# 1 Bayesian analysis of Faraday rotation for the SKA

## 1.1 Estimation of flux densities

As a preliminary analysis we look at the estimation of flux density $S(\nu)$ of a radio source (assumed unresolved). Assuming for the moment that the flux density is constant in the band, the total energy received is

$$E = S(\nu)A\Delta t\Delta\nu \tag{1}$$

where $A$ is the collecting area, $\Delta\nu$ the bandwidth and $\Delta t$ the observation period. In the same period we receive a noise power from the receiver at temperature $T$ of

$$\delta E^2 = k^2T^2\Delta t\Delta\nu \tag{2}$$

the signal-to-noise ratio is therefore

$$\frac{E}{\delta E} = S(\nu)\frac{A\sqrt{\Delta t}}{kT}\sqrt{\Delta}\nu \tag{3}$$

where we have separated the parts that depend on the sky, the telescope and the design of the spectrometer. This formula provides the basis for defining the likelihood:

$$\Pr(S(\nu)|S_0(\nu)) = \sqrt{\frac{\Delta\nu}{2\pi\Delta S^2}}\exp\left(-\frac{(S(\nu)-S_0(\nu))^2\Delta\nu}{2\Delta S^2}\right) \tag{4}$$

where $S_0(\nu)$ is the expected flux. In this formula $\Delta S(\nu)$ is the overall sensitivity (in Jy $\mathrm{Hz}^{1/2}$) of the system at frequency $\nu$. At frequencies where the system has no sensitivity $\Delta S$ is infinite. Perhaps we might consider using its inverse? I've left it like this at the moment.

Suppose now that we are to infer parameters from a number of channels at frequencies $\nu_i$. The log-likelihood is simply

$$\log\Pr(\{S(\nu_i)\}|\{S_0(\nu_i)\}) = -\tfrac{1}{2}\sum_i \log(2\pi\Delta S_i^2/\Delta\nu_i) - \tfrac{1}{2}\sum_i \frac{(S(\nu_i)-S_0(\nu_i))^2\Delta\nu_i}{\Delta S_i^2} \tag{5}$$

All the predictive power for estimating parameters is contained in the second term in the above formula. If we assume for the moment that we have good spectral resolution , we can conveniently express it as an integral over frequency:

$$-\tfrac{1}{2}\sum_i \frac{(S(\nu_i)-S_0(\nu_i))^2\Delta\nu_i}{\Delta S_i^2} \approx -\tfrac{1}{2}\int \mathrm{d}\nu\ \frac{(S(\nu)-S_0(\nu))^2}{\Delta S^2(\nu)} \tag{6}$$

The first term in (5) does not contain the data or the parameters, so does not affect the posterior. However, it cannot be expressed as an integral over frequency, so that the details of the spectrometer configuration do affect the evidence. This is a general feature of systems where data are averaged before measurement. A simple example is explained in the Appendix.

## 1.2 Parameter estimation of Faraday rotation

Conventions for polarisation parameters are from Thompson, Moran & Svenson, 1986. Suppose we have a source with spectral index $\alpha$, with total polarisation $P_0$ at standard frequency $\nu_0$, a polarisation angle $\phi_\infty$ at infinite frequency, and rotation measure $R$. We wish to estimate these 4 parameters given measurement of $Q(\nu)$ and $U(\nu)$, with a telescope configuration defined as above by a sensitivity function $\Delta S^2(\nu)$. In what follows we assume that the these sensitivities refer directly to measurements of $U$ and $Q$.

The rotation at frequency $\nu$ is

$$\phi(\nu) = Rc^2\nu^{-2} + \phi_\infty \tag{7}$$

and is related to the Stokes parameters via

$$\frac{U(\nu)}{Q(\nu)} = \tan 2\phi(\nu) \tag{8}$$

The total polarisation at frequency $\nu$ is related to the spectral index via

$$(Q(\nu)^2 + U(\nu)^2)^{1/2} = P_0 \left(\frac{\nu}{\nu_0}\right)^{-\alpha} \tag{9}$$

Hence the complete model is:

$$Q_0(\nu) = P_0 \left(\frac{\nu}{\nu_0}\right)^{-\alpha} \cos 2(Rc^2\nu^{-2} + \phi_\infty); \quad U_0(\nu) = P_0 \left(\frac{\nu}{\nu_0}\right)^{-\alpha} \sin 2(Rc^2\nu^{-2} + \phi_\infty) \tag{10}$$

where the subscripts on $Q$ and $U$ indicate that they are the model for the data.

The relevant part of the log-likelihood is then

$$-\tfrac{1}{2} \int d\nu \, \frac{((Q(\nu) - Q_0(\nu))^2 + (U(\nu) - U_0(\nu))^2}{\Delta S^2(\nu)} \tag{11}$$

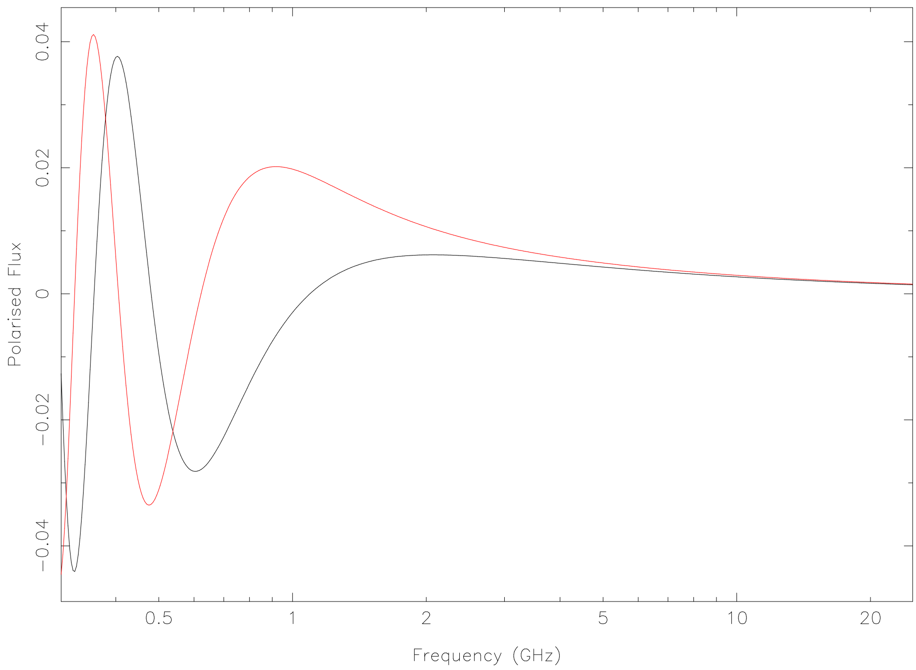
where we have assumed that the sensitivities in the two polarisation are the same. This analysis has ignored the effects of finite bandwidth for the individual channels which could, in extreme cases, reduce the overall polarisation. For a rotation measure of 200 rads $m^{-1}$, the bandwidth should be less than $\approx 200(\nu/GHz)^3$ MHz.

The SKA will have 700 MHz coverage from 300 MHz to 1 GHz, and 4 GHz coverage above 1 GHz, so we will assume that it starts at 1 GHz for the moment. We will be probing clusters at redshifts of about 2, so the effective rotation measure is reduced by about a factor of 4. Figure 1 shows the expected polarised signal as a function of frequency.

Paul Alexander has provided the following sensitivity information for the SKA. These figures relate to one hour of observation.

| Band | Bandwidth | $\sigma_I$ in 1hr |
|---|---|---|
| 0.3GHz — 1 GHz | 700 MHz | 1.4 $\mu$Jy |
| 1 GHz — 4 GHz | 1 GHz | 0.13 $\mu$Jy |
| 4 GHz — 25 GHz | 4 GHz | 0.08 $\mu$Jy |

Since I do not have information about the precise frequencies available, I have assumed that we use the whole band, and have scaled the sensitivities downwards accordingly.



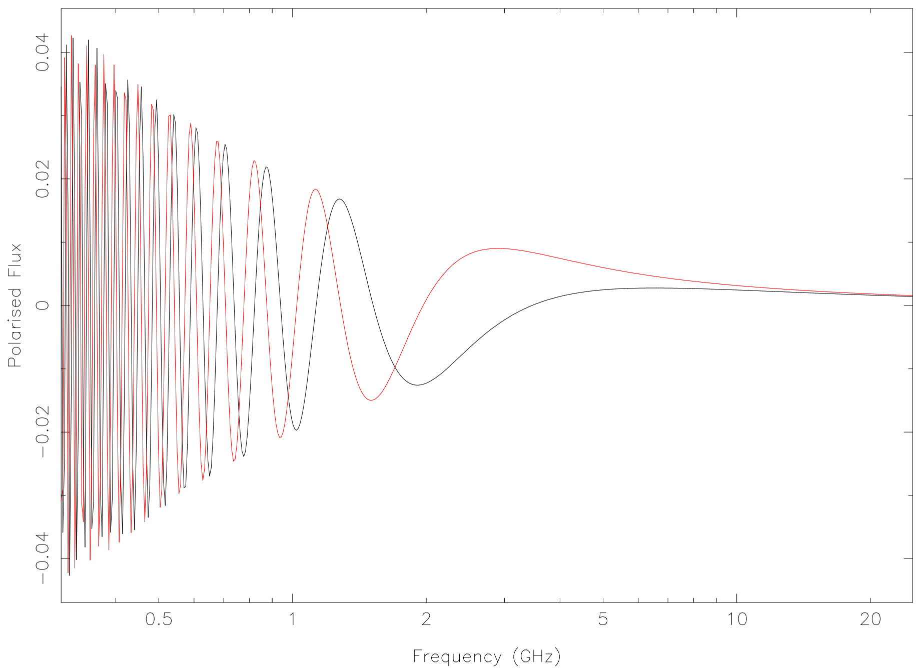




Figure 1: Simulated values of the Stokes parameters $Q$ and $U$ for rotation measures of 5 and 50. The spectral index was 0.7.

## 1.3 Simulation of parameter estimation using BayeSys

We have now assembled everything we need to use the MCMC program BayeSys to simulate the estimation process. For this problem BayeSys will use a flat prior in a 4-dimensional hypercube, using uniform random numbers $0 < r_i < 1$. We need to map

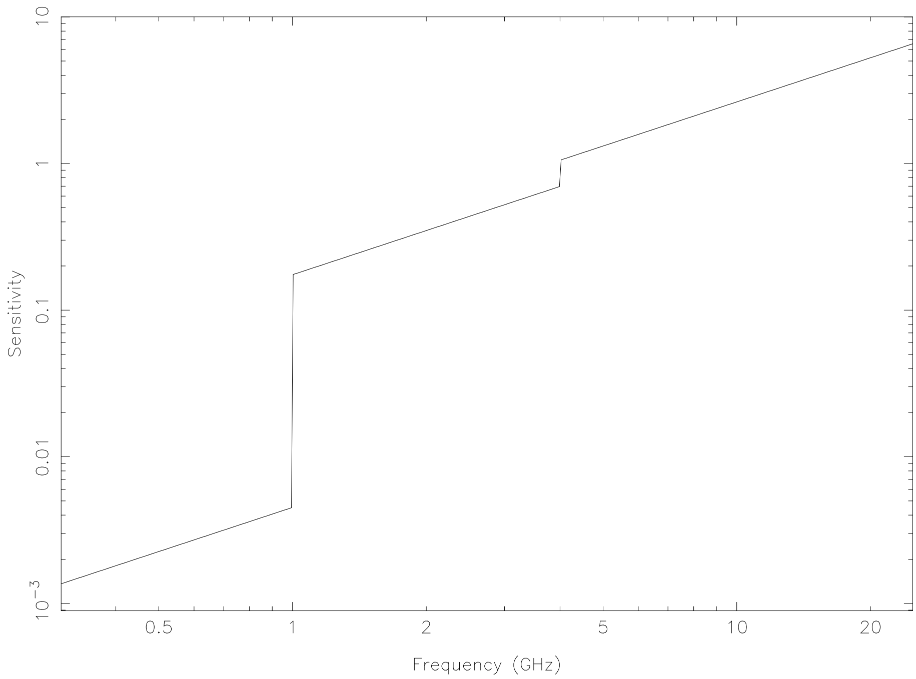



Figure 2: Frequency dependence of the SKA sensitivity, measured in $\mu$ Jy$^{-2}$ GHz$^{-1}$.

these 4 random numbers into our parameter space. This is fairly standard and we use the usual mappings:

1. Spectral index (Cauchy + offset): $\alpha = \alpha_0 + \delta\alpha \tan^{-1}(\pi(r - \frac{1}{2}))$.
2. Polarisation (John Skilling special): $P_0 = P_{00}r/(1 - r)$ (good general form for positive parameters).
3. Polarisation angle (uniform): $\phi_\infty = \pi r$.
4. Rotation measure (Cauchy ): $\alpha = \delta R \tan^{-1}(\pi(r - \frac{1}{2}))$.

The constants in these assignments need to be finalised in the light of the expected values of the parameters, but will not affect the predictions very much.

The simulation showed that the rotation measure can be measured very reliably in source of about 1 $\mu$Jy polarised flux (about a $4\sigma$ detection).