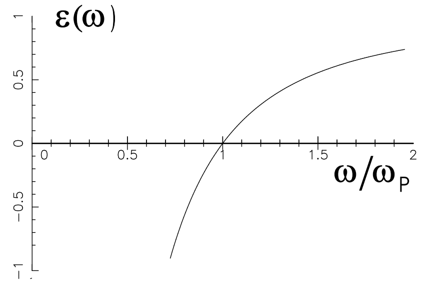
- Define *plasma frequency*: $\omega_p^2 = \dfrac{Ne^2}{m_e \epsilon_0}$
- Refractive index: $n = \sqrt{\epsilon} = \left(1 - \dfrac{\omega_p^2}{\omega^2}\right)^{1/2}$

Waves in Plasma III

- Plasma frequency: $$\omega_p^2 = \frac{Ne^2}{m_e \epsilon_0}$$

- Characteristic frequency of a plasma — determines the electromagnetic properties.

- $\epsilon(\omega)$ for transverse waves



- $\epsilon(\omega) < 1$ always

  but is *negative* for $\omega < \omega_p$

- $\Rightarrow$ refractive index is *imaginary* for frequencies below ω_p.

- $\nu_p = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{Ne^2}{m_e \epsilon_0}} \approx 9\sqrt{N}$ Hz.

- For ionosphere $N \approx 10^{12}$ m^{-3}, $\Rightarrow \nu_p \approx 10$ MHz.

Dispersion of Plasma Waves

- Conditions above plasma frequency:

$$c' = \frac{c}{n} = \frac{c}{\sqrt{1 - \omega_p^2/\omega^2}}$$

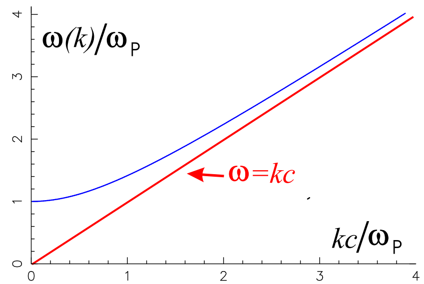
$\Rightarrow$ waves are *dispersive*.

- Different frequencies travel with different speeds.

- Phase velocity is $v_\phi = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \omega_p^2/\omega^2}} > c$

- Group velocity is $v_g = \frac{d\omega}{dk} < c$

- Summarised by *dispersion relation* $\omega(k)$



$$\omega^2(k) = \omega_p^2 + k^2 c^2$$

$$v_\phi v_g = \frac{\omega}{k}\frac{d\omega}{dk} = c^2$$

EVANESCENT PLASMA WAVES

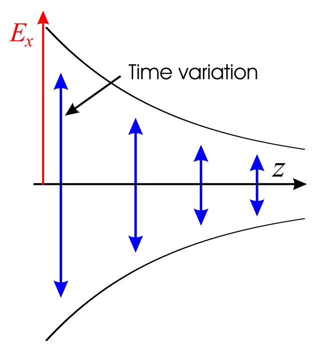
- Below plasma frequency dielectric constant $\epsilon < 0$

  $\Rightarrow$ refractive index $n = \sqrt{\epsilon}$ is imaginary.

- $$n = \pm i\beta; \quad \beta = \sqrt{|\epsilon|}; \quad k = \omega n/c$$

$$\begin{aligned} E_x &= E_0 \exp i(kz - \omega t) = E_0 \exp i\omega \left(\frac{nz}{c} - t\right) \\ &= E_0 \exp\left(-\frac{\omega\beta z}{c}\right) \exp(-i\omega t) \end{aligned}$$

- No longer a travelling wave — decays with z.



- No variation of phase with z

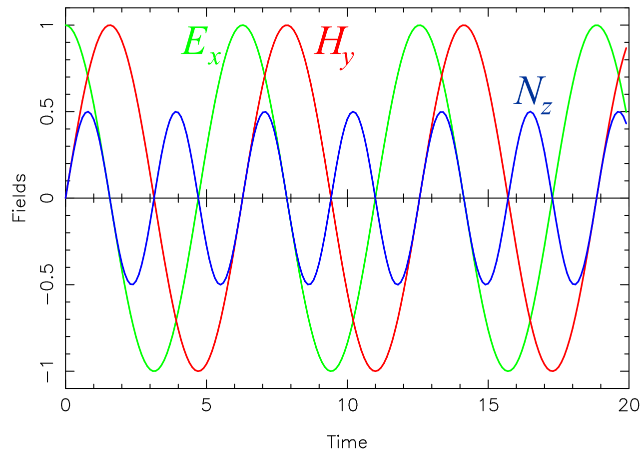
  all parts of wave in phase

EVANESCENT WAVES

- Below ω_p transverse plasma waves are *evanescent*.

- Magnetic field: $H_y = \sqrt{\frac{\epsilon\epsilon_0}{\mu_0}} E_x = \frac{i\beta E_x}{Z_0} \quad (\mu = 1)$.

- $\boldsymbol{H}$ is $\pi/2$ behind $\boldsymbol{E}$ in phase.



- Poynting vector $\boldsymbol{N} = \boldsymbol{E} \times \boldsymbol{H}$:

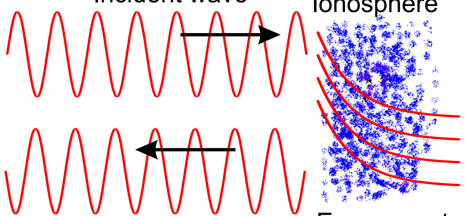
$$N_z = E_x H_y \quad \propto \quad \Re\left(e^{-i\omega t}\right) \quad \Re\left(ie^{-i\omega t}\right)$$

$$\cos\omega t \qquad \sin\omega t$$

$\Rightarrow$ mean energy flux is zero.

Waves in Plasma

- Waves incident on a plasma having frequencies below ω_p are reflected.



- Long wave radio transmissions are reflected from the Earth's ionosphere and bounce around the Earth.
- At $\omega = \omega_p$ all the plasma oscillates together.
- Longitudinal plasma oscillations also occur at ω_p

Waves in Conducting Media

- Constitutive relations for isotropic medium:

$$\begin{aligned} \boldsymbol{D} &= \epsilon\epsilon_0 \boldsymbol{E} \\ \boldsymbol{B} &= \mu\mu_0 \boldsymbol{H} \\ \boldsymbol{J} &= \sigma \boldsymbol{E} \end{aligned}$$

- Maxwell 4:

$$\nabla\times\boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla\times\boldsymbol{H} = \sigma\boldsymbol{E} + \epsilon\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

- Look for oscillating solutions: $\boldsymbol{E}, \boldsymbol{H} \propto e^{-i\omega t}$

$$\begin{aligned} \nabla\times\boldsymbol{H} &= \left(-i\omega\epsilon\epsilon_0 + \sigma\right)\boldsymbol{E} \\ &= -i\omega\epsilon_0 \left(\epsilon + \tfrac{i\sigma}{\omega\epsilon_0}\right)\boldsymbol{E} \end{aligned}$$

- Effective dielectric constant is $\left(\epsilon + \dfrac{i\sigma}{\omega\epsilon_0}\right)$

Waves in Conducting Media II

- Effective dielectric constant ϵ' is *complex*

$$\epsilon' = \left(\epsilon + \frac{i\sigma}{\omega\epsilon_0}\right)$$

(c.f. plasma — dielectric constant is real.)

- Typical values: Copper $\sigma = 5 \times 10^7\ \Omega^{-1}$ m^{-1}

$$\frac{\sigma}{\omega\epsilon_0} \approx \frac{10^{18}}{\nu} \quad (\nu = \frac{\omega}{2\pi})$$

- So for light waves $(\nu \leq 10^{15})$ and all lower frequency waves the real part of ϵ' is negligible.

- For metals we can approximate: $\epsilon' = \dfrac{i\sigma}{\omega\epsilon_0}$ (ignores displacement current).

- Refractive index is *complex*:

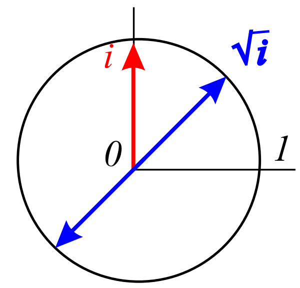
$$n = \sqrt{\epsilon'\mu} = \sqrt{\frac{\sigma\mu i}{\omega\epsilon_0}}$$

Waves in Metals

- Refractive index is *complex*:

$$n = \sqrt{\epsilon'\mu} = \sqrt{\frac{\sigma\mu i}{\omega\epsilon_0}}$$

$$= \pm\frac{(1+i)}{\sqrt{2}}\sqrt{\frac{\sigma\mu}{\omega\epsilon_0}}$$



- Wave travelling in the $+z$ direction $\propto \exp i(kz - \omega t)$

$$k = \frac{\omega}{c/n} = \sqrt{\frac{\sigma\mu}{2\omega\epsilon_0}}\frac{\omega}{c}(1+i) = \sqrt{\frac{\sigma\mu\mu_0\omega}{2}}(1+i)$$

- Characteristic length $\delta = \sqrt{\dfrac{2}{\sigma\mu\mu_0\omega}}$ is the *skin depth*.

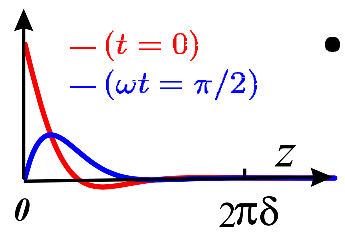
$$\Rightarrow E_x = E_0 \exp i\left(\frac{z}{\delta} - \omega t\right) e^{-z/\delta}$$

↑ travelling wave ↑ decay in z

- Wave travelling in the negative z direction also decays in the direction of propagation.

Properties of Waves in Conductors

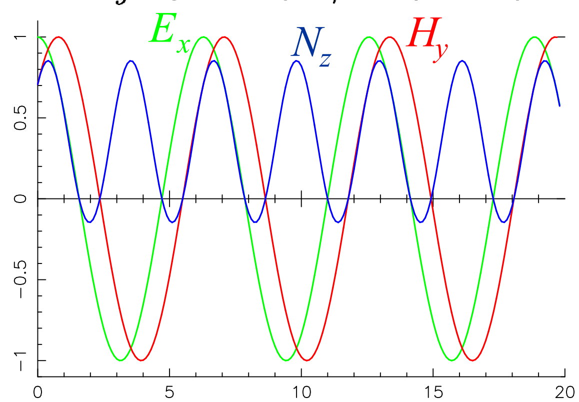
- Real part refractive index $n_r \equiv \sqrt{\frac{\mu\sigma}{2\omega\epsilon_0}}$ for Copper at $\nu = \omega/2\pi$=100 MHz is $\approx 7 \times 10^4$.
- $\Rightarrow$ skin depth at 100 MHz ($\lambda \approx 3$ m) is $\delta \approx 6\ \mu$m.



- Attenuation in z is very severe. Decays by factor of $e^{-2\pi} \approx 1/535$ in every wavelength.

- $H_y = \sqrt{\frac{\epsilon'\epsilon_0}{\mu\mu_0}} E_x = \sqrt{\frac{i\sigma}{\omega\mu\mu_0}} = \frac{n_r}{\mu Z_0}(1+i)E_x$

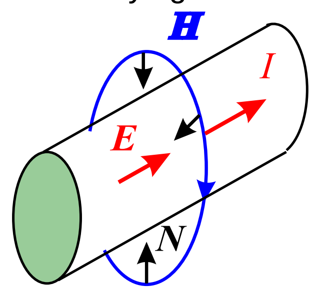
  $\Rightarrow H_y$ lags E_x by $\pi/4$ in phase (wave $\propto e^{-i\omega t}$).



- Poynting vector has mean in the positive z direction.
  $\Rightarrow$ Ohmic loss, Joule heating.

Skin Effect

- Wire carrying current I oscillating at frequency ω:



- Poynting vector $\boldsymbol{N} = \boldsymbol{E}\times\boldsymbol{H}$ points perpendicular to, *and into* the surface of the conductor.
- $\Rightarrow$ Flow of energy into the wire.

- Treat as travelling wave in the surface of the conductor.
- For plane surface

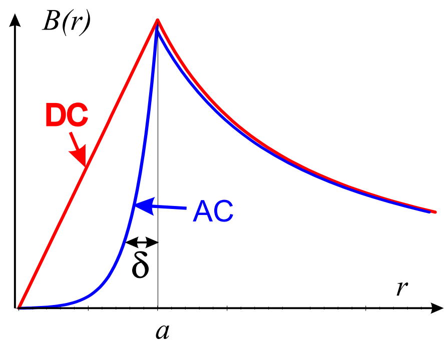
$$E_x = E_0 \exp i\left(\frac{z}{\delta} - \omega t\right) e^{-z/\delta}$$

$$J_x = J_0 \exp i\left(\frac{z}{\delta} - \omega t\right) e^{-z/\delta}$$

- $\Rightarrow$ Amplitude of surface wave decays into the wire.

SKIN EFFECT II

- Plane surface — skin depth $\delta = \sqrt{\dfrac{2}{\sigma \mu \mu_0 \omega}}$
- Oscillating currents confined to surface. Skin depth gets thinner as the frequency increases.



- Qualitatively the same effect for a circular wire: $\boldsymbol{J}$ confined to depth $\approx \delta$ near the surface.
- $\Rightarrow$ No magnetic field in the middle.

RESISTANCE OF WIRE AT HIGH FREQUENCY

- Suppose radius of wire is $a \gg \delta$
- We can approximate the circular case by "unwrapping" the cylindrical surface into a plane surface of width $\approx 2\pi a$.
- Total current in wire is

$$I = \int \mathrm{d}S \, J_x \approx 2\pi a \int \mathrm{d}z J_x(z)$$

- Take $z = 0$ on the surface and let other limit $\to \infty$:

$$I = 2\pi a J_0 e^{-i\omega t} \int_0^\infty \mathrm{d}z \exp\left(\frac{z}{\delta}(-1+i)\right)$$

$$= 2\pi a J_0 e^{-i\omega t} \frac{\delta}{(1-i)} \frac{1+i}{1+i} = \pi a J_0 e^{-i\omega t} \delta(1+i)$$

- Real part of current: $\Re(I) = \pi a J_0 \delta \left(\cos \omega t + \sin \omega t\right)$
- $\Rightarrow \langle I^2 \rangle = \pi^2 a^2 J_0^2 \delta^2$

Resistance of Wire at High Frequency II

- Power dissipated per unit volume: $\boldsymbol{J}\cdot\boldsymbol{E} = \left\langle J^2 \right\rangle / \sigma$.

$$\Re(J_x) = J_0\, e^{-z/\delta} \cos\left(z/\delta - \omega t\right)$$

$$\Rightarrow \frac{\left\langle J^2 \right\rangle}{\sigma} = \frac{J_0^2}{2\sigma}\, e^{-2z/\delta}$$

- Total power dissipated per unit length of wire

$$\frac{2\pi a J_0^2}{2\sigma} \int_0^\infty \mathrm{d}z\, e^{-2z/\delta} = \frac{\pi a J_0^2 \delta}{2\sigma}$$

- Resistance per unit length:

$$\frac{\langle \text{Power dissipated} \rangle}{\langle I^2 \rangle} = \frac{\pi a J_0^2 \delta}{2\sigma\, \pi^2 a^2 J_0^2 \delta^2} = \frac{1}{2\pi a \delta \sigma}$$

- Effective cross-section area of wire $= 2\pi a\delta$
  — just as if current were flowing uniformly in skin depth δ

Electromagnetic Waves in Media — Summary

	μ	ϵ	n	$Z = Z_0\sqrt{\frac{\mu}{\epsilon}}$
Dielectric medium	μ	ϵ	real	real
Free electrons	μ	$1 - \frac{\omega_\mathrm{p}^2}{\omega^2}$	a) real $(\omega > \omega_\mathrm{p})$ b) imag $(\omega < \omega_\mathrm{p})$	a) real $(\omega > \omega_\mathrm{p})$ b) imag $(\omega < \omega_\mathrm{p})$
Conduction electrons	μ	$\approx \frac{i\sigma}{\omega\epsilon_0}$	complex $(=n_\mathrm{r}(1+i))$	complex