Part IB Physics B

Classical Dynamics

S.F. Gull

University of Cambridge
Department of Physics

Michaelmas Term 2011 and Lent Term 2012

Synopsis

This course builds on the ideas introduced in Part IA, using the machinery of vector calculus taught in Part IA Mathematics. The main areas covered are orbits, rigid body dynamics, normal modes and continuum mechanics (elasticity and fluids).

Newtonian mechanics, frames of reference. Review of Part IA mechanics: many-particle system, internal and external forces and energy. Central forces, motion in a plane. Non-inertial frames, rotating frames, centrifugal and Coriolis forces. Examples.

Orbits. Effective potential and radial motion, bound and unbound orbits. Inverse-square law orbits, circular and elliptic, Kepler's laws. Escape velocity, transfer orbits, gravitational slingshot. Hyperbolic orbits, angle of scattering, repulsive force. Two-body problem, reduced mass. General features of three-body problem.Brief treatment of tidal effects in gravitational systems.

Rigid body dynamics. Instantaneous motion of a rigid body, angular velocity and angular momentum, moment of inertia tensor, principal axes and moments. Rotational energy, inertia ellipsoid. Euler's equations, free precession of a symmetrical top, space and body frequencies. Forced precession, gyroscopes.

Introduction to Lagrangian mechanics. Generalised coordinates. Hamilton's principle and Lagrange's equations. Symmetries and conservation laws. Conservation of the Hamiltonian for time-independent systems.

Normal modes. Analysis of many-particle system in terms of normal modes. Degrees of freedom, matrix notation, zero-frequency and degenerate modes. Continuum limit, wave equation. Standing waves, energy and normal modes. Motion in three dimensions, modes of molecules.

Elasticity. Hooke's law, Young's modulus, Poisson's ratio. Bulk modulus, shear modulus, stress tensor, principal stresses. strain tensor. Elastic energy. Torsion of cylinder. Bending of beams, bending moment, boundary conditions. Euler strut. Brief treatment of elastic waves. Energy flow in waves.

Fluid dynamics. Continuum fields, material derivatives, relation to particle paths and streamlines. Mass conservation, incompressibility. Convective derivative and equation of motion. Bernoulli's theorem, applications. Velocity potential, applications: sources and sinks; flow past a sphere and cylinder; vortices; Magnus effect. Viscosity, Couette and Poiseuille flow. Reynolds number, lamina and turbulent flow.

Books

Classical Mechanics, Barger V D and Olsson M G (McGaw-Hill, 1995).

Classical Mechanics, Kibble T W B and Berkshire F H (Imperial College 2004).

Principles of Dynamics, Greenwood D T (Prentice & Hall 1988).

Mechanics, Landau L D and Lifshitz E M, (Pergamon, 1976)

For elasticity

Lectures on Physics, Feynman R P, Leighton R B and Sands S L (Addison Wesley 1964).

Theory of Elasticity, Landau L D and Lifshitz E M,(Pergamon, 1976)

For fluids

Fluid Dynamics for Physicists, Faber T E, (Cambridge, 1995).

Lectures on Physics, Feynman R P, Leighton R B and Sands S L (Addison Wesley 1964).

Contents

1 Mathematics — Review

1.1 Coordinate systems

Cartesian	Cylindrical polars	Spherical polars
(x, y, z)	(ρ, ϕ, z)	(r, θ, ϕ)

$$\begin{array}{lllll} x & = & \rho \cos\phi & = & r \sin\theta \cos\phi \\ y & = & \rho \sin\phi & = & r \sin\theta \sin\phi \\ z & = & z & = & r \cos\theta \end{array}$$

1.2 Vectors

$\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$.

$\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$ are unit vectors along x, y, z axes.

Addition: $\mathbf{a} + \mathbf{b}$.

Scalar multiplication: $\lambda\mathbf{a}$.

Scalar product: $\mathbf{a}{\cdot}\mathbf{b} = a_x b_x + a_y b_y + a_z b_z = \mid a \mid \mid b \mid \ \cos\theta$. θ is the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$.

Vector product: $\mathbf{a}{\times}\mathbf{b}$ = a vector of magnitude $|a|\,|b|\ \sin\theta$ in direction perpendicular to $\mathbf{a}$ and $\mathbf{b}$; $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}{\times}\mathbf{b}$ are a right-hand set.

$\mathbf{a}{\times}\mathbf{b} = (a_y b_z - a_z b_y,\ \ldots,\ \ldots)$.

Triple scalar product: $\mathbf{a}{\cdot}(\mathbf{b}{\times}\mathbf{c}) = (\mathbf{a}{\times}\mathbf{b}){\cdot}\mathbf{c}$ etc = volume of parallelepiped from $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Triple vector product: $\mathbf{a}{\times}(\mathbf{b}{\times}\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$.

$$\text{del} = \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

$$\text{grad}\,\phi = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \text{ — a vector.}$$

$$\text{div}\,\mathbf{A} = \nabla{\cdot}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \text{ — a scalar.}$$

$$\text{curl}\,\mathbf{A} = \nabla{\times}\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z},\ \ldots,\ \ldots\right) \text{ — a vector.}$$

$\text{curl}\,(\text{grad}\,\phi) = 0$ always.

$\text{div}\,(\text{curl}\,\mathbf{A}) = 0$ always.

$\text{curl}\,\text{curl}\,\mathbf{A} = \text{grad}\,(\text{div}\,\mathbf{A}) - \nabla^2\mathbf{A}$.

1.3 Matrices

1.3.1 Geometrical view

Linear relation between vectors: $\mathbf{r}' = \mathbf{M}\mathbf{r}$. Two situations:

1. Linear transformation of space (rotation, deformation etc of an object). Point P at $\mathbf{r}$ in a body is displaced to $\mathbf{r}'$. Columns of $\mathbf{M} \equiv$ where unit vectors along the original axes are moved to.

2. Change of axes. $\mathbf{r}'$ and $\mathbf{r}$ refer to the *same* vector (same magnitude and direction in space) in new and old coordinate systems. Columns of $\mathbf{M} \equiv$ old $\hat{\mathbf{e}}$'s (unit vectors along axes) in new coordinates; those of $\mathbf{M}^{-1} \equiv$ new $\hat{\mathbf{e}}$'s in old coordinates.

1.3.2 Orthogonal matrices

preserve orthogonality of axes; scalar products of different columns are zero. If in addition the scalar product of any column with itself is 1, then

$$\begin{pmatrix} \text{———} \\ \text{———} \\ \text{———} \end{pmatrix} \begin{pmatrix} | & | & | \\ | & | & | \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{pmatrix} = \mathbf{I}, \qquad (. = 0)$$

or $\mathbf{M}'\mathbf{M} = \mathbf{I}$, or $\mathbf{M}^{-1} = \mathbf{M}'$ [$\mathbf{M}'$ is the transpose of $\mathbf{M}$].

1.3.3 Tensors

Given $\mathbf{a} = \mathbf{K}\mathbf{b}$, with $\mathbf{a}$ and $\mathbf{b}$ vectors and $\mathbf{K}$ a matrix, a *tensor* is the relationship between $\mathbf{a}$ and $\mathbf{b}$ thought of as magnitudes and directions in space without regard to components *wrt* specific axes [a physicist's view — see IB Maths for a formal treatment; we need to enlarge it later]. For given (3D) axes, $\mathbf{K}$ needs 9 numbers

$$a_i = \sum_j K_{ij} b_j.$$

The tensor is usually written K_{ij}.

Rotate axes by $\mathbf{M}$, giving $\mathbf{a}' = \mathbf{M}\mathbf{a}$, $\mathbf{b}' = \mathbf{M}\mathbf{b}$. Then

$$\mathbf{M}^{-1}\mathbf{a}' = \mathbf{K}\mathbf{M}^{-1}\mathbf{b}',$$

or

$$\mathbf{a}' = \underbrace{\mathbf{M}\mathbf{K}\mathbf{M}^{-1}}_{\equiv \mathbf{K}'}\mathbf{b}'.$$

The tensor is the relationship expressed by $\mathbf{K}$ and $\mathbf{K}'$, which are physically equivalent, though their components are different.

Ex. 1 *Stress in a material.* $\mathbf{b}$ *= vector area of a surface within the material,* $\mathbf{a}$ *= force across that surface;* $\mathbf{a}$ *is linear with ('proportional to')* $\mathbf{b}$ *though not necessarily parallel to it. The 'proportionality' is the stress tensor.*

1.3.4 Eigen vectors, eigen values

An eigen vector of $\mathbf{M}$ is a vector $\mathbf{e}$ such that $\mathbf{Me} = \lambda\mathbf{e}$, ($\lambda$ scalar); λ is an eigen value of $\mathbf{M}$. Eigen values are obtained by solving the *characteristic equation* $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$; there are n of them for an $n{\times}n$ matrix.

If $\mathbf{M}$ is symmetrical, eigen values are all real.

1. If all λ's different, eigen vectors $\mathbf{e}_i$ are unique (in direction) and mutually orthogonal.
2. If some λ's are the same, there is a choice of $\mathbf{e}_i$'s, but you can always *find* orthogonal $\mathbf{e}_i$'s [if $\mathbf{e}_a$ and $\mathbf{e}_b$ are eigen vectors with the same λ, then so is any linear combination $\alpha\mathbf{e}_a + \beta\mathbf{e}_b$].

Wrt $\mathbf{e}$'s as axes, matrix is *diagonal*, the diagonal elements being the λ's.

1.3.5 Quadratic forms

e.g. $F = ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz$. It can be written $F = \mathbf{r}'\mathbf{Mr}$, with

$$\mathbf{M} = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}.$$

$F =$ constant is an ellipsoid/hyperboloid/conicoid in (x, y, z) space.

$$\operatorname{grad} F = \begin{pmatrix} 2ax + 2dy + 2ez \\ 2dx + 2by + 2fz \\ 2ex + 2fy + 2cz \end{pmatrix} = 2\mathbf{Mr}$$

and is $\perp$ surface of constant F. Axes of conicoid are when F-surface $\perp$ $\mathbf{r}$, or grad$F \parallel \mathbf{r}$, or $\mathbf{Mr} = \lambda\mathbf{r}$; i.e. axes are eigen vectors of $\mathbf{M}$. *Wrt* these axes $F = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2$.

1.4 Rotations

$\mathbf{r} \to \mathbf{r}' = \mathbf{Mr}$ with $\mathbf{M}$ an orthogonal matrix.

Ex. 2 $\mathbf{M} = \begin{pmatrix} \cos\psi & -\sin\psi & . \\ \sin\psi & \cos\psi & . \\ . & . & 1 \end{pmatrix}$ *rotates by ψ about the z-axis.*

$\mathbf{M}' = \mathbf{M}^{-1} \equiv$ rotation by $-\psi$ (see §1.3.2 above). $\mathbf{M}$ is not symmetrical. The eigen values (not necessarily real) are $1, e^{\pm i\psi}$. The eigen vector for the one real eigen value $\equiv$ the rotation axis.

Euler angles: 3 real numbers are needed to specify a rotation, e.g. (θ, ϕ) of rotation axis in spherical polars, and $\psi =$ angle of rotation about that axis [NB: not a symmetrical set — beware of different conventions].

Two successive rotations give

$$\mathbf{r}'' = \mathbf{M}_2\mathbf{r}' = \underbrace{\mathbf{M}_2\mathbf{M}_1}\,\mathbf{r} \equiv \mathbf{M}_3\mathbf{r}.$$

$\mathbf{M}_2\mathbf{M}_1 \neq \mathbf{M}_1\mathbf{M}_2$ in general.

$\mathbf{M}$'s for *infinitesimal* rotations *do* commute. Define $\boldsymbol{\psi} =$ vector of magnitude ψ (Euler) in direction of clockwise rotation axis; suppose $|\,\psi\,| \ll 1$. Rotation by $\boldsymbol{\psi}$ moves any $\mathbf{r}$ by $\mathbf{dr} = \boldsymbol{\psi}\times\mathbf{r}$. So

$$\mathbf{Mr} = \mathbf{r} + \mathbf{dr} = \mathbf{r} + \boldsymbol{\psi}\times\mathbf{r}.$$

Two rotations give

$$\begin{aligned} \mathbf{M}_2(\mathbf{M}_1\mathbf{r}) &= \mathbf{M}_1\mathbf{r} + \boldsymbol{\psi}_2\times(\mathbf{M}_1\mathbf{r}) \\ &= \mathbf{r} + \boldsymbol{\psi}_1\times\mathbf{r} + \boldsymbol{\psi}_2\times(\mathbf{r} + \boldsymbol{\psi}_1\times\mathbf{r}) \\ &= \mathbf{r} + (\boldsymbol{\psi}_1 + \boldsymbol{\psi}_2)\times\mathbf{r} + \underbrace{\boldsymbol{\psi}_2\times(\boldsymbol{\psi}_1\times\mathbf{r})}_{\approx 0,\ |\,\psi\,| \ll 1}. \end{aligned}$$

Therefore $\mathbf{M}_3 = \mathbf{M}_2\mathbf{M}_1 \equiv$ a rotation by vector $\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1 \equiv \boldsymbol{\psi}_3$, and $\mathbf{M}_2\mathbf{M}_1 = \mathbf{M}_1\mathbf{M}_2$.

In detail

$$\mathbf{M} = \begin{pmatrix} 1 & -\psi_z & \psi_y \\ \psi_z & 1 & -\psi_x \\ -\psi_y & \psi_x & 1 \end{pmatrix}.$$

Angular velocity $\boldsymbol{\omega} = d\boldsymbol{\psi}/dt$ similarly. $\boldsymbol{\omega}_3 = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 =$ vector resultant. $\boldsymbol{\psi}$'s here are necessarily infinitesimal.

2 Coordinate systems

2.1 Fixed frames

In Cartesians, the equation of motion of a particle is

$$m\ddot{\mathbf{r}} = \mathbf{F}; \quad \text{or} \quad m\ddot{x} = F_x \text{ etc.}$$

Consider first cylindrical polars; ignore z-motion *pro tem.*

$$\mathbf{r} = \rho\hat{\mathbf{e}}_\rho$$

where $\boldsymbol{\sigma}_\rho$, $\hat{\mathbf{e}}_\phi$ and $\hat{\mathbf{e}}_z$ are unit vectors in the directions of increasing ρ, ϕ, z; $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ change as the particle moves.

$$\dot{\mathbf{r}} = \dot\rho\hat{\mathbf{e}}_\rho + \rho\dot{\hat{\mathbf{e}}}_\rho.$$

As the particle moves from say P to P' in $\mathrm{d}t$, $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ rotate by $\mathrm{d}\phi$. Elementary geometry gives

$$\mathrm{d}\hat{\mathbf{e}}_\rho = \mathrm{d}\phi\,\hat{\mathbf{e}}_\phi \quad \text{or} \quad \dot{\hat{\mathbf{e}}}_\rho = \dot\phi\hat{\mathbf{e}}_\phi,$$

and similarly

$$\dot{\hat{\mathbf{e}}}_\phi = -\dot\phi\hat{\mathbf{e}}_\rho,$$

giving

$$\dot{\mathbf{r}} = \underbrace{\dot\rho}_{\text{radial}}\,\hat{\mathbf{e}}_\rho + \underbrace{\rho\dot\phi}_{\text{transverse}}\,\hat{\mathbf{e}}_\phi. \qquad \text{velocities}$$

Similarly

$$\begin{aligned}
\ddot{\mathbf{r}} &= \ddot\rho\hat{\mathbf{e}}_\rho + \dot\rho\underbrace{\dot{\hat{\mathbf{e}}}_\rho}_{\dot\phi\hat{\mathbf{e}}_\phi} + \dot\rho\dot\phi\hat{\mathbf{e}}_\phi + \rho\ddot\phi\hat{\mathbf{e}}_\phi + \rho\dot\phi\underbrace{\dot{\hat{\mathbf{e}}}_\phi}_{-\dot\phi\hat{\mathbf{e}}_\rho} \\
&= \underbrace{(\ddot\rho - \rho\dot\phi^2)}_{\text{radial}}\,\hat{\mathbf{e}}_\rho + \underbrace{(2\dot\rho\dot\phi + \rho\ddot\phi)}_{\text{transverse}}\,\hat{\mathbf{e}}_\phi. \qquad \text{accelerations}
\end{aligned}$$

The z-motion is independent: $(\ddot{\boldsymbol{r}})_z$ is just $\ddot{z}\hat{\mathbf{e}}_z$ since $\dot{\hat{\mathbf{e}}}_z = 0$.

Alternative derivation (for cylindrical coordinates): Treat the 2D (x, y) vector as the complex number $r = x + iy = \rho e^{i\phi}$. Express $\ddot{r} = \mathrm{d}^2(\rho e^{i\phi})/\mathrm{d}t^2$ as $(a_\rho + ia_\phi)e^{i\phi}$; a_ρ and a_ϕ are then the radial and transverse accelerations.

Spherical polars can be treated by putting $\mathbf{r} = r\hat{\mathbf{e}}_r$, and expanding $\dot{\mathbf{r}}$ etc with $\dot{\hat{\mathbf{e}}}_r$ put in terms of $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$.

2.2 Frames in relative motion

Suppose we have a frame S_0 in which $m\ddot{\mathbf{r}}_0 = \mathbf{F}$, with $\mathbf{F}$ ascribed to known physical causes. What is the apparent equation of motion in a moving frame S?

Ex. 3 *Suppose* $\mathbf{r} = \mathbf{r}_0 - \mathbf{R}(t)$.

Axes remain parallel and $t = t_0$ (as always in classical physics).

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_0 - \ddot{\mathbf{R}}.$$

For the special case $\ddot{\mathbf{R}} = 0$ (i.e. steady motion between frames), $m\ddot{\mathbf{r}} = m\ddot{\mathbf{r}}_0 = \mathbf{F}$, i.e. the *same* equation of motion (Galilean transformation).

For general $\mathbf{R}$,

$$m\ddot{\mathbf{r}} = m\ddot{\mathbf{r}}_0 - m\ddot{\mathbf{R}}$$

or the *apparent force* in S includes both the actual force ($m\ddot{\mathbf{r}}_0$, as in S_0) and a *fictitious force* $-m\ddot{\mathbf{R}}$. Fictitious forces are

1. associated with *accelerated* frames, and
2. proportional to *mass*.

Question: Is gravity a fictitious force? Answer (according to Einstein and general relativity): Yes — space is *curved*, i.e. non-linear, and equivalent to acceleration in some sense.

2.3 Rotating frames

Suppose now S rotates *wrt* S_0 at steady $\boldsymbol{\omega}$. Let

$$\mathbf{r}_0 = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$$

where the $\hat{\mathbf{e}}$'s are unit vectors along the axes in S; $\mathbf{r} = (x, y, z)$ is the apparent position in S. Because of the rotation

$$\dot{\hat{\mathbf{e}}}_x = \boldsymbol{\omega}\times\hat{\mathbf{e}}_x,$$
$$\ddot{\hat{\mathbf{e}}}_x = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\hat{\mathbf{e}}_x).$$

Therefore

$$\begin{aligned}
\ddot{\mathbf{r}}_0 &= \ddot{x}\hat{\mathbf{e}}_x + 2\dot{x}\dot{\hat{\mathbf{e}}}_x + x\ddot{\hat{\mathbf{e}}}_x + \ldots + \ldots \ (y \text{ and } z \text{ terms similarly}) \\
&= \ddot{x}\hat{\mathbf{e}}_x + 2\dot{x}(\boldsymbol{\omega}\times\hat{\mathbf{e}}_x) + x[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\hat{\mathbf{e}}_x)] + \ldots + \ldots \ (y \text{ and } z \text{ terms}) \\
&= \ddot{\mathbf{r}} + 2\boldsymbol{\omega}\times\mathbf{v} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r});
\end{aligned}$$

here $\ddot{\mathbf{r}}$, $\mathbf{v}$ and $\mathbf{r}$ are the apparent acceleration, velocity and position in S ($\ddot{\mathbf{r}} = \ddot{x}\hat{\mathbf{e}}_x + \ddot{y}\hat{\mathbf{e}}_y + \ddot{z}\hat{\mathbf{e}}_z$ etc). We can rewrite this in its usual form as

$$m\ddot{\mathbf{r}} = \mathbf{F} \underbrace{-\ 2m\boldsymbol{\omega}\times\mathbf{v}}_{\text{Coriolis force}} \quad \underbrace{-\ m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})}_{\text{centrifugal force}}$$

where $m\ddot{\mathbf{r}}$, the apparent force in S, is given in terms of the physically known force $\mathbf{F}$ and two fictitious forces, Coriolis and centrifugal, both expressed in terms of apparent coordinates $\mathbf{v}$ and $\mathbf{r}$. The centrifugal force can easily be shown to be $+m\omega^2\mathbf{r}'$, where $\mathbf{r}'$ is the position vector in S *wrt* an origin at the nearest point on the rotation axis.

Alternative derivations:

1. Operators. For any vector **A**

$$\left[\frac{\mathrm{d}\mathbf{A}}{dt}\right]_{\text{in } S_0} = \left[\frac{\mathrm{d}\mathbf{A}}{dt}\right]_{\text{in } S} + \boldsymbol{\omega}\times\mathbf{A}.$$

   Apply this to $\left[\frac{d}{dt}\right]_{\text{in } S_0}\mathbf{r}$, giving

$$[m\ddot{\mathbf{r}}_0]_{\text{in } S_0} = m\left(\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]_{\text{in } S} + \boldsymbol{\omega}\times\right)\left(\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]_{\text{in } S} + \boldsymbol{\omega}\times\right)\mathbf{r}.$$

2. Complex numbers, since essentially 2D.

3 Mechanics — Review

Development from Newton's Laws.

$$m_a\ddot{\mathbf{r}}_a = \mathbf{F}_a;$$

$a = 1, N$ for the a'th of N particles. A rigid body is a special case.

3.1 Overall motion

$$\sum_a m_a\ddot{\mathbf{r}}_a = \sum_a \mathbf{F}_a = \sum_a \mathbf{F}_{a0} + \sum_a\sum_b \mathbf{F}_{ab}$$

where $\mathbf{F}_{a0}$ is the *external* force on particle a and $\mathbf{F}_{ab}$ is the force on a due to b. Since $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$ by Newton's 3rd Law, the $\sum_a\sum_b$ term above sums to zero.

Put $M = \sum_a m_a$ and $M\mathbf{R} = \sum_a m_a\mathbf{r}_a$. $\mathbf{R}$ is the position of the Centre of Mass. Then

$$M\ddot{\mathbf{R}} = \sum_a \mathbf{F}_{a0} = \mathbf{F}_0$$

i.e. the Centre of Mass moves as if it were a particle of mass M acted upon by the resultant external force $\mathbf{F}_0$.

In terms of momentum

$$\dot{\mathbf{p}}_a = \mathbf{F}_a; \quad \dot{\mathbf{P}} = \mathbf{F}_0$$

with $\mathbf{P}$ = total momentum.

3.2 Moments

Def$^{\mathbf{n}}$. 1 Couple, torque: $\mathbf{G} = \mathbf{r}\times\mathbf{F}$.

Def$^{\mathbf{n}}$. 2 Angular momentum: $\mathbf{J} = \mathbf{r}\times\mathbf{p}$.

Since $\dot{\mathbf{p}}_a = \mathbf{F}_a$,

$$\sum_a \mathbf{r}_a\times\dot{\mathbf{p}}_a = \sum_a \mathbf{r}_a\times\mathbf{F}_a.$$

Expand RHS:

$$\text{RHS } = \sum_a \mathbf{r}_a\times\mathbf{F}_{a0} + \underbrace{\sum_a\sum_b \mathbf{r}_a\times\mathbf{F}_{ab}}_{\sum\sum_{a<b}\underbrace{(\mathbf{r}_a-\mathbf{r}_b)\times\mathbf{F}_{ab}}_{=0}}.$$

The latter term is zero since $\mathbf{F}_{ab}$ is assumed to be along the line between a and b.

The LHS for one particle is

$$\dot{\mathbf{J}}_a = \frac{d}{dt}(\mathbf{r}_a\times\mathbf{p}_a) = \underbrace{\dot{\mathbf{r}}_a\times\mathbf{p}_a}_{\text{zero, since } m\dot{\mathbf{r}} = \mathbf{p}} + \mathbf{r}_a\times\dot{\mathbf{p}}_a.$$

For the system of particles

$$\dot{\mathbf{J}} = \sum_a \dot{\mathbf{J}}_a = \sum_a \mathbf{r}_a\times\dot{\mathbf{p}}_a = \text{RHS } = \sum_a \mathbf{r}_a\times\mathbf{F}_{a0} = \mathbf{G}_0;$$

$\mathbf{G}_0$ is the resultant $\mathbf{G}$ from all *external* forces.

3.3 Choice of origin

Suppose we move the origin by a constant $\mathbf{a}$, giving new coordinates $\mathbf{r}'$ with $\mathbf{r} = \mathbf{r}' + \mathbf{a}$. Then $\dot{\mathbf{r}} = \dot{\mathbf{r}}'$ and §3.1 unaffected. What about $\mathbf{J}$? For one particle $\mathbf{J}_a = \mathbf{J}'_a + \mathbf{a}\times\mathbf{p}_a$, or for the system

$$\mathbf{J} = \mathbf{J}' + \sum_a \mathbf{a}\times\mathbf{p}_a = \mathbf{J}' + \mathbf{a}\times\mathbf{P},$$

i.e. $\mathbf{J}$ depends on the choice of origin *unless* $\mathbf{P} = 0$.

Def$^{\mathbf{n}}$. 3 Intrinsic angular momentum: J *in the frame in which* $\mathbf{P} = 0$ *(zero-momentum, or Centre of Mass frame). It is independent of origin.*

Similarly $\mathbf{G} = \mathbf{G}' + \mathbf{a}\times\mathbf{F}$.

3.4 Energy

Def$^{\mathbf{n}}$. 4 Work done: *force × distance moved* $\|$ *force = change in* **energy**.

For a single particle

$$\mathbf{F}\cdot\mathbf{dr} = m\ddot{\mathbf{r}}\cdot\mathbf{dr} = m\underbrace{\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}}_{\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})}dt$$

or

$$\mathbf{F}\cdot\mathbf{dr} = d(\tfrac{1}{2}mv^2).$$

Def$^{\mathbf{n}}$. 5 Kinetic energy: $T \equiv \frac{1}{2}mv^2$.

Work done on particle = change in kinetic energy.

For a system of particles

$$\begin{aligned} dT &= \sum_a dT_a = \sum_a \mathbf{F}_a\cdot\mathbf{dr}_a \\ &= \sum_a \mathbf{F}_{a0}\cdot\mathbf{dr}_a + \sum\sum_{a<b}\mathbf{F}_{ab}\cdot(\mathbf{dr}_a - \mathbf{dr}_b) \end{aligned}$$

where we have used $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$. We can write the ab-term as $-\mathcal{F}_{ab}d\mid\mathbf{r}_a - \mathbf{r}_b\mid$, where $\mathcal{F}_{ab}$ has magnitude $= |F_{ab}|$ and is positive if force is attractive, negative if repulsive.

Def$^{\mathbf{n}}$. 6 Potential energy, internal energy: *U, such that*

$$dU = \sum\sum_{a<b}\mathcal{F}_{ab}\,d|\,\mathbf{r}_a - \mathbf{r}_b\,|\,.$$

Note the zero of $U = \int dU$ is undefined. It is often taken with $U = 0$ with particles at infinite separation, giving negative U for a system of particles with attractive forces. For a rigid body $dU = 0$ since $\mid\mathbf{r}_a - \mathbf{r}_b\mid$ is fixed.

Def$^{\mathbf{n}}$. 7 Energy: $E = T + U$.

As defined above

$$dE = dT + dU = \sum_a \mathbf{F}_{a0}\cdot\mathbf{dr}_a.$$

The RHS term is the work done by external forces; it can be incorporated into U if desired.

3.5 Galilean transformation

Go from frame S to S' with $\mathbf{r} = \mathbf{r}' + \mathbf{V}t$; $\mathbf{V}$ steady; $t = t'$.

3.5.1 Momentum

$$\mathbf{p} = \mathbf{p}' + m\mathbf{V}; \quad \mathbf{P} = \mathbf{P}' + M\mathbf{V};$$

i.e. $\mathbf{P}$ in S and $\mathbf{P}'$ in S' change together (or remain steady together if there if no external force). If $\mathbf{P}' = 0$, then S' is the *zero-momentum* or *Centre of Mass* frame.

3.5.2 Angular momentum

$$\mathbf{J} = \sum_a (\mathbf{r}'_a + \mathbf{V}t)\times(\mathbf{p}'_a + m_a\mathbf{V}).$$

There are 4 terms. The 4th is $\mathbf{V}\times\mathbf{V} = 0$. The others give

$$\mathbf{J} = \mathbf{J}' + \mathbf{V}t\times\mathbf{P}' + \underbrace{\sum_a \mathbf{r}'_a\times m_a\mathbf{V}}_{\displaystyle\sum_a (m_a\mathbf{r}'_a)\times\mathbf{V} = M\mathbf{R}'\times\mathbf{V}}.$$

Thus if S' is the zero-momentum frame, $\mathbf{P}' = 0$ and

$$\underset{\text{in } S}{\mathbf{J}} \quad = \quad \underset{\text{intrinsic}}{\mathbf{J}'} \quad + \underset{\text{motion of C of M in } S}{\underbrace{M\mathbf{R}'\times\mathbf{V}}}.$$

3.5.3 Energy

$$\begin{aligned} T &= \sum_a \tfrac{1}{2} m_a v_a^2 = \tfrac{1}{2} m_a (\mathbf{v}'_a + \mathbf{V})\cdot(\mathbf{v}'_a + \mathbf{V}) \\ &= T' + \underbrace{\sum_a m_a\mathbf{v}'_a}_{(=0,\ \text{if } S' = \text{zero-momentum frame})}\cdot\mathbf{V} + \tfrac{1}{2}MV^2. \end{aligned}$$

or

$$T = \text{KE in zero-momentum frame} + \tfrac{1}{2}MV^2.$$

4 Orbits

Motion of a particle in a central force field: $\mathbf{F} \parallel \mathbf{r}$; $\mathbf{F}\hat{\mathbf{e}}_r F(r)$. Potential energy $U(r)$; $F = -dU/dr$. Immediate features

1. Motion confined to the plane defined by $\mathbf{r}$ and $\mathbf{v}$.
2. No couple from central force, therefore constant $\mathbf{J}$. $\mathbf{J}$ is $\perp$ plane of orbit. $|\,J\,| = mr^2\dot{\phi}$ $= m \times$ (twice rate of describing area). Kepler's 2nd Law.

4.1 Radial equation

$$m(\ddot{r} - r\dot{\phi}^2) = -\frac{dU}{dr}.$$

Put $\dot{\phi} = J/mr^2$ (J constant). Then

$$m\ddot{r} = -\frac{dU}{dr} + \frac{J^2}{mr^3} = -\frac{dU'}{dr}$$

where

$$U' = \text{ effective } U \equiv U + \frac{J^2}{2mr^2}.$$

An alternative derivation is via energy: $E = U + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) = \frac{1}{2}m\dot{r}^2 + U'$.

4.2 Power-law force

Let $F = -Ar^n$; A positive, so force is attractive; $n =$ index, with common cases $n = +1$ (2D SHM) and $n = -2$ (gravity, electrostatics).

$$U' = \frac{Ar^{n+1}}{n+1} + \frac{J^2}{2mr^2}.$$

4.2.1 Nearly circular orbits

as oscillations/perturbations about r_0. Taylor expansion of U' gives

$$U' = U'_{min} + (r - r_0)\left(\frac{dU'}{dr}\right)_{r_0} + \tfrac{1}{2}(r - r_0)^2\left(\frac{d^2U'}{dr^2}\right)_{r_0} + \cdots$$

dU'/dr is zero at U'_{min} giving

$$\frac{dU'}{dr} = +Ar^n - \frac{J^2}{mr^3} = 0 \text{ at } r_0;$$

$$\begin{aligned}\frac{d^2U'}{dr^2} &= nAr^{n-1} + \frac{3J^2}{mr^4} \\ &= \frac{(n+3)J^2}{mr_0^4} \text{ at } r_0.\end{aligned}$$

Therefore SHM equation

$$m\ddot{r} + \underbrace{\frac{(n+3)J^2}{mr_0^4}(r-r_0)}_{\text{1st-order Taylor of } dU'/dr} = 0;$$

i.e. SHM about r_0 with angular frequency

$$\omega_p = \sqrt{n+3}\,\frac{J}{mr_0^2}.$$

How does ω_p of the perturbation compare with ω_c of the circular orbit at r_0? $\omega_c = \dot{\phi} = J/mr_0^2$. Therefore $\omega_p = \sqrt{n+3}\,\omega_c$. The common cases are

1. $n = 1$. Force proportional to r, i.e. SHM. $\omega_p = 2\omega_c$, giving a central ellipse (Lissajous figure).
2. $n = -2$. Inverse square force. $\omega_p = \omega_c$, giving an ellipse with a focus at $r = 0$ (planetary orbit).

General n gives non-commensurate ω_p and ω_c and non-repeating orbits.

4.3 Inverse square orbits

$F = -A/r^2$, $U = -A/r$. Minimum of U' is at $r_0 = J^2/Am$ = radius of circular orbit of given J.

$$\text{Energy:} \quad E = \tfrac{1}{2}m\dot{r}^2 + \underbrace{\frac{J^2}{2mr^2}}_{\frac{Ar_0}{2r^2}} - \frac{A}{r}.$$

$$\text{Angular momentum:} \quad J = mr^2\dot{\phi}.$$

4.3.1 Shape of orbit

We may write

$$\mathbf{J} \times \dot{\mathbf{v}} = -A\dot{\hat{\mathbf{e}}}_r$$

since the vectors $\mathbf{J}$, $\dot{\mathbf{v}}$ and $\dot{\hat{\mathbf{e}}}_r$ have magnitudes $mr^2\dot{\phi}$, A/mr^2 and $\dot{\phi}$ respectively and are mutually perpendicular; the sign is obtained by inspection. Since $\mathbf{J}$ is constant the equation may be integrated to give

$$\mathbf{J} \times \mathbf{v} + A(\hat{\mathbf{e}}_r + \mathbf{e}) = 0$$

where $\mathbf{e}$ is a vector integration constant. Taking the dot-product of this equation with $\mathbf{r}$ gives

$$\underbrace{\mathbf{J} \times \mathbf{v} \cdot \mathbf{r}}_{= \ \mathbf{J} \cdot \mathbf{v} \times \mathbf{r} = -J^2/m} + A(r + \mathbf{e} \cdot \mathbf{r}) = 0.$$

Therefore

$$r(1 + \mathbf{e} \cdot \hat{\mathbf{e}}_r) = r(1 + e\cos\phi) = \frac{J^2}{mA} = r_0. \tag{1}$$

This is the polar equation of a conic with *focus* at $r = 0$ (Kepler's 1st Law). The major axis is in the direction of $\mathbf{e}$; e is the eccentricity: 0, <1, 1 or >1 for a circle, ellipse, parabola or hyperbola. r_0 is the *semi-latus-rectum*, i.e. half the width at the position of the focus.

To get the energy take the scalar product of $A\mathbf{e} = -(\mathbf{J} \times \mathbf{v} + A\hat{\mathbf{e}}_r)$ with itself (note that $\mathbf{J}$ and $\mathbf{v}$ are perpendicular)

$$\begin{aligned} A^2e^2 &= J^2v^2 + 2\underbrace{\mathbf{J} \times \mathbf{v} \cdot \hat{\mathbf{e}}_r}\, A + A^2 \\ &\qquad\qquad = \mathbf{J} \cdot \mathbf{v} \times \hat{\mathbf{e}}_r = -J^2/mr \end{aligned}$$

Therefore

$$A^2(e^2 - 1) = J^2\left(v^2 - \frac{2A}{mr}\right) = \frac{2EJ^2}{m}$$

where E is the total energy. The major axis of the orbit is given by

$$2a = r_0\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{2r_0}{1-e^2} = -\frac{A}{E}$$

i.e. $E = -A/2a$ independent of eccentricity.

4.3.2 Time in orbit

The integral of $\dot{\phi}(t)$ to give $\phi(t)$ is insoluble in elementary functions. We *can* get the period, from the area and $d(\text{area})/dt = J/2m$. To get the area, turn equation (1) into Cartesians, to give

$$\left(\frac{x + x_0}{a}\right)^2 + \frac{y^2}{b^2} = 1$$

with

$$a = \frac{r_0}{1-e^2},\quad b = \frac{r_0}{\sqrt{1-e^2}}\,,\quad x_0 = ea.$$

The area is πab, so that

$$\text{Period} = \frac{\pi ab}{J/2m} = \frac{\pi r_0^2}{(1-e^2)^{3/2}}\frac{2m}{(r_0Am)^{1/2}} = 2\pi\left(\frac{ma^3}{A}\right)^{1/2},$$

i.e. proportional to $a^{3/2}$ independent of e; Kepler's 3rd Law.

4.4 Unbound inverse square orbits

Hyperbolic orbits. They can occur with an attractive force and E positive, or a repulsive force and any E, e.g. between two positively charged nuclei. The algebra is similar to that of the ellipse with some changes of sign; $e > 1$. b is the *impact parameter*; if the orbiting body is initially at infinity with positive speed, b is how close to the central body it would pass if it were not deviated by the force.

Important results in practice are

1. $J = mr^2\dot{\phi} = mbv_\infty = mr_cv_c$ where c suffix refers to the moment of closest approach.
2. Energy equation gives r_c and v_c in terms of b and v_∞.

3. The asymptotes occur when r becomes infinite, i.e. $1 + e\cos\phi_\infty = 0$ in Eqn. (2). $\cos\phi_\infty = -1/e$ (ϕ_∞ is between 90° and 180°, say $180° - \psi_\infty$). Geometry of $\hat{\mathbf{e}}_r$ and $\mathbf{e}$ gives $|\hat{\mathbf{e}}_r + \mathbf{e}| = \tan\psi_\infty$ and the magnitudes of the two terms in Eqn. (1) give
$$\tan\psi_\infty = \frac{Jv_\infty}{A} = \frac{mv_\infty^2 b}{A}.$$
The total deviation during the traverse is $180° - 2\psi_\infty$.

   For a repulsive orbit, with force $= B/r^2$, B positive, the angle of deviation is similarly
$$180° - 2\arctan\frac{mv_\infty^2 b}{B}.$$
The angle of the symptotes can also be simply derived from the $x-$momentum.

5 Rigid body dynamics

Basic equations:

$M\ddot{\mathbf{R}} = \mathbf{F}_0$; centre of mass moves as if it were a particle of mass M under action of the resultant external force.

$\dot{\mathbf{J}} = \mathbf{G}_0$; d/d$t$ of angular momentum = total external couple.

Reminder: $\mathbf{J}$ and T in general frame = $\mathbf{J}$ and T in zero-momentum frame + contribution from motion of M as if a particle. In this section we consider only force-free motion in the zero-momentum frame.

5.1 J and ω

$$\begin{aligned}\mathbf{J} &= \sum \mathbf{r}{\times}\mathbf{p} = \sum \mathbf{r}{\times}m(\boldsymbol{\omega}{\times}\mathbf{r}) \\ &= \sum mr^2\boldsymbol{\omega} - \sum m\mathbf{r}(\underbrace{\boldsymbol{\omega}\cdot\mathbf{r}}_{\omega_x x + \omega_y y + \omega_z z}).\end{aligned}$$

Therefore $\mathbf{J}$ is proportional to $\boldsymbol{\omega}$ but not necessarily parallel to $\boldsymbol{\omega}$, i.e. a matrix/tensor relationship $\mathbf{J} = \mathbf{I}\,\boldsymbol{\omega}$. In detail

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}.$$

$\mathbf{I}$ is the Moment of Inertia matrix (or tensor).

Also

$$\begin{aligned} T &= \sum \tfrac{1}{2}m(\boldsymbol{\omega}{\times}\mathbf{r})\cdot(\boldsymbol{\omega}{\times}\mathbf{r}) \\ &= \sum \tfrac{1}{2}m\boldsymbol{\omega}\cdot\mathbf{r}{\times}(\boldsymbol{\omega}{\times}\mathbf{r}) = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{J}.\end{aligned}$$

The Moment of Inertia tensor is symmetrical. There are therefore 3 real eigen values I_1 to I_3 and 3 mutually perpendicular eigen vectors $\mathbf{e}_1$ to $\mathbf{e}_3$. *Wrt* eigen-vector axes,

$$\mathbf{I}' = \begin{pmatrix} I_1 & \cdot & \cdot \\ \cdot & I_2 & \cdot \\ \cdot & \cdot & I_3 \end{pmatrix}.$$

$J_1 = I_1\omega_1$ etc, or

$$\mathbf{J} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}.$$

$$T = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2).$$

I_1 to I_3 are the *Principal Moments of Inertia* = moments of inertia in the ordinary sense about the eigenvector axes; the eigenvector axes are called the *Principal Axes.* In $\boldsymbol{\omega}$-space, a surface of constant $T = T(\boldsymbol{\omega})$ is an ellipsoid, the *Moment of Inertia Ellipsoid*; the axes of the ellipsoid have length $\propto I_i^{-1/2}$. The ellipsoid is regarded as fixed to the body.

In $\boldsymbol{\omega}$-space,

$$\operatorname{grad} T = \left(\frac{\partial T}{\partial\omega_1}, \frac{\partial T}{\partial\omega_2}, \frac{\partial T}{\partial\omega_3}\right) = (I_1\omega_1, I_2\omega_2, I_3\omega_3) \equiv \mathbf{J}$$

i.e. $\mathbf{J}$ is $\perp$ surface of constant T at $\boldsymbol{\omega}$.

5.1.1 Principal axes

They are fixed to the body and must be $\perp$ each other, whatever the shape of the body. Rotationally bodies come in three types

1. Spherical Tops; all I's equal, e.g. sphere, cube. $\mathbf{J} = I\boldsymbol{\omega}$ with a *scalar* I. $\mathbf{J} \parallel \boldsymbol{\omega}$ in this case. Rotationally the body is isotropic, with the same I about any axis.
2. Symmetrical Tops; $I_1 = I_2 \neq I_3$, e.g. many simple molecules. Subgroups are oblate (lens or disc shaped) and prolate (cigar shaped). $\mathbf{e}_3$ axis unique; $\mathbf{e}_1$ and $\mathbf{e}_2$ anywhere in plane $\perp$ $\mathbf{e}_3$.
3. Asymmetrical Tops; all three I's different. Principal axes unique.

5.1.2 Elementary properties

1. No one I_i can be larger than the sum of the other two. Thus (*wrt* x, y, z along principal axes)
$$I_1 + I_2 = \sum m(y^2 + z^2 + x^2 + z^2) = I_3 + 2\sum mz^2 \geq I_3.$$
A special case is a flat lamina, $z = 0$, for which $I_1 + I_2 = I_3$, the Theorem of Perpendicular Axes.

   Ex. 4 *Disc, mass M, radius a.*
$$I_3 = \sum m(x^2 + y^2) = \int_0^a 2\pi r dr \frac{M}{\pi a^2} r^2 = \tfrac{1}{2} Ma^2.$$
$I_1 = I_2 =$ moment of inertia about a diameter $= \frac{1}{4}Ma^2$.

2. Theorem of Parallel axes. For I about an axis *not* through centre of mass, say $\mathbf{a}$ away and parallel to a principal axis.
$$I = \sum m(\mathbf{r} + \mathbf{a}) \cdot (\mathbf{r} + \mathbf{a}) = I_0 + Ma^2 + 2\left(\sum m\mathbf{r}\right) \cdot \mathbf{a} = I_0 + Ma^2,$$
since $\sum m\mathbf{r}$ is zero when $\mathbf{r}$ is *wrt* C of M. [Vectors $\mathbf{r}$ etc here are all taken as 2D projections in plane perpendicular to I axis.]

5.2 Free motion

$\mathbf{F} = 0$, $\mathbf{G} = 0$, i.e. an isolated body spinning freely. $\mathbf{J}$ is constant. What about $\boldsymbol{\omega}$? Answer is trivial if $\mathbf{J}$ and $\boldsymbol{\omega}$ are along a principal axis. Otherwise behaviour is *much* more complicated: the direction of $\boldsymbol{\omega}$ varies both in space and *wrt* the body; for asymmetrical tops, the magnitude of $\boldsymbol{\omega}$ varies too. The variations are called *free precession*, to distinguish them from the *forced* precession produced by an external couple.

The problem of handling free precession may be treated in three quite different ways:

1. Poinsot's geometrical approach.
2. Euler's equations = the equations of motion in a coordinate frame moving with the body.
3. Lagrange's appoach, which gives the equations of motion *wrt* fixed axes.

The 2nd and 3rd of these generalise readily to include forced motion, i.e. with external couples.

5.3 Poinsot's treatment

For free precession (no external couple) only. Constant **J** and $T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{J}$ implies constant ω_J = component of $\boldsymbol{\omega}$ in the **J**-direction. Thus the $\boldsymbol{\omega}$-vector OP, as it varies, keeps its tip P in a plane $\perp$ **J**; the plane is the *invariable plane.* Since **J** is $\perp$ the surface of the ellipsoid at P, it follows that the ellipsoid is *tangential* to the invariable plane at P. The crucial step in Poinsot's argument is to say that since the instanteous motion is a rotation about OP, P is instanteously at rest, and the ellipsoid therefore *rolls*, rather than slides, on the invariable plane.

5.3.1 Symmetrical top

Ellipsoid is a spheroid about the 3-axis. Consider it prolate, with $I_3 < I_1$. Because of the symmetry the vector OP remains at a fixed angle to **J** as the spheroid rolls and also to the 3-axis. OP sweeps out a cone in space, the *space-cone*, P tracing a circle, the *herpolhode* on the invariable plane. OP also sweeps out a cone, the *body-cone*, in the moving ellipsoid; the circular track of P on the ellipsoid is the *polhode.* The overall motion can be visualised as a rolling of the body-cone, to which the body is fixed, about the space cone, the instanteous $\boldsymbol{\omega}$-vector being along the line of contact. The vectors **J**, $\boldsymbol{\omega}$ and **3** remain coplanar and their plane rotates about **J**.

Let θ_s and θ_b be the half-angles of the space-cone and body-cone, and $\theta = \theta_s + \theta_b$ the angle between **J** and the 3-axis. Take the 2-axis in the **J-3** plane, and the 1-axis perpendicular to it. Then

$$\boldsymbol{\omega} = (0, \omega\sin\theta_b, \omega\cos\theta_b);$$

$$\mathbf{J} = (0, J\sin\theta, J\cos\theta) = (0, I_1\omega_2, I_3\omega_3).$$

We have put $I_2 = I_1$. Since I_3 is less than I_1 for our example of a prolate spheroid, $|\,J\,| < I_1\ |\omega|$. The angles are given by

$$\sin\theta = \frac{J_2}{J} = \frac{I_1\omega_2}{J} = \frac{I_1\omega}{J}\sin\theta_b.$$

For a prolate spheroid, $\theta > \theta_b$ since $I_1\omega > J$.

$$\sin\theta_s = \frac{|\,\mathbf{J}\times\boldsymbol{\omega}\,|}{J\omega} = \frac{(I_1 - I_3)\omega_2\omega_3}{J\omega}.$$

and is positive for a prolate spheroid.

Two frequencies are important

1. The 'Space Frequency' $\Omega_s \equiv$ the rotation speed of the plane of $\boldsymbol{\omega}$ and **3** about **J**. It is the 'free precession' speed. To get Ω_s, put $\boldsymbol{\omega} =$ '$\boldsymbol{\omega}_3$' $+ \Omega_s$, where

   '$\boldsymbol{\omega}_3$' is $\|$ 3-axis [NB *it is* **not** ω_3]. '$\boldsymbol{\omega}_3$' represents a rotation about the symmetry axis and is of no consequence for precession of the axis.

   Ω_s is $\|$ **J** and gives the rotation of the **J-3** plane about **J**.

   By simple geometry

   $$\omega_2\ (= J\sin\theta/I_1) = \Omega_s\sin\theta,$$

   or $\Omega_s = J/I_1$.

2. The 'Body Frequency' $\Omega_b \equiv$ the rate at which OP describes its cone about the 3-axis. It can be obtained from the rolling of the two cones: the angular frequencies are proportional to (radius of rim)$^{-1}$.

$$\frac{\Omega_b}{\Omega_s} = \frac{\sin\theta_s}{\sin\theta_b} = \frac{(I_1 - I_3)\omega_2\omega_3}{J\omega} / \frac{\omega_2}{\omega}$$

or

$$\Omega_b = \frac{I_1 - I_3}{I_1}\omega_3.$$

Ex. 5 *The Earth; force-free motion. Slightly oblate.* **J** *slightly inclined to symmetry axis.*

$(I_3 - I_1)/I_1 \equiv \beta \approx \frac{1}{300}$. Oblate; θ_s is negative *cf* above treatment. Angles tiny. $\theta_s \approx -\beta\theta_b \approx -\theta_b/300$. Space-cone tiny and *inside* body-cone, which swings round it each day. In ≈ 300 days, $\boldsymbol{\omega}$ moves in a cone round **3**, giving a latitude change: *Polar wander*, or *Chandler wobble* [actually 427-day period, and irregular; the Earth is not rigid and its deformations affect the arguments]. **J-3** angle is about 0.1 arcsec.

5.3.2 Asymmetrical top

Poinsot's construction still applies. But the ellipsoid is triaxial and the angles between **J**, $\boldsymbol{\omega}$ and the 3-axis depend on its orientation; they vary as the ellipsoid rolls. The three vectors **J** , $\boldsymbol{\omega}$ and **3** are not normally coplanar. Simple qualitative results can be seen in the particular cases of **J** nearly parallel to an axis.

1. **J** nearly $\|$ to the longest (**3**) or shortest (**1**) axis of the ellipsoid. $\boldsymbol{\omega}_J$ is near to its largest and smallest possible values in such cases, and the ellipsoid has to be with its long axis roughly $\perp$ the invariable plane or its flat surface $\|$ to it respectively. In either case these is a limited range of orientations as the ellipsoid rolls, and the polhodes are small loops around the axis near to **J**.

2. **J** and $\boldsymbol{\omega}$ initially close to the intermediate axis (**2**) of the ellipsoid. The intermediate value of $\boldsymbol{\omega}_J$ gives a wide range of possible orientations, and rotation initially about the intermediate axis will deviate from it very substantially.

Hence rotations about the longest or shortest axes are stable to small deviations, those about the intermediate axis unstable.

5.4 Euler's equations

Euler's equations are of the motion in the moving frame of the body (= S say; let S_0 = fixed frame). For any vector **A**

$$\left[\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right]_{\text{in } S_0} = \left[\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right]_{\text{in } S} + \boldsymbol{\omega}\times\mathbf{A}.$$

So

$$\mathbf{G} = \left[\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}\right]_{\text{in } S} + \boldsymbol{\omega}\times\mathbf{J}.$$

J in $S = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$. Therefore

$$G_1 = I_1\dot{\omega}_1 + \underbrace{\omega_2 I_3\omega_3 - \omega_3 I_2\omega_2}_{(I_3 - I_2)\omega_2\omega_3}$$

and G_2, G_3 similarly (*Euler's equations*). If $\mathbf{G} = 0$, then

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$$

and two others.

5.4.1 Symmetrical top

As before, take it prolate: $I_3 < I_1 = I_2$. Euler's equations are

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_1 - I_3)\omega_2\omega_3, \\ I_1\dot{\omega}_2 &= (I_3 - I_1)\omega_1\omega_3, \\ I_3\dot{\omega}_3 &= 0. \end{aligned}$$

A 'descriptive' solution is as follows: Let $\mathbf{J}$ be in the 2-3 plane at $t = 0$, i.e.

$$\mathbf{J} = (0, I_1\omega_2, I_3\omega_3)$$

at $t = 0$ [NB: $\mathbf{J}$ is constant in space, but axes change]. Euler gives, at $t = 0$,

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_1 - I_3)\omega_2\omega_3, \\ I_1\dot{\omega}_2 &= 0, \\ I_3\dot{\omega}_3 &= 0. \end{aligned}$$

i.e. $\mathbf{d\boldsymbol{\omega}}$ in dt is in 1-direction, ⊥ plane of **3**, $\mathbf{J}$ etc. [NB: $\boldsymbol{\omega}+\mathbf{d\boldsymbol{\omega}}$ means where we should look after time dt to find the new rotation axis; *not* that the line of atoms along $\boldsymbol{\omega}$ has moved to this position (they are at rest if on the rotation axis)].

To get frequencies, consider $\mathbf{d\boldsymbol{\omega}}$ in plane $\perp \mathbf{J}$ [i.e. look down the $\mathbf{J}$ -axis]. In dt it moves an angular distance $\Omega_s dt$ around $\mathbf{J}$. The radius of the circle traced out by $\boldsymbol{\omega}$ is $\omega \sin\theta_s$, giving

$$\begin{aligned} \Omega_s &= \frac{I_1 - I_3}{I_1}\omega_2\omega_3 / \underbrace{\frac{(I_1 - I_3)\omega_2\omega_3}{J}}_{\omega \sin\theta_s} \\ &= J/I_1 \end{aligned}$$

as above.

To get the body frequency, look down the 3-axis. $\Omega_b dt = d\omega_1/\omega_2 = d\omega_1/\omega\sin\theta_b$. Therefore

$$\Omega_b = \frac{I_1 - I_3}{I_1}\omega_3,$$

also as above.

Note: Euler's equations say how $\boldsymbol{\omega}$ moves *wrt* 1-2-3 axes. At any time, axes rotate with current $\boldsymbol{\omega}$. You need to combine both effects to get the motion in space.

A more formal solution to Euler's equations can be found by noting that they give ω_3 constant, and coupled SHM equations for ω_1 and ω_2. Thus

$$\begin{aligned} \dot{\omega}_1 &= \underbrace{\frac{I_1 - I_3}{I_1}\omega_3}_{\equiv \Omega} \omega_2 \equiv \Omega\omega_2, \\ \dot{\omega}_2 &= -\Omega\omega_1, \end{aligned}$$

giving

$$\ddot{\omega}_1 = -\Omega^2\omega_1,$$

i.e. SHM at Ω, and similarly for ω_2. This gives circular motion in the 1-2 plane at $\Omega \equiv \Omega_b$, the total motion (body-cone) being the resultant of that with constant ω_3.

5.4.2 Asymmetrical top

Euler's equations can be used to derive the behaviour of an asymmetrical top for **J** near to an axis, say the 3-axis. Put

$$\omega_1 = \omega_{10} e^{i\Omega t},\ \omega_2 = \omega_{20} e^{i\Omega t},\ \text{ both } \ll \omega_3.$$

Then Euler's equations give

$$\Omega^2 = \frac{(I_1 - I_3)(I_2 - I_3)}{I_1 I_2} \omega_3^2.$$

If the I_3 above is the largest or the smallest of the three values, Ω^2 is positive, implying oscillatory ω_1 and ω_2; $\boldsymbol{\omega}$ undergoes small stable oscillations about the 3-axis.

If I_3 is the middle of the three values, Ω^2 is negative, implying exponential behaviour of ω_1 and ω_2, i.e. rotation about the intermediate axis is unstable.

5.5 Lagrange

The most direct and systematic treatment of the motion *wrt* fixed axes is via Lagrangian Mechanics. Here we obtain the same equations by inspection.

We consider the motion of a symmetrical top, at first isolated, and then supported at its base under gravity. Let (θ, ϕ) be the spherical polar coordinates of the symmetry (3) axis and χ the angle of rotation of the top about the 3-axis (*Euler angles*). Let the 1-axis be horizontal; the vertical axis is in the 2-3 plane. Instantaneously

$$\boldsymbol{\omega} = (\omega_1,\ \omega_2,\ \omega_3) = (\dot\theta, \dot\phi \sin\theta, \dot\chi + \dot\phi \cos\theta).$$

χ is measured *wrt* the moving z-3 plane.

$$\mathbf{J} = (I_1\dot\theta,\ I_1\dot\phi\sin\theta,\ I_3(\dot\chi + \dot\phi\cos\theta)).$$

I_1 is *wrt* the stationary point within the body (C of Mass for the isolated body, base if supported under gravity). The gravitational couple **G**, if present, is in the 1-direction.

There are three constants of motion, giving three equations

1. $J_3 = J_\chi = I_3(\dot\chi + \dot\phi\cos\theta)$ is constant; this is from Euler's equation for $\dot\omega_3$ with $G_3 = 0$ and $I_1 = I_2$, implying $\dot J_3 = I_3\dot\omega_3 = 0$.
2. $J_z = J_\phi = J_3\cos\theta + J_2\sin\theta = J_\chi\cos\theta + I_1\dot\phi\sin^2\theta$ is constant; this is because $G_z = 0$.
3. $E = T + U$ is constant.

The first two equations enable us to express $\dot\phi$ and $\dot\chi$ in terms of the momentum constants J_ϕ and J_χ and the angle θ:

$$\begin{aligned} \dot\chi &= J_\chi/I_3 - \dot\phi\cos\theta, && (1) \\ \dot\phi &= \frac{J_\phi - J_\chi\cos\theta}{I_1\sin^2\theta}. && (2) \end{aligned}$$

5.5.1 Force-free motion

The body is isolated in free space. Take the **J** direction as defining 'vertical'. Then

$J_\phi = J$ since total **J** is along ϕ-axis.

$J_\chi = J\cos\theta$.

Therefore

$$\begin{aligned}\dot\phi &= \frac{J(1-\cos^2\theta)}{I_1\sin^2\theta} = \frac{J}{I_1} = \Omega_s,\\ \dot\chi &= \frac{J\cos\theta}{I_3} - \frac{J}{I_1}\cos\theta = I_3\omega_3\left(\frac{1}{I_3}-\frac{1}{I_1}\right)\\ &= \frac{I_1-I_3}{I_1}\omega_3 = \Omega_b.\end{aligned}$$

The above constitutes a third derivation of the space and body frequencies Ω_s and Ω_b. [To my mind, the relationships between the different treatments are not at all obvious; text books can be baffling since few books give more than one treatment].

5.5.2 With external couple

Hereafter we consider the body supported at its base. I_1 is about the support, which is at h from the C of Mass.

Equations (1) and (2) give $\dot\phi$ and $\dot\chi$ as known functions of θ. Once θ is known as a function of time, ϕ and χ may in principle be found by integration.

θ may be found from the energy equation

$$E = \tfrac{1}{2}I_1(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \tfrac{1}{2}I_3(\dot\chi + \dot\phi\cos\theta)^2 + mgh\cos\theta = \text{ constant.}$$

Substitutions from equations (1) and (2) give

$$\underbrace{E - \frac{J_\chi^2}{2I_3}}_{E'} = \tfrac{1}{2}I_1\dot\theta^2 + \underbrace{\frac{(J_\phi - J_\chi\cos\theta)^2}{2I_1\sin^2\theta} + mgh\cos\theta}_{U'(\theta)}\,. \tag{3}$$

The problem is now solved in principle, since $\dot\theta$ is a known function of θ and may be integrated to give $\theta(t)$; ϕ and χ then follow. The integrations involve elliptic integrals and are well beyond the scope of IB mathematics.

The important physical results can be seen without resource to the full mathematics by treating the θ motion as oscillation in an effective potential U' (*cf* §4.2.1 for the similar treatment of orbits).

Steady *precession* arises when θ is at the 'equilibrium' position (minimum of U') and constant; it is physically a special case and requires $E' = U'_{\text{min}}$. Equations (1) and (2) then show that $\dot\phi$ and $\dot\chi$ are constant. The motion thus consists of steady rotation by $\dot\chi$ about the symmetry axis, with the axis itself steadily precessing by $\dot\phi$ at constant θ to the vertical.

If E' is slightly larger than the minimum of U', then the θ motion can be treated as approximate SHM by Taylor expansion of U' about the minimum. The oscillations of θ give *nutation*, i.e. oscillations in $\dot\chi$ and $\dot\phi$ about the steady precession values.

5.5.3 Precession

The condition for steady precession is $dU'/d\theta = 0$, which gives an equation for θ in terms of the constants J_ϕ and J_χ. After some algebra, it may be rewritten (using equation (2)) as

$$\dot{\phi}^2 I_1 \cos\theta - \dot{\phi} J_\chi + mgh = 0. \tag{4}$$

Equation (4) is useful since it gives us the steady precession speed $\dot{\phi}$ as a function of inclination θ. It is a quadratic, with solutions

$$\dot{\phi} = \frac{J_\chi \pm \sqrt{J_\chi^2 - 4I_1 mgh\cos\theta}}{2I_1\cos\theta}.$$

These shows that if $\cos\theta$ is positive (i.e. the top is standing above its base and not hanging below it), $\dot{\phi}$ is impossible unless

$$J_\chi^2 \geq 4I_1 mgh\cos\theta.$$

Thus steady precession requires the top to be spinning fast enough. In the 'gyroscopic limit' (very large $\mathbf{J}$ from rapid rotation about the symmetry axis) $J_\chi^2 \gg mghI_1$ and the inequality is enormously oversatisfied.

The quadratic for $\dot{\phi}$ shows there are two possible precession frequencies for given θ; in the gyroscopic limit they are

$\dot{\phi} \approx mgh/J_\chi$, independent of θ; this is the 'slow precession', as derived in IA.

$\dot{\phi} \approx J_\chi/I_1\cos\theta$, independent of $\mathbf{G}$; this is the 'free precession' as derived above by Poinsot or Euler; $\mathbf{J}$ is entirely in the z-direction and $\dot{\phi} \equiv \Omega_s$.

5.5.4 Nutation

The analysis of nutation about precession at general θ, even in the gyroscopic limit, is algebraically laborious. The case of nutation of a horizontal gyroscope is reasonably straightforward.

Ex. 6 *Nutation of a gyroscope, with axis horizontal and supported at one end.*

$\theta_0 = 90°$. Put $\theta = \pi/2 + \epsilon$. For small ϵ, $\cos\theta \approx -\epsilon$, $\sin\theta \approx 1 - \epsilon^2/2$. Then

$$U'(\theta) = \text{ constant } + \epsilon\left(\frac{J_\phi J_\chi}{I_1} - mgh\right) + \epsilon^2\left(\frac{J_\chi^2}{2I_1} + \frac{J_\phi^2}{2I_1}\right) + \cdots$$

for power series expansion in ϵ. The term $\propto \epsilon$ is zero at θ_0; therefore

$$J_\phi = \frac{mghI_1}{J_\chi}; \quad \dot{\phi} = \frac{mgh}{J_\chi}.$$

The gyroscope condition is $J_\chi^2 \gg mghI_1$ and hence $\gg J_\phi^2$. The ϵ^2-term gives the 'restoring force' term in U' and hence the equation of motion as follows

$$U' = \text{ constant } + \epsilon^2\frac{J_\chi^2}{2I_1},$$

$$I_1\ddot{\epsilon} + \frac{J_\chi^2}{I_1}\epsilon = 0.$$

This gives SHM in ϵ at $\Omega \equiv \Omega_s = J_\chi / I_1$.

It is instructive, and relatively easy, to derive this simple result from first principles.

Another limiting case that can be handled relatively easily is the conical pendulum, i.e. as above but with $I_3 = 0$. There is no gyroscopic action and the pendulum can only hang below the support (for motion at steady θ). Note that *cf* conventional treatments, θ is measured from the upward rather than downward vertical.

6 Normal modes

The dynamics of n-parameter systems, or 'coupled systems', e.g.:

A molecule of N atoms, treated as N point-nuclei; $n = 3N$

Two LRC circuits with mutual inductance; $n = 2$

Waves on strings, in solids etc; n very large, e.g. 10^{20+}

We deal first with the basic theory for N similar masses, taking as a worked example two masses in 1D motion.

6.1 Basic theory

6.1.1 Equations of Motion

$$m\ddot{x}_i = F_i;\ i = 1, n$$

$i \equiv$ one cartesian component of one particle; or

$$m\ddot{\mathbf{x}} = \mathbf{F}$$

in nD space.

Suppose system is elastic, i.e. there is a potential energy $U \equiv U(x_1, \dots x_n)$

$$F_i = -\frac{\partial U}{\partial x_i}.$$

Equilibrium position $\equiv$ minimum of U. We study oscillations about U_{min}. Take origin in nD space at minimum of U. Taylor expansion gives

$$U(\mathbf{x}) = U_0 \quad + \textstyle\sum_j \underbrace{\left(\frac{\partial U}{\partial x_j}\right)_0}_{=0} x_j \quad + \sum_j \sum_k \frac{1}{2} \underbrace{\left(\frac{\partial^2 U}{\partial x_j \partial x_k}\right)_0}_{\equiv b_{jk}} x_j x_k + \cdots$$

The U_0 term is irrelevant (the arbitrary zero of PE); the next term is zero, since $\partial U/\partial x_j = 0$ at the minimum of U. The really important term is the quadratic term. Note that $b_{kj} = b_{jk}$.

$$U(\mathbf{x}) = \sum_j \sum_k \tfrac{1}{2} b_{jk} x_j x_k = \tfrac{1}{2}\mathbf{x}'\mathbf{b}\,\mathbf{x};$$

in the matrix notation, $\mathbf{x}'$ is the transpose of the nD column matrix $\mathbf{x}$. The force F_i comes from the derivative $-\partial U/\partial x_i$ of this quadratic form and is (see § 1.3.5 if not immediately obvious)

$$F_i = -\frac{\partial U}{\partial x_i} = -\sum_j b_{ij} x_j; \quad \text{or} \quad \mathbf{F} = -\text{grad}\, U = -\mathbf{b}\,\mathbf{x}.$$

The equations of motion are thus

$$m\ddot{\mathbf{x}} = -\mathbf{b}\,\mathbf{x}.$$

In simple cases, the equations would often be written down directly, rather than obtained from the expression for U.

Ex. 7 *Two equal masses m at l and $2l$ from end of a light spring of total length $3l$ stretched between fixed supports; consider just motion parallel to length of spring.*

Measure x_1, x_2 from equilibrium position. Equations of motion are

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$

$$m\ddot{x}_2 = -k(x_2 - x_1) - kx_2$$

where k is the force constant for each third of the spring. Or

$$m\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\underbrace{k\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{\equiv \mathbf{b}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$U = \frac{1}{2}k(\text{extension})^2$ for each spring; i.e.

$$U = \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}k(x_1 - x_2)^2 + \tfrac{1}{2}kx_2^2 = k\left[x_1^2 + x_2^2 - x_1x_2\right]$$

or $\frac{1}{2}\sum_i \sum_j b_{ij}x_ix_j$ as expected.

6.1.2 Oscillations

Def$^{\mathbf{n}}$. 8 Normal mode: *An oscillation when all parts of a system oscillate at the* **same** *frequency, i.e. the same $e^{i\omega t}$ time-variation everywhere.*

A normal mode implies a constant amplitude and phase relation between the x_i's, i.e. between different parts of the system. Put

$$\mathbf{x} = \mathbf{x}_0 e^{i\omega t},$$

$$\ddot{\mathbf{x}} = -\omega^2\mathbf{x},$$

$$m\omega^2\mathbf{x} = \mathbf{b}\,\mathbf{x}.$$

Therefore possible $m\omega^2$'s for normal modes are the *eigen values of* $\mathbf{b}$; there are n of them. The eigen vectors $\mathbf{e}_j, j = 1, n$ give the actual *modes*, i.e. the amplitude and phase relations between different parts of the system [the theory gives only the *direction* of $\mathbf{e}_j$ in nD space; the magnitude, = overall amplitude, is indeterminate].

Ex. 7 *(contd.)*

$$x_i = x_{i0}e^{i\omega t}; \quad i = 1, 2$$

$$-m\omega^2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -k\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

To get eigen values, rewrite as

$$\begin{pmatrix} 2 - \dfrac{m\omega^2}{k} & -1 \\ -1 & 2 - \dfrac{m\omega^2}{k} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0. \tag{1}$$

For a non-trivial solution for x_1, x_2, the determinant of the 2×2 matrix must be zero. Therefore

$$\left(2 - \frac{m\omega^2}{k}\right)^2 - 1 = 0,$$

or

$$\omega^2 = \frac{k}{m} \text{ or } \frac{3k}{m}.$$

The eigen vectors in this case are

1. $\omega^2 = k/m$. Either of the equations in (1) gives $x_1 - x_2 = 0$, or $x_1 = x_2$, i.e. masses oscillate in phase with equal amplitude.
2. $\omega^2 = 3k/m$. Equations (1) give $-x_1 - x_2 = 0$, or $x_1 = -x_2$, i.e. masses oscillate in antiphase with equal amplitude.

6.1.3 Orthogonality

Matrix **b** is symmetrical, therefore its eigen vectors are orthogonal in nD space:

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0 \quad \text{if } i \neq j.$$

1. If ω_i's are all different, $\mathbf{e}_i$'s are unique.
2. If some ω_i's are the same, then there is a choice of $\mathbf{e}_i$'s, but you can always *find* orthogonal $\mathbf{e}_i$'s [if $\mathbf{e}_a$ and $\mathbf{e}_b$ are eigen vectors with the same ω, then so is any linear combination $\alpha\mathbf{e}_a + \beta\mathbf{e}_b$].

The $\mathbf{e}_i$'s are the 'natural axes' in nD space. With respect to them

$$U = \sum_i \sum_j \tfrac{1}{2} b'_{ij} x'_i x'_j$$

with

$$\mathbf{b}' = m \begin{pmatrix} \omega_1^2 & \cdot & \cdot & \\ \cdot & \omega_2^2 & \cdot & \\ \cdot & \cdot & \omega_3^2 & \\ & & & \searrow \end{pmatrix},$$

i.e. **b** is *diagonal.* $U = \sum \frac{1}{2} m\omega_i^2 x_i'^2 = \sum U$ for each normal mode separately.

Ex. 7 *(contd.)*

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \mathbf{e}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Take new coordinates *wrt* eigen-vector axes

$$x_s = \frac{1}{\sqrt{2}}(x_1 + x_2); \quad x_d = \frac{1}{\sqrt{2}}(x_1 - x_2);$$

or

$$x_1 = \frac{1}{\sqrt{2}}(x_s + x_d); \quad x_2 = \frac{1}{\sqrt{2}}(x_s - x_d).$$

Then

$$\begin{aligned} U &= k[x_1^2 + x_2^2 - x_1x_2] \\ &= k\frac{1}{2}\left[(x_s + x_d)^2 + (x_s - x_d)^2 - (x_s + x_d)(x_s - x_d)\right] \\ &= k\frac{1}{2}(x_s^2 + 3x_d^2); \end{aligned}$$

as expected, the cross-terms cancel. The equations of motion are

$$m\ddot{x}_s = -kx_s$$

$$m\ddot{x}_d = -3kx_d$$

i.e. the motions are *uncoupled*, giving independent oscillations, of x_s with $m\omega^2 = k$, and x_d with $m\omega^2 = 3k$.

6.1.4 Energy

Kinetic energy $= T = \sum_i \frac{1}{2}m_iv_i^2 = \frac{1}{2}m \mid \dot{\mathbf{x}} \mid^2$; $\dot{\mathbf{x}}$ is the velocity vector in nD space. Now | vector | is independent of choice of orthogonal axes, and so

$$T = \frac{1}{2}m \mid \dot{\mathbf{x}}' \mid^2 = \sum T \text{ for each mode;}$$

[$\dot{\mathbf{x}}'$ is *wrt* eigen-vector axes].

Ex. 7 *(contd.)*

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) = \frac{1}{2}m(\dot{x}_s^2 + \dot{x}_d^2).$$

$$\text{Energy} = T + U = \underbrace{\frac{1}{2}m\dot{x}_s^2 + \frac{1}{2}kx_s^2}_{\substack{x_s \text{ mode} \\ m\omega_s^2 = k}} + \underbrace{\frac{1}{2}m\dot{x}_d^2 + \frac{3}{2}kx_d^2}_{\substack{x_d \text{ mode} \\ m\omega_d^2 = 3k}}.$$

In oscillation

1. $\overline{T} = \overline{U}$ for each mode (overline $\equiv$ time-average).
2. $E = \sum E$ over modes.

Ex. 8 *Two pendulums, weakly coupled; each in 1D motion in parallel planes*

Modes of oscillation are

1. In phase with equal amplitude θ_s
2. In antiphase with equal amplitude θ_d

By inspection these are the normal modes — each oscillates at a single ω, and for a 2-parameter system there can only be 2 modes. ω_s, for the first mode, is the same as for uncoupled pendulums; ω_d is a little faster.

Suppose initially θ_1 (of pendulum 1) $= \theta_0$, $\theta_2 = 0$ and both are stationary [NB: 4 initial conditions]. At $t = 0$,

$$\theta_s = \theta_d = \frac{1}{\sqrt{2}}\theta_0$$

and at later time

$$\theta_s = \frac{\theta_0}{\sqrt{2}}e^{i\omega_s t}, \quad \theta_d = \frac{\theta_0}{\sqrt{2}}e^{i\omega_d t},$$

giving beats at the difference frequency; when $(\omega_d - \omega_s)t = \pi$, pendulum 2 has large amplitude and pendulum 1 is stationary. The system can be viewed as

1. Two oscillators, with weak coupling and periodic interchange of energy between them at the beat frequency; or
2. One 2D system, with 2 modes and no transfer of energy between modes.

6.2 Extensions

6.2.1 Masses different

Ex. 9 *CO_2 molecule; linear with masses M (carbon) and m (oxygen); 1D motion along line of nuclei*

$$\mathbf{b} = k\begin{pmatrix} 1 & -1 & . \\ -1 & 2 & -1 \\ . & -1 & 1 \end{pmatrix}$$

and

$$\underbrace{\begin{pmatrix} m_1 & . & . \\ . & m_2 & . \\ . & . & m_3 \end{pmatrix}}_{\equiv \mathbf{m}} \ddot{\mathbf{x}} = -\mathbf{b}\,\mathbf{x} \tag{2}$$

for three general masses. If all of system oscillates as $e^{i\omega t}$, then $\mathbf{m}\omega^2\mathbf{x} = \mathbf{b}\,\mathbf{x}$, and the ω's are given by

$$\det(\mathbf{m}\omega^2 - \mathbf{b}) = 0. \tag{3}$$

These are not strictly eigen value equations, though they may be easy enough to handle.

The usual way of making use of the previous theory is via *mass-weighted coordinates.* Introduce $\xi_i = x_i\sqrt{m_i}$. Then

$$\sqrt{m_i}\ddot{\xi}_i = -\sum_j b_{ij}\frac{\xi_j}{\sqrt{m_j}}.$$

On dividing by $\sqrt{m_i}$,

$$\ddot{\xi}_i \;=\; -\sum_j \underbrace{\frac{b_{ij}}{\sqrt{m_i m_j}}}_{\text{symmetrical; eigen values give } \omega^2} \xi_j.$$

We have scaled our original x_1-x_2-x_3 space parallel to the 1-2-3 axes so that the same $\mid \dot{\xi} i \mid \equiv$ same kinetic energy. Eigen vectors become orthogonal and the theory of § 6.1 applies.

Ex. 9 *(contd.)*

The modes for CO_2 are

1. $\omega^2 = 0,\ \mathbf{e}_1 = (1, 1, 1)$, i.e. translational motion of molecule as a whole.
2. $\omega^2 = k/m,\ \mathbf{e}_2 = (1, 0, -1)$, i.e. carbon stationary, oxygens in antiphase.
3. $\omega^2 = k(2m + M)/Mm,\ \mathbf{e}_3 = (1, -2m/M, 1)$, i.e. oxygens in phase with each other and in antiphase with carbon; no motion of C of Mass.

The modes could probably have been guessed by inspection.

6.2.2 Extended masses

N-parameter systems involving rigid bodies will often have equations of motion similar to those of (2) but with the LHS matrix **m** non-diagonal; the coordinates represented by the vector **x** will often be angular instead of linear, or a mix of both. These can be transformed into the simpler type used in § 6.1 by suitable change of coordinates (see Riley). Simple cases are best treated by handling the equivalent equation (3) directly.

6.3 Many parameter systems

Ex. 10 *Standing waves on a string, tension F, mass ρ per unit length; total length l, anchored on x-axis at $x = 0, x = l$; motion in xy-plane. Assume small angle deviations, i.e. $F \cos\theta \approx F$ everywhere.*

6.3.1 Equations of motion

Motion of an element dx is

$$\rho dx\ddot{y} = F(\theta_2 - \theta_1) = F\frac{\partial\theta}{\partial x}dx.$$

$\theta \approx \partial y/\partial x$, therefore

$$\rho\ddot{y} = F\frac{\partial^2 y}{\partial x^2},$$

the non-dispersive wave equation.

6.3.2 Normal modes

Normal modes imply a solution of the form $y = y(x)e^{i\omega t}$, so

$$-\omega^2\rho y(x) = F\frac{\partial^2 y}{\partial x^2},$$

which is the SHM equation, with solution

$$y(x) = A_k \cos(kx + \phi_k)$$

in which

A_k and ϕ_k are arbitrary constants,

$\omega^2\rho = Fk^2$ — the dispersion relation, and

v = velocity of wave $= \omega/k = \sqrt{F/\rho}$, and is independent of ω, k.

For a string of length l, the boundary conditions that $y = 0$ at $x = 0$ and $x = l$ give

$$y(x) = B_k \sin(kx)$$

with no cos term, and $kl = n\pi$. Put $k = nk_0, \omega = n\omega_0$, where k_0 and ω_0 are the values for the lowest mode, $k_0 = \pi/l$, $\omega_0 = \frac{\pi}{l}\sqrt{F/\rho}$. The general $y(x,t)$ is then

$$y = \sum y_n = \sum_{n=1}^{\infty} \sin(nk_0x)\underbrace{\left[C_n\cos(n\omega_0t) + S_n\sin(n\omega_0t)\right]}_{\equiv\, t_n(t)}.$$

y can be thought of as a vector in ∞D space, one dimension for each value of x. Each n of the above is one normal mode, with C_n and S_n the amplitudes of the in-phase and quadrature components. $y_n = \sin(nk_0x)$ is the nth *eigen function*, equivalent to an eigen vector in ∞D space.

6.3.3 Energy

The orthogonality relations between the eigen vectors/functions are

$$\mathbf{y}_n \cdot \mathbf{y}_m = \int_0^l \sin(nk_0x)\sin(mk_0x)dx = 0,$$

and similarly

$$\int_0^l \cos(nk_0x)\cos(mk_0x)dx = 0,$$

for $n \neq m$.

$U = \int_0^l F(ds - dx)$, where dx is the original length of an element of string and ds the length it is stretched to by the wave. Simple geometry gives

$$U = \int_0^l \tfrac{1}{2}F\left(\frac{\partial y}{\partial x}\right)^2 dx;$$

$$\frac{\partial y}{\partial x} = \sum_n nk_0\cos(nk_0x)t_n(t).$$

The expression for U is of the form $\int(\sum\text{term})^2$. But by orthogonality the cross-terms in $(\sum\text{term})^2$ go to zero on integration. We are thus just left with $\int\sum(\text{term}^2)$. The result is

$$U = \tfrac{1}{4}Fl\sum n^2k_0^2t_n^2$$

using the fact that $< \cos^2 >= \frac{1}{2}$ in x-integration. The time-average of U is

$$\overline{U} = \tfrac{1}{8}Flk_0^2\sum n^2(C_n^2 + S_n^2)$$

using $< t_n^2 >= \frac{1}{2}(C_n^2 + S_n^2)$. Note that $U = \sum_n U_n$.

The kinetic energy is

$$T = \int \tfrac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 dx$$

and after similar evaluations gives

$$\overline{T} = \tfrac{1}{8}\rho l \omega_0^2 \sum n^2 (C_n^2 + S_n^2).$$

As $Fk_0^2 = \rho\omega_0^2$, $\overline{U} = \overline{T}$. Also $E = T + U$ is constant for each mode; thus for the C_n-term,

$$E \quad = \quad \tfrac{1}{4} F l k_0^2 n^2 C_n^2 \left[\underbrace{\cos^2(n\omega_0 t)}_{U\text{-part}} + \underbrace{\sin^2(n\omega_0 t)}_{T\text{-part}}\right].$$

7 Elasticity — Fundamentals

Hooke's law: *'Ut tensio, sic vis'* (his words).

7.1 Basic ideas

Ex. 11 *A wire of length l, width w, uniform cross-sectional area A. Force F stretches it to length $l + \delta l$.*

Def$^{\mathbf{n}}$. 9 Stress: *τ (or $\sigma, S, \ldots$) $\equiv F/A$.*

Def$^{\mathbf{n}}$. 10 Strain: *e (or $u, \ldots$) $\equiv \delta l/l$.*

Def$^{\mathbf{n}}$. 11 Young's modulus: *E (or $Y, \ldots$) $\equiv$ stress/strain.*

Note

1. F is the force across *any* transverse cut; it is the same for all cuts since the piece of wire between any two cuts is in equilibrium.
2. Stress and strain are *local* as well as global properties. Young's modulus is independent of dimensions and is a property of the material.
3. Young's Modulus is stress/strain defined *for this specific geometry.*

The width can also change, by δw.

Def$^{\mathbf{n}}$. 12 Poisson's ratio: *σ (or $\nu, \ldots$) $\equiv -(\delta w/w)/(\delta l/l)$.*

As with Young's modulus, Poisson's ratio is defined for this specific geometry.

We need to generalise to full 3D geometry (and in principle anisotropic media).

7.2 Stress

Force and area are really vectors $\mathbf{F}$ and $\mathbf{A}$. They need not be parallel.

Ex. 12 *Take Ex. 11 and take an oblique cut.* $\mathbf{A}$ *is perpendicular to the cut and of magnitude* $A_0 \sec\theta$ *(A_0 is the ordinary transverse area).* $\mathbf{F}$ *is still the same, parallel to the length. With axes, x parallel to the length, y and z transverse,*

$$F_x = \tau A_x; \;\; F_y = F_z = 0$$

i.e. a matrix *or* tensor *relationship between vectors* $\mathbf{F}$ *and* $\mathbf{A}$.

We can see that the relation between $\mathbf{F}$ and $\mathbf{A}$ must in general be a matrix as follows. Consider a polyhedron-shaped element in a material in equilibrium under uniform stress. $\mathbf{F}$ on polyhedron $= 0 = \sum_a \mathbf{F}_a$ exerted over its faces. Also $\mathbf{A} = \sum_a \mathbf{A}_a$ over faces $= 0$ for a closed polyhedron (? IA result; if not prove simply by summing each Cartesian component = projected area). This is true for *any* polyhedron = *any* set of $\mathbf{A}_a$, and implies that $\mathbf{F}_a$ must be linear with, i.e. proportional to, $\mathbf{A}_a$ though not necessarily parallel to it; or

$$F_i = \sum_j \tau_{ij} A_j.$$

Def$^{\mathbf{n}}$. 13 Stress: $\tau_{ij} \equiv$ *force-component in the i-direction per unit area of face in a cut normal to the j-direction*

The matrix τ_{ij} is *symmetric*, i.e. $\tau_{ij} = \tau_{ji}$. [Any antisymmetric part of $\boldsymbol{\tau}$ would give a couple and hence a rotation, so must be zero in equilibrium. In fact it must be zero even in non-equilibrium, since the couple would give infinite angular acceleration for an infinitesimal element (prove this)]. Therefore *any* $\boldsymbol{\tau}$ has

3 real eigen-values

3 mutually perpendicular eigen-vectors, with respect to which as axes $\boldsymbol{\tau}$ is diagonal

i.e. *any* stress is equivalent the three mutually perpendicular longitudinal stresses.

7.3 Strain

Strain refers to the *distortion* of the material. Let $\mathbf{u}(\mathbf{r})$ be the displacement vector, i.e. the distortion is such as to move $\mathbf{r}$ to $\mathbf{r} + \mathbf{u}(\mathbf{r})$. Uniform linear distortion implies $\mathbf{u}$ proportional but not necessarily parallel to $\mathbf{r}$, i.e. a matrix/tensor relation. Put

$$\mathbf{r} + \mathbf{u} = \mathbf{M}\mathbf{r} = (\mathbf{I} + \mathbf{m})\mathbf{r};\;\; \mathbf{u} = \mathbf{m}\,\mathbf{r};$$

$$|\,m_{ij}\,| \ll 1.$$

$\mathbf{M}$, $\mathbf{m}$ are matrices; $\mathbf{I}$ is the unit matrix.

Consider distortion in xy-plane. Points A at 100 and B at 010 move by (m_{xx}, m_{yx}, m_{zx}) and (m_{xy}, m_{yy}, m_{zy}) respectively. The longitudinal strain in the x-direction is $m_{xx}/1 = m_{xx}$. There is a *transverse* strain in the xy-plane (ignore z-motions *pro tem*), measured by the angles that OA and OB (O is the origin) move from the axes:

$$OA \text{ moves anticlockwise by } \theta_x = m_{yx}$$

$$OB \text{ moves clockwise by } \theta_y = m_{xy}$$

to first order, equivalent to

1. a *rotation* of the whole body by $\frac{1}{2}(\theta_x - \theta_y)$. This is of *no* interest in relation to the deformation and the stresses which cause it.

   Note: rotation by small θ gives $\mathbf{u} = \mathbf{m}\,\mathbf{r} = \boldsymbol{\theta} \times \mathbf{r}$ with $\mathbf{m}$ antisymmetrical.

2. a *shear*, defined by the angle by which right-angles change. In this case $90° \to 90° - (\theta_x + \theta_y)$, or shear angle $= m_{yx} + m_{xy}$.

Def$^{\mathbf{n}}$. 14 Strain: $e_{ij} \equiv \frac{1}{2}(m_{ij} + m_{ji})$,

i.e. the symmetrical part of $\mathbf{m}$.

$e_{xy} =$ half the shear angle in the xy-plane.

If the $\mathbf{u}$ variation is non-uniform, take a Taylor expansion

$$u_i(\mathbf{r}) = \underbrace{u_i(\mathbf{r}_0)}_{\text{bodily shift}} + \underbrace{\sum_j \frac{\partial u_i}{\partial r_j}(r_j - r_{j0})}_{\text{first-order distortion}} + \cdots$$

Therefore the relevant m_{ij} is $\partial u_i/\partial r_j$. e_{ij} is symmetric. As with $\boldsymbol{\tau}$ it has 3 real eigen values and 3 mutually perpendicular eigen-vectors, with respect to which as axes it is diagonal. *Any* distortion is therefore $\equiv$ three mutually perpendicular compressions or expansions (plus an overall rotation).

7.4 General elastic modulus

The strain tensor/matrix is a linear function of the stress tensor/matrix. Or

$$\tau_{ij} = \sum_k \sum_l \lambda_{ijkl} e_{kl}.$$

λ_{ijkl} is the general elastic modulus and is a property of the material; it is a *fourth-rank tensor*, with 81 separate components (many are equal because of symmetries).

7.5 Isotropic materials

$\boldsymbol{\tau} \equiv$ three longitudinal stresses along the eigen vectors of $\boldsymbol{\tau}$. $\mathbf{e} \equiv$ three longitudinal strains along the eigen vectors of $\mathbf{e}$. Since there are no other preferred directions, the eigen vector directions of $\mathbf{e}$ *must* be those of $\boldsymbol{\tau}$.

With axes in the eigen vector directions, we can say, as in § 7.1,

$$e_{xx} = \frac{1}{E}\left[\tau_{xx} - \sigma(\tau_{yy} + \tau_{zz})\right] \tag{1}$$

with e_{yy} and e_{zz} equivalents. These equations contain *all* the physics. There are just *two* independent numbers, e.g. E and σ, in the 81 components of λ_{ijkl} for an isotropic material. Two other elastic moduli, logically equivalent to E and σ, are often convenient.

7.5.1 Bulk modulus

Take the stress as a uniform pressure, i.e. all three longitudinal τ's are equal to $-p$. Then

$$e_{xx} = e_{yy} = e_{zz} = -\frac{p}{E}(1 - 2\sigma).$$

$$\frac{\delta V}{V} = \text{volume strain} = e_{xx} + e_{yy} + e_{zz} = e_v.$$

[? obvious from $V = xyz$, $\delta V = \ldots$].

Def$^{\mathbf{n}}$. 15 Bulk modulus: *B (or $K, \ldots$) $\equiv -p/$(volume strain),*

or

$$\text{stress} = -p \quad = \quad \underbrace{\frac{E}{3(1-2\sigma)}}_{B}\, e_v.$$

7.5.2 Shear modulus

Take stress of form $\tau_{xx} = -\tau_{yy} = \tau$; $\tau_{zz} = 0$. Then

$$e_{xx} = \frac{\tau}{E}(1 + \sigma) = -e_{yy};\ e_{zz} = 0.$$

A square element $PQRS$ at 45° to x and y axes becomes distorted. Simple arguments show that *wrt* $x'y'$ axes at 45°:

Stress on PQ face is transverse $\tau_{x'y'}$ and equal to τ_{xx}.

Shear strain = departure of angle PQR from 90° $= 2e_{xx} = \theta_{sh}$.

Def$^{\text{n}}$. 16 Shear modulus: *G (or n, $\mu, \ldots$) $\equiv$ (transverse stress)/(shear strain),*

or

$$\tau = \underbrace{\frac{E}{2(1+\sigma)}}_{n} \theta_{sh}$$

Note: B and n *must* be positive; therefore $-1 \le \sigma \le \frac{1}{2}$ (in practice $0 \le \sigma \le \frac{1}{2}$).

7.5.3 Other forms

Equations (1) may be inverted, by algebra, to give

$$\tau_{xx} = \underbrace{\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}}_{M_l} \left[e_{xx} + \frac{\sigma}{1-\sigma}(e_{yy} + e_{zz}) \right]. \tag{2}$$

M_l is important as the longitudinal elastic modulus when transverse *strains* are zero, as e.g. when a large flat sheet is compressed, or for a longitudinal sound wave in a solid. As we shall see, and would expect [why?], it is larger than Young's modulus.

A way through the algebra is first to sum the equations of (1), giving

$$E\,\mathrm{tr}(\mathbf{e}) = (1-2\sigma)\,\mathrm{tr}(\boldsymbol{\tau}); \text{ or } \mathrm{tr}(\boldsymbol{\tau}) = 3B\,\mathrm{tr}(\mathbf{e}).$$

Here *tr* is the *trace* of the matrix, i.e. the sum of the diagonal terms; it is independent of a change in orthogonal axes (see IA maths), e.g. $\mathrm{tr}(\mathbf{e})$ is the fractional volume change. The above result, obtained in § 7.5.1 for equal longitudinal stresses along the eigen-vector axes, is thus more general and holds e.g. even when there are shear stresses present. Put

$$Ee_{xx} = \tau_{xx}(1+\sigma) - \sigma\,\mathrm{tr}(\boldsymbol{\tau}).$$

Therefore

$$\tau_{xx} = \underbrace{\frac{E}{1+\sigma}}_{2n} \left[e_{xx} + \frac{\sigma}{1-2\sigma}\,\mathrm{tr}(\mathbf{e}) \right].$$

Since $\tau_{xy} = 2ne_{xy}$ etc. for shear terms if present, the general relation between the $\boldsymbol{\tau}$ and $\mathbf{e}$ matrices is

$$\boldsymbol{\tau} = 2G\mathbf{e} + \frac{E\sigma}{(1+\sigma)(1-2\sigma)}\mathbf{I}\,\mathrm{tr}(\mathbf{e}),$$

[sometimes written $2\mu\mathbf{e} + \lambda\mathbf{I}\,\mathrm{tr}(\mathbf{e})$, with λ and μ known as the *Lamé coefficients.* Feynman II argues illuminatingly that the $\boldsymbol{\tau}$–$\mathbf{e}$ relation *must* be of this form for an isotropic material].

Hence

$$\tau_{xx} = \frac{E}{1+\sigma} \left[(1 + \frac{\sigma}{1-2\sigma})e_{xx} + \frac{\sigma}{1-2\sigma}(e_{yy} + e_{zz}) \right],$$

as in (2).

$$\begin{aligned} M_l &= \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \\ &= E\left[\frac{1}{3(1-2\sigma)} + \frac{2}{3(1+\sigma)}\right] \quad \text{by partial fractions} \\ &= B + \tfrac{4}{3}G. \end{aligned}$$

7.6 Voigt notation

Treat $\boldsymbol{\tau}$, $\mathbf{e}$ as 6D vectors e.g. T_α [use capital letter equivalents and greek suffices, e.g. $\alpha = 1, 6$].

Def$^{\mathbf{n}}$. 17 $T_\alpha \equiv (\tau_{xx}, \tau_{yy}, \tau_{zz}, \tau_{yz}, \tau_{xz}, \tau_{xy})$.

Def$^{\mathbf{n}}$. 18 $E_\alpha \equiv (e_{xx}, e_{yy}, e_{zz}, 2e_{yz}, 2e_{xz}, 2e_{xy})$.

E_4 to E_6 are the strain angles; note the *factors of 2* in the definition.

Def$^{\mathbf{n}}$. 19 General elastic modulus: $\Lambda_{\alpha\beta}$ *such that* $T_\alpha = \sum_\beta \Lambda_{\alpha\beta} E_\beta$.

Λ has 36 components. For an isotropic material $T_4 = nE_4$ and similarly for T_5 and T_6. Voigt notation simplifies handling of work and energy; change of axes in 3D space entails complicated changes in Voigt 6D-space.

7.7 Work, energy

Work done = force × distance. In general terms

$$dW = (\text{stress} \times \text{area}) \times (d(\text{strain}) \times \text{length}).$$

For a single component of stress and equivalent strain (proportional to it)

$$\text{Work done} = \text{energy in stressed material} = \tfrac{1}{2}\,\text{stress} \times \text{strain} \times \text{volume}.$$

Ex. 13 *Wire stretched by u under force F. Work done* $= \frac{1}{2}Fu = \frac{1}{2}A\tau\, el$.

Ex. 14 *Block of volume xyz sheared by F_{xy} forces; material along y-axis moves through θ. Work done*$=\frac{1}{2}F_{xy}xz\, y\theta = \frac{1}{2}T_6\, xyz\, E_6$ *in Voigt notation.*

Let W be the work per unit volume. Then

$$dW = \sum_i \sum_j \tau_{ij} de_{ij} = \sum_\alpha T_\alpha dE_\alpha.$$

[Note that the factor of 2 in the definition of the shear components of E_α takes account of the xy terms appearing twice in the former sum (as e_{xy} and e_{yx}) and only once in the latter (as E_6)].

We evaluate W in two stages

7.7.1 dW is a 'perfect differential'

i.e.

$$\int_A^B dW = W_B - W_A$$

is independent of the path between in initial (A) and final (B) configurations of T_α and E_α. This is because the deformations are elastic and energy is conserved. Standard maths then gives

$$\frac{\partial^2 W}{\partial E_\alpha \partial E_\beta} = \frac{\partial^2 W}{\partial E_\beta \partial E_\alpha},$$

or

$$\frac{\partial T_\alpha}{\partial E_\beta} = \frac{\partial T_\beta}{\partial E_\alpha},$$

or $\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha}$, giving at most 21 independent components (the most general type of anisotopic materials have this many elastic constants).

7.7.2 Integration of W

Let the initial E_α's all be zero (unstressed) and grow *proportionately* to the final state (i.e. steady growth in the *same* direction in 6D Voigt-space); this is a specific path of integration, but we have just shown that all paths give the same answer.

$$dW = \sum_\alpha T_\alpha dE_\alpha = \sum_\alpha (\sum_\beta \Lambda_{\alpha\beta} E_\beta) dE_\alpha,$$

or

$$W = \sum_\alpha \sum_\beta \tfrac{1}{2} \Lambda_{\alpha\beta} E_\alpha E_\beta,$$

since each of the 36 terms summed is of the form $\int_0^1 f E_\beta \, df E_\alpha$, where f is the fraction of the final value. The non-Voigt form is

$$W = \sum_i \sum_j \sum_k \sum_l \tfrac{1}{2} \lambda_{ijkl} e_{ij} e_{kl}.$$

8 Elasticity — Applications

8.1 Simple examples

Ex. 15 *Long thin cylindrical tube, wall thickness $t \ll$ radius r; pressure p inside.*

Take cylindrical coordinates; in these coordinates all stresses and strains are longitudinal (tensors diagonal). The circumferential stress $\tau_{\phi\phi}$ is obtainable from the equilibrium of half the cylinder sliced lengthways. Force on half of cylinder gives

$$2t\tau_{\phi\phi} = 2rp; \quad \text{or} \quad \tau_{\phi\phi} = p\,\frac{r}{t}.$$

The lengthways stress τ_{zz} is got from the force on the closed ends:

$$2\pi rt\tau_{zz} = \pi r^2 p; \quad \text{or} \quad \tau_{zz} = p\,\frac{r}{2t}.$$

The radial stress $\tau_{\rho\rho}$ will depend on the depth within the wall, but will be of the order of p and so tiny *cf* $\tau_{\phi\phi}$ and τ_{zz}. Therefore the circumferential change $e_{\phi\phi}$ is

$$e_{\phi\phi} = \frac{1}{E}\left(\tau_{\phi\phi} - \sigma\tau_{zz}\right) = \frac{pr}{Et}\left(1 - \tfrac{1}{2}\sigma\right).$$

e_{zz} is found analogously.

Ex. 16 *Torsion of a solid cylindrical wire, twisted through an angle ϕ in length l.*

Consider annular element of radius r and thickness $t \ll r$. $\theta =$ shear angle $= r\phi/l$. Therefore force per unit area at a cross-section is $n\theta$. Couple, of all forces acting at a cross-section of the annular element, is

$$C = r\,G\theta\,2\pi rt = \frac{2\pi r^3 tG}{l}\phi.$$

For a *solid* wire, put t equal to dr and integrate:

$$C = \frac{\pi r^4 G}{2l}\phi.$$

8.2 Bending of beams

Consider a thin beam bent into an arc of radius R (which may vary down the beam). The inner side of the beam will be compressed, the outer expanded. A sheet in the middle, the *neutral axis*, will retain its original length.

Def$^{\mathbf{n}}$. 20 Bending moment: *couple produced by stress forces at a transverse cut across the beam.*

For an isolated beam in equilibrium, the bending moment must be the same all along the beam and equal to the external couples applied at its ends.

Measure ξ across the beam from the neutral axis. A strand of material at ξ of original length $R\theta$ will be stretched to $(R+\xi)\theta$ by the bending. Therefore

$$\text{strain} = dl/l = \xi/R$$

or the bending moment, obtained by integrating across the area,

$$B = \int \underbrace{E\frac{\xi}{R}dA}_{\text{force on dA}}\,\xi = \frac{E}{R}\int \xi^2 dA.$$

$\int \xi^2 dA$ is the 'moment of area' about the neutral axis, and is a measure of the stiffness, which therefore goes as (linear size)^4.

Ex. 17 *Light rod, length l, clamped horizontally at one end, weight W at the other. Assume small angular deviations.*

Take x along rod, $y(x)$ = *downwards* displacement. For a rectangular cross-section a high and b wide, and the neutral axis in the middle,

$$I = \int \xi^2 dA = \frac{a^3 b}{12}.$$

Consider equilibrium from x to the end. Total B on this piece of beam = 0. The bending moment B at x is therefore given by

$$B = W(l - x).$$

For this geometry $y'' = 1/R$, so

$$EIy'' = W(l-x);$$

$$EIy' = W\left(lx - \frac{x^2}{2} + C_1\right); \qquad C_1 = 0, \text{ since } y' = 0 \text{ at } x = 0.$$

$$EIy = W\left(\frac{lx^2}{2} - \frac{x^3}{6} + C_0\right); \qquad C_0 = 0, \text{ since } y = 0 \text{ at } x = 0.$$

Ex. 18 *The same rod, but bending under its own weight.*

Let $w(x)dx$ be the weight of length dx; no W at end. Let $F(x)$ be the vertical force exerted across any transverse section (like y, measure it downwards).

Consider equilibrium of dx. Stress forces are F down (at x) and $F + dF$ up (at $x + dx$). Therefore

$$dF = w(x)dx; \quad dF/dx = w(x).$$

Consider now the couple. The vertical stress forces produce a couple Fdx. This must just balance the *difference* in the bending moments at the two ends of dx, i.e.

$$dB = Fdx; \quad dB/dx = F(x).$$

Therefore

$$\begin{aligned} EIy'' &= B(x), \\ EIy''' &= F(x), \\ EIy'''' &= w(x). \end{aligned}$$

If w is uniform, and equal to say W/l, then

$$EIy''' = \int w dx = \frac{W}{l}x + C_3 = W\left(\frac{x}{l} - 1\right)$$

using $F = 0$ at $x = l$.

$$EIy'' = W\left(\frac{x^2}{2l} - x + C_2\right) = W\left(\frac{x^2}{2l} - x + \frac{l}{2}\right)$$

using $B = 0$ at $x = l$.

$y'(x)$ and $y(x)$ follow from further integrations, with $y' = y = 0$ at $x = 0$.

8.3 Euler strut

A rod of length l is subject to a compressional force F parallel to its length. Imagine F increasing slowly. For small F the rod is compressed but stays straight. When F exceeds a certain value, the rod 'bows' and departs markedly from being straight. The onset of bowing is sudden; why, and at what force?

8.3.1 Equilibrium analysis

If the rod is 'bowed', the moment of the end force F at a point on the rod displaced to y is Fy. This must balance the bending moment if in equilibrium, giving (with due regard for signs)

$$EIy'' + Fy = 0$$

i.e. an SHM equation for $y(x)$ with solution

$$y = y_0 \sin\sqrt{\frac{F}{EI}}\,x$$

($y = 0$ at $x = 0$, so the cos term can be omitted). The wavelength $\lambda = 2\pi\sqrt{\frac{EI}{F}}$; there will be a physically possible solution other than $y = 0$ if $\lambda/2$ fits into l. In detail

$$y = y_0 \sin\sqrt{\frac{F}{EI}}\,x \text{ and must be zero at } x = l.$$

1. If $\sqrt{\frac{F}{EI}} < \pi$, $y_0 = 0$ is the only possibility. The rod does not bow.
2. If $\sqrt{\frac{F}{EI}} = \pi$, $y = 0$ whatever y_0, i.e. bowing is equally possible at *any* amplitude. The rod is in neutral equilibrium as between different values of y_0.
3. If $\sqrt{\frac{F}{EI}} > \pi$, $y_0 = 0$ is an *unstable* solution. The easiest way of seeing this is via energy.

8.3.2 Energy

Take rod of form $y = y_0 \sin kx$; $kl = \pi$. We evaluate

(a) Energy stored in bent beam. Energy $= \int B d\theta = \frac{1}{2}B\theta$ for each piece ($B \propto \theta$). Put $\theta = dx/R = y''dx$ for each length dx and $B = EIy''$; integrate over x

$$W_1 = \int \tfrac{1}{2}EI(y'')^2 dx = \tfrac{1}{4}EIy_0^2k^4 l.$$

(b) Work done by forces at ends as the ends move in towards one another. $W_2 = F\times$(arc minus chord).

$$ds = dx\sqrt{1+y'^2} \approx dx(1+\tfrac{1}{2}y'^2)$$

giving

$$W_2 = \tfrac{1}{4}Fy_0^2k^2l.$$

The important result is that

$$\frac{W_1}{W_2} = \frac{EIk^2}{F} \text{ independent of } y_0.$$

We can draw up a table showing which of W_1 and W_2 is the larger for each of the three situations listed above

Case:	1	2	3
Larger:	W_1	$W_1 = W_2$	W_2
$y = 0$ equilibrium:	stable	neutral	unstable

When the force is sufficient to produce bowing, y_0 increases until the small-angle approximations in the above treatment cease to be valid; the rod takes an equilibrium position determined by the fuller (and more complicated) equations.

9 Elastic waves

9.1 Basic theory

We apply the results of previous sections to the non-equilibrium situation. We have the following chain of relationships, each of whose links we need to think about

$$\begin{array}{ccccccccc} \mathbf{u}(\mathbf{r}) & \longleftrightarrow & e_{ij} & \longleftrightarrow & \tau_{ij} & \longleftrightarrow & \mathbf{F} & \longleftrightarrow & \ddot{\mathbf{u}} \\ \text{displacement} & & \text{strain} & & \text{stress} & & \text{on dV} & & \end{array}$$

Wave motion can occur when the stress forces generated by a pattern of displacements $\mathbf{u}(\mathbf{r})$ are sufficient to maintain the pattern.

9.1.1 F and $\ddot{u}$

We suppose (a) isotropic and homogeneous materials, and (b) strains small ($\ll 1$), implying a constant *density* and thefore allowing us to write

$$\mathbf{F} \text{ on } dV \;= \rho dV \ddot{\mathbf{u}}$$

with ρ constant.

9.1.2 τ and F

We need the force on element $dV = dxdydz$ resulting from the stress forces on all its faces. The x-component of the force on the x-facing faces is $-\tau_{xx}dydz$ on the face at x and $+[\tau_{xx} + (\partial\tau_{xx}/\partial x)dx]dydz$ on the face at $x + dx$, giving a net contribution to F_x of $(\partial\tau_{xx}/\partial x)dV$. The y-facing and z-facing faces give contributions to F_x of $(\partial\tau_{xy}/\partial y)dV$ and $(\partial\tau_{xz}/\partial z)dV$ similarly, giving a total F_x per unit volume of

$$F_x/dV = \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}.$$

[More elegant derivation: The i-th component of force on area $\mathbf{dA}$ can be written as the scalar product $\boldsymbol{\tau}_i \cdot \mathbf{dA}$, where $\boldsymbol{\tau}_i$ is the *vector* in the i-th row of the matrix $\boldsymbol{\tau}$. F_i on a volume is then given by

$$F_i = \int \boldsymbol{\tau}_i \cdot \mathbf{dA} = \int \operatorname{div} \boldsymbol{\tau}_i \, dV$$

using the divergence theorem. This is equivalent to an F_i of $\operatorname{div}\boldsymbol{\tau}_i$ per unit volume — the same result as above.]

9.1.3 τ, e and u

We have

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and also the results of § 7.5 relating $\mathbf{e}$ and $\boldsymbol{\tau}$. We postpone discussion of the general $\mathbf{u}(\mathbf{r})$ relationship in favour of dealing first with the two important simple cases (which in fact represent all the physics).

9.2 Plane waves

We shall suppose the waves are travelling the x-direction, i.e. put $\partial/\partial y = \partial/\partial z = 0$.

9.2.1 Transverse waves

or 'Shear waves' ('S waves'). We suppose $\mathbf{u}$ is in the y-direction, i.e.

$$\mathbf{u} = (0, u_y, 0); \quad u_y = u_y(x,t).$$

The y-component of force is

$$F_y/dV = \frac{\partial \tau_{yx}}{\partial x}$$

and

$$\tau_{yx} = G\,2e_{yx} = G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = G\frac{\partial u_y}{\partial x}.$$

Therefore

$$\rho \ddot{u}_y = \frac{\partial}{\partial x}\left(G\frac{\partial u_y}{\partial x}\right) = G\frac{\partial^2 u_y}{\partial x^2}.$$

This is the non-dispersive wave equation for waves travelling with velocity

$$v_t = \sqrt{G/\rho} = \sqrt{M_t/\rho}.$$

M_t is here used to denote the modulus for transverse waves (it is equal to G).

9.2.2 Longitudinal waves

or 'Pressure waves' ('P waves'). Here we suppose $\mathbf{u}$ is in the x-direction, i.e.

$$\mathbf{u} = (u_x, 0, 0); \quad u_x = u_x(x,t).$$

The relevant force is

$$F_x/dV = \frac{\partial \tau_{xx}}{\partial x}.$$

With a longitudinal wave, there is no y or z motion and hence $e_{yy} = e_{zz} = 0$. The $\boldsymbol{\tau}$–$\mathbf{e}$ relationship is thus simply $\tau_{xx} = M_l e_{xx}$ (see equation (2) of § 7.5). Therefore

$$\rho \ddot{u}_x = \frac{\partial \tau_{xx}}{\partial x} = \frac{\partial}{\partial x}\left(M_l e_{xx}\right) = M_l\frac{\partial^2 u_x}{\partial x^2}.$$

This is the non-dispersive wave equation for waves of velocity

$$v_l = \sqrt{M_l/\rho}.$$

9.2.3 Relative velocities

As already noted $M_l = B + \frac{4}{3}nG$. So $M_l > M_t$; longitudinal waves are faster than transverse. The ratio

$$\left(\frac{v_l}{v_t}\right)^2 = \frac{M_l}{M_t} = \frac{B + 4G/3}{n} = \frac{B}{G} + \frac{4}{3} = \frac{2-2\sigma}{1-2\sigma}$$

where the final expression has been obtained by putting B and G in terms of E and σ (E cancels and the velocity ratio depends only on σ).

9.2.4 Fluids

The characteristic property is that a shear stress cannot be maintained; shear strain e_{xy} can exist without any τ_{xy}. Formally one can put $G = 0$, $E = 0$, $\sigma = \frac{1}{2}$. B is *the* one elastic constant. The consequences for wave motion are

1. Longitudinal waves: $M_l = B$, $v_l = \sqrt{B/\rho}$.
2. Transverse waves: $M_t = 0$, $v_t = 0$. Physically this implies infinite wavelength, or uniform steady $\dot{u}_y$ with no restoring force.

Note that for longitudinal waves in a solid, the modulus M_l is more than just B; there is a *shear* as well.

9.2.5 General $u(r)$

We can write, using the results of § 7.5 and the notation of § 9.1.2

$$\rho \ddot{u}_x = \operatorname{div}(\boldsymbol{\tau}_x) = 2G\operatorname{div}(\mathbf{e}_x) + \lambda \operatorname{div}(\mathbf{I}_x \operatorname{tr}(\mathbf{e})).$$

$\mathbf{I}_x \operatorname{tr}(\mathbf{e})$ is the vector in the top row of the matrix $\mathbf{I}\operatorname{tr}(\mathbf{e})$ and is $(\operatorname{tr}(\mathbf{e}), 0, 0)$; div of it is

$$\operatorname{div}(\mathbf{I}_x \operatorname{tr}(\mathbf{e})) = \frac{\partial}{\partial x}\operatorname{tr}(\mathbf{e}) = \frac{\partial}{\partial x}(e_{xx} + e_{yy} + e_{zz})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = \frac{\partial}{\partial x}\operatorname{div}\mathbf{u}.$$

We also need

$$2\operatorname{div}(\mathbf{e}_x) = 2\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$= \nabla^2 u_x + \frac{\partial}{\partial x}\operatorname{div}(\mathbf{u}).$$

Therefore

$$\rho \ddot{u}_x = G\nabla^2 u_x + (G + \lambda)\frac{\partial}{\partial x}\operatorname{div}\mathbf{u}$$

or

$$\rho \ddot{\mathbf{u}} = G\nabla^2 \mathbf{u} + (G + \lambda)\operatorname{grad}\operatorname{div}\mathbf{u}.$$

This equation is in terms of the Lamé coefficient

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)} = B - \tfrac{2}{3}G;$$

$$G + \lambda = B + \frac{G}{3}.$$

Transverse waves arise when $\operatorname{div}\mathbf{u} = 0$ (no volume change; the planes of material slide sideways). Longitudinal or irrotational waves arise when $\operatorname{curl}\mathbf{u} = 0$.

9.3 Waves on a plate

Take the plate as in the xy-plane; suppose the thickness $\ll \lambda$. The boundary condition is that there is no *force* at the surface; $\tau_{xz} = \tau_{yz} = \tau_{zz} = 0$. The arguments are similar to those for 3D but with a different $\boldsymbol{\tau}$–$\mathbf{e}$–$\mathbf{u}$ relationship. The results, for waves travelling in the x-direction, are

1. $\mathbf{u}$ in y-direction: transverse waves with $v_t = \sqrt{G/\rho}$ as in the 3D case.
2. $\mathbf{u}$ in x-direction: longitudinal waves with relevant modulus $M_l{}' = E/(1-\sigma^2)$. Unlike the 3D case, e_{zz} is non-zero; the wave produces variations of thickness of the plate.
3. $\mathbf{u}$ in z-direction: *bending* of the plate — see transverse waves on rods below.

9.4 Waves on rods

9.4.1 Torsional waves

Cylindrical rod. From § 8.1, the couple C across surface at x is given by

$$G = \frac{\pi r^4 G}{2} \frac{\partial \phi}{\partial x}.$$

The net C on a disc of rod of length dx is

$$dG = \frac{\partial G}{\partial x} dx = \frac{\pi r^4 G}{2} \frac{\partial^2 \phi}{\partial x^2} dx$$

which is equal to

$$dI\,\ddot{\phi} = \tfrac{1}{2} \rho \pi r^2 dx\, r^2\, \ddot{\phi}$$

giving the non-dispersive wave equation

$$\rho \ddot{\phi} = G \frac{\partial^2 \phi}{\partial x^2}$$

and velocity $v = \sqrt{G/\rho}$.

9.4.2 Transverse waves

The theory of bending beams (see § 8.2) gives

$$EI \frac{\partial^4 y}{\partial x^4} = w(x)$$

where $w(x)dx$ is the external sideways force on dx that would produce equilibrium with the beam bent to $y(x)$. The equation of motion for non-equilibrium $y(x)$ and no external w is therefore

$$EIy'''' = -\rho \ddot{y}$$

where ρ is the mass per unit length along the rod. For a wave solution of the form $y = y_0 e^{i(\omega t - kx)}$,

$$EIk^4 = \rho \omega^2.$$

The waves are *dispersive* and this is the *dispersion relation*; $v = \omega/k$ depends on ω.

For SHM time-variation $y = y(x)e^{i\omega t}$, the space-variation is given by

$$y'''' = k^4 y; \quad (k \text{ as above}).$$

This is a fourth-order differential equation. The general solution has 4 arbitrary constants; one possible form of it is

$$y = A\cos kx + B\sin kx + C\cosh kx + D\sinh kx,$$

the values of A, B, C and D being determined by four boundary conditions.

Ex. 19 *Free rod of length 2l. What are the frequencies of the symmetrical modes of oscillation?*

At the ends, the force F and bending moment G across an interface are both zero. So $y''' = y'' = 0$ (see § 8.2) at $x = \pm l$. For symmetrical modes $B = D = 0$, and substitution of the $x = \pm l$ conditions gives

$$G = 0: \quad 0 = -Ak^2\cos kl + Ck^2\cosh kl$$

$$F = 0: \quad 0 = +Ak^3\sin kl + Ck^3\sinh kl$$

or (for non-zero A and C)

$$\tan kl = -\tanh kl.$$

Sketches of $\tan kl$ and $-\tanh kl$ give solutions at $kl = 0$ and then slightly greater than $\frac{3\pi}{4}$, $\frac{7\pi}{4}$ etc.

9.5 Propagation, impedance

Reminder of standard IA results. Consider longitudinal waves in 3D case.

$$u_x = u_0 e^{i(\omega t - kx)}; \quad \dot{u}_x = i\omega u_x.$$

Also

$$e_{xx} = \frac{\partial u_x}{\partial x} = -iku_x;$$

$$\tau_{xx} = M_l e_{xx} = -ikM_l u_x$$

so that $\tau_{xx} \propto \dot{u}_x$.

Def$^{\mathbf{n}}$. 21 Impedance: *$Z \equiv$ 'force'/('response velocity'), force and response being such that rate of energy transfer = force × response velocity.*

In this case, 'force' $= \tau_{xx}$ and response velocity $= \dot{u}_x$. Therefore

$$Z = \left|\frac{\tau_{xx}}{\dot{u}_x}\right| = \frac{kM_l}{\omega} = \frac{M_l}{v_l} = \sqrt{M_l \rho}.$$

$$\text{Rate of work/m}^2 = |\tau_{xx}\dot{u}_x| = \frac{\tau_{xx}^2}{Z} = \dot{u}_x^2 Z.$$

Energy: $U = \frac{1}{2}$ stress × strain per unit volume $= \frac{1}{2}M_l e_{xx}^2$ per unit volume. $T = \frac{1}{2}\rho\dot{u}_x^2$ per unit volume. As can easily be shown

1. $\overline{U} = \overline{T}$.
2. $\overline{E} = \overline{T + U} = \frac{1}{2} M_l k^2 u_0^2$.
3. $\overline{E}$ × velocity = energy flow rate.

Ex. 20 *Torsional waves on cylindrical rod.*

$\phi(x,t) = \phi_0 e^{i(\omega t - kx)}$. The response velocity is $\dot{\phi} = i\omega\phi$ and the 'force' is C. The impedance is therefore $Z = \pi r^4 G k / 2\omega$.

9.6 Reflection of waves — Not for Examination

Take 3D waves, two media with plane boundary at $z = 0$.

9.6.1 Normal incidence

Boundary conditions are

1. u, and hence $\dot{u}$, continuous; therefore $u_i + u_r = u_t$ (i, r and t are for incident, reflected and transmitted waves).
2. Force on medium 1 from medium 2 = minus force on 2 from 1; therefore τ_{jx} continuous for each j-th component of force.

These give $Z_1(\dot{u}_i - \dot{u}_r) = Z_2 \dot{u}_t$, or

$$\frac{u_r}{u_i} = -\frac{\tau_r}{\tau_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2}.$$

9.6.2 Oblique incidence

Much more complicated. Take solid/vacuum interface (solid at negative z) and a P wave incident from below from direction $\mathbf{k} = \mathbf{k}_i$ in the xz-plane. We can write down u for incident and reflected waves (time factor $e^{i\omega t}$ understood)

$$\mathbf{u}_i = \underbrace{\frac{u_i}{k}(k_x, 0, k_z)}_{\text{vector, length } u_i,\ \|\ \mathbf{k}} \ \underbrace{e^{-i(k_x x + k_z z)}}_{\text{wave } \|\ \mathbf{k}}$$

$$\mathbf{u}_r = \frac{u_r}{k}(k_x, 0, -k_z)\, e^{-i(k_x x - k_z z)}.$$

These are based on the Snell's Law arguments that the boundary conditions are repetitive at $2\pi/\omega$ in t and $2\pi/k_x$ in x, and that u_r must be similarly repetitive. $k_x^2 + k_z^2 = \omega^2/v_l^2$ is the same for both, and so k_z for u_r is $-k_z$ for u_i.

The boundary condition is that there is no force on the face ⊥ z; $\tau_{xz} = \tau_{yz} = \tau_{zz} = 0$ at $z = 0$. The τ_{xz} condition implies that $2e_{xz} = \partial u_z/\partial x + \partial u_x/\partial z$ must be zero at $z = 0$, and substitution from the above formulae gives $u_i = u_r$. But the τ_{zz} condition, by similar algebra requires $u_i = -u_r$. Therefore $u_i = u_r = 0$ and a pair of waves of this type is not physically possible.

What happens is that the reflection produces an S wave as well; it has the same ω, k_x, but a different velocity v_t and different total $\mathbf{k}$, say $\mathbf{k}'$, with a different z-component k_z'. The disturbance may be written

$$\mathbf{u}_s = \frac{u_s}{k'}(k_z', 0, k_x)\, e^{-i(k_x x - k_z' z)};$$

note that the $\mathbf{u}$-vector is in the direction *perpendicular* to that of propagation. The wave may be called an SV wave (S wave with u in vertical plane).

Snell's Law, from common k_x for all waves, gives

$$k_x = k' \sin\theta_s = k \sin\theta_p$$

with $k' = \omega/v_t$, $k = \omega/v_l$; since $v_t < v_l$, $\theta_s < \theta_p$ and the S wave direction is nearer to the normal.

Matching the boundary conditions with the three waves is algebraically tedious. I give a table of impedances for incident and reflected P and SV waves.

	$\tau_{xz}/\dot{u}$	$\tau_{zz}/\dot{u}$
P_i	$-A \equiv -\dfrac{2k_xk_z}{\omega k}n$	$-B \equiv -\dfrac{k_z^2 + sk_x^2}{\omega k}M_l$
P_r	$+A$	$-B$
SV_i	$+C \equiv \dfrac{k_z'^2 - k_x^2}{\omega k'}n$	$-D \equiv -\dfrac{k_xk_z'(1-s)}{\omega k'}M_l$
SV_r	$+C$	$+D$

Here $s = \sigma/(1-\sigma)$ and arises from

$$\tau_{zz} = M_l\left(e_{zz} + \frac{\sigma}{1-\sigma}e_{xx}\right) = M_l\left(\frac{\partial u_z}{\partial z} + s\frac{\partial u_x}{\partial x}\right).$$

Note that k_z reverses sign in going from incident to reflected.

The case we are considering, of $\mathrm{P}_i \rightarrow \mathrm{P}_r + \mathrm{SV}_r$, has two boundary conditions, giving

$$\begin{aligned} \tau_{xz} = 0: &\quad -Au_i + Au_r + Cu_s = 0 \\ \tau_{zz} = 0: &\quad -Bu_i - Bu_r + Du_s = 0 \end{aligned}$$

and hence the two ratios (or reflection coefficients) u_s/u_i and u_r/u_i.

9.6.3 Oblique incidence — qualitative summary

1. An incident P wave generates *two* reflected waves, a P_r and an SV_r wave; the SV wave is at a different angle to the interface. The process is called *mode conversion.*
2. Similarly $\mathrm{SV}_i \rightarrow \mathrm{SV}_r + \mathrm{P}_r$.
3. The P wave is nearer to glancing than the SV. If an incident SV wave is sufficiently near to glancing, $\sin\theta_p$ can be > 1 and the P wave is *evanescent.*

4. Incident SH waves (horizontally polarised, u in y-direction) are reflected without mode conversion; SH$_i$ →SH$_r$. This is because τ_{yz} is the only non-zero stress component at the boundary, and one reflected wave is sufficient to balance it.

5. With *two* media, mode conversion gives two reflected and two refracted waves: P$_i$ →P$_r$+SV$_r$+P$_t$+SV$_t$ and SV$_i$ similarly. SH waves give a single reflected and refracted wave.

Note that the behaviour is in general more complicated than for electromagnetic waves, basically because both longitudinal and transverse waves are possible.

9.6.4 Rayleigh waves

Evanescent P waves from SV$_i$ →SV$_r$+P$_r$ with sufficiently glancing SV$_i$ have k_z' imaginary and exponential decay of u_p with distance from the surface. They have some similarities with EM waves in the second medium on ‘total internal reflection’. In particular they are driven by the incident wave and cannot exist in isolation. But mode conversion allows a possibility which has no parallel in electromagnetism, namely that you *can* get a self-sustaining wave with an SV and a P wave, *both* evanescent.

If k_z and k_z' are both imaginary, then from the impedance table, A and D are imaginary, and B and C are real. The boundary conditions with just P$_r$ and SV$_r$ waves give

$$\begin{aligned} \tau_{xz} = 0: &\quad Au_p + Cu_s = 0 \\ \tau_{zz} = 0: &\quad -Bu_p + Du_s = 0 \end{aligned}$$

with a non-zero solution if $AD = -BC$. These are *Rayleigh* waves, or *surface* waves. For given k_x, the $AD = -BC$ condition gives unique but different k_z and k_z' (the SV and P waves penetrate to different depths) and a unique ω. The waves turn out to be non-dispersive with a velocity slightly less than v_t.

These, and all other forms of elastic waves, are important in seismology.