Part IB Physics B 2009
Answers to Classical Dynamics and Fluids Examples 1

These worked answers to the Dynamics problems are offered to students and supervisors as a guide, and I hope they contain some generally useful hints. In some cases, I have indulged in a little research that goes beyond the stated problem. Where possible I have included illustrative diagrams.

Health Warning

These solutions must be used with caution.

- The worked answers will be useless to you unless you have already made a very serious attempt to solve the problem yourself and have discussed it with your supervisor. If you consult the solutions earlier, they will just act as 'spoilers'.
- You must remember that there is often more than one way to solve a physics problem. I have sometimes suggested alternatives, but your supervisors will know many others.
- The most difficult part of a physics problem is knowing where to start. It is therefore quite possible that I taken too much for granted at the beginning of some problems. If so, please let me know.
- I have often given the numerical answers to several more decimal places than is really justified, so that you can check your calculations (and mine) carefully. I have used values of the physical constants taken from the formula book that you will use in the Examination.

Please let me me know about any errors, typos and suggestions for improvement.

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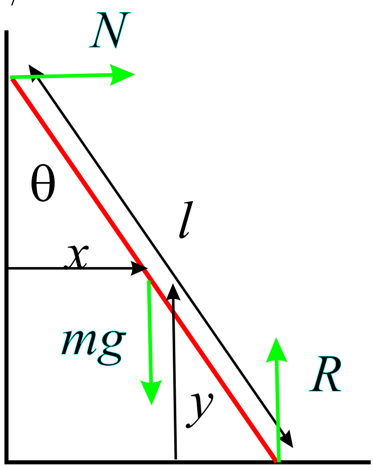
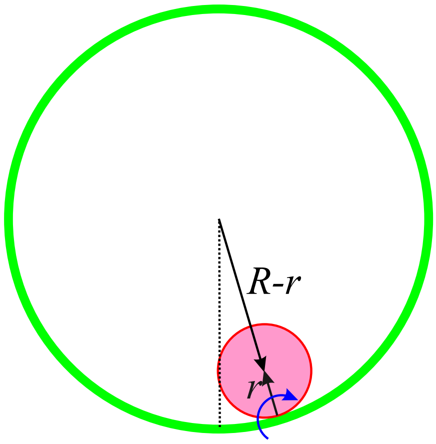
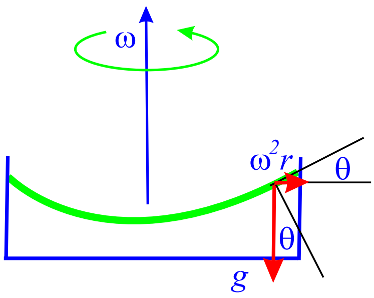
1. The moment of inertia of the cylinder is $\frac{1}{2}ma^2$, so its initial kinetic energy is $\frac{1}{4}ma^2\omega^2$. The friction $F(t)$ slows the rotation and increases the linear velocity v: $m\dot{v} = F$; $I\dot{\omega} = -Fa$ so that $I\omega(t) + mav(t)$ is a constant $(= I\omega)$. The friction stops at ω' when the cylinder rolls $v' = a\omega'$, so $\frac{1}{2}ma^2\omega' + ma\omega' = \frac{1}{2}ma^2\omega$, so $\omega' = \frac{1}{3}\omega$. The energy is evaluated as $\frac{1}{4}ma^2(\omega')^2 + \frac{1}{2}ma^2(\omega')^2 = \frac{1}{12}ma^2\omega^2$, i.e. 2/3 has been lost.



2. The coordinates of the centre of mass are $x = \frac{1}{2}l\sin\theta$; $y = \frac{1}{2}l\cos\theta$ and the velocities are $\dot{x} = \frac{1}{2}l\cos\theta\ \dot{\theta}$; $\dot{y} = -\frac{1}{2}l\sin\theta\ \dot{\theta}$. The kinetic energy is $T = \frac{1}{2}m(\dot{x}^2) + \dot{y}^2 + \frac{1}{2}I\dot{\theta}^2$, where the moment of inertia is $I = \frac{1}{12}ml^2$. We find $T = \frac{1}{6}ml^2\dot{\theta}^2$.
   The potential energy is $V = \frac{1}{2}mgl\cos\theta$.
   Using the energy method $\dfrac{\mathrm{d}}{\mathrm{d}t}(T+V) = 0$ we derive the equation of motion $\ddot{\theta} = \dfrac{3g}{2l}\sin\theta$.
   The story doesn't quite end there, because the reaction force at the wall must eventually go negative, so the ladder will actually lose contact with the wall. This is easy to prove by Newtonian methods, and indeed by Lagrangian methods if you know what you're doing... We find $N = \frac{3}{4}mg\sin\theta(3\cos\theta - 2\cos\theta_0)$ where θ_0 is the initial angle when the ladder was at rest.

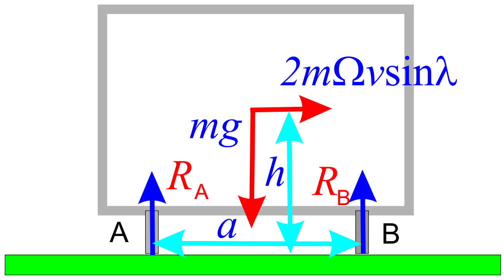


3. The coordinates of the centre of mass are $x = (R-r)\sin\theta$; $y = -(R-r)\cos\theta$. As θ varies we see that the velocity of the centre of mass is $(R-r)\dot{\theta}$. As the cylinder rolls, the velocity of its centre of mass must just be ωr, so $\omega = \dfrac{(R-r)}{r}\dot{\theta}$ as required.
   The kinetic energy is $T = \frac{1}{2}I\omega^2 + \frac{1}{2}mr^2\omega^2 = \frac{3}{4}mr^2\omega^2 = \frac{3}{4}m(R-r)^2\dot{\theta}^2$.
   The potential energy is $V = -mg(R-r)\cos\theta$.
   Using the energy method the EoM is $\ddot{\theta} + \dfrac{2g}{3(R-r)}\sin\theta = 0$.
   The angular frequency of small oscillations is $\omega^2 = 2g/3(R-r)$.



4. The centrifugal force per unit mass is $\omega^2 x$ and the surface satisfies $\tan\theta = \mathrm{d}y/\mathrm{d}x = \omega^2 x/g$. We get $y = \omega^2 x^2/2g$ and using the formula given we get $\omega^2 = g/2f = 1.566$ rad s^{-1}.

5. Taking moments about the left-hand wheel:
   $R_B a = \frac{1}{2}mga + 2hmv\Omega\sin\lambda$
   so that $R_B - \frac{1}{2}mg = 2hmv\Omega\sin\lambda/a$.
   The reaction $R_A = mg - R_B$ so the difference is $4hmv\Omega\sin\lambda/a$. Comparing this to the average reaction $\frac{1}{2}mg$ we get a fractional difference $R_B/R_A \approx 1 + 8hv\Omega\sin\lambda/ag$.



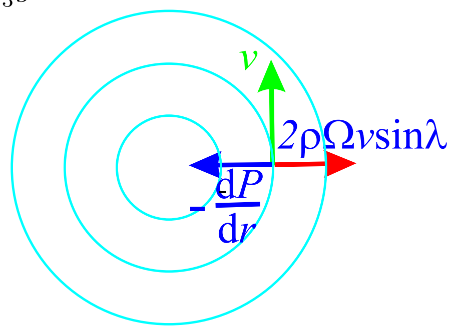
6. **Method 1** In coordinate system used in lecture $v = gt$ and $m\ddot{x} = 2mgt\Omega\cos\lambda$ and so $x = \frac{1}{3}g\Omega t^3$ for a point on the equator. Using $h = \frac{1}{2}gt^2$ we get $x = \frac{1}{3}\Omega\sqrt{8h^3/g} = 0.245455$ m to the East. That's using the sidereal day of 23.9344696 hr, of course.

   Method 2 The stone falls from height H above the Earth's surface at radius R, rotating at angular velocity Ω *wrt* the inertial frame. Its angular momentum is $\Omega(R+H)^2$ and is constant. At height h its angular velocity is $\omega = \Omega(R+H)^2/(R+h)^2$ and its relative velocity *wrt* an observer rotating with the Earth is

$$(\omega - \Omega)(R+h) = \Omega R(1 + H/R)^2(1 + h/R)^{-1} - \Omega(R+h) \approx 2\Omega(H-h)$$

   Using $H - h = \frac{1}{2}gt^2$ we recover the above formula $x = \frac{1}{3}gt^3$.

7. The winds circulate anticlockwise, approximately parallel to the isobars. The force on a unit volume of air due to the pressure gradient $-\mathrm{d}P/\mathrm{d}r$ (inwards) and the Coriolis force $2\rho v\Omega\sin\lambda$ is outwards.



   The equation of motion for a unit volume of air at radius r in a frame rotating with the Earth is

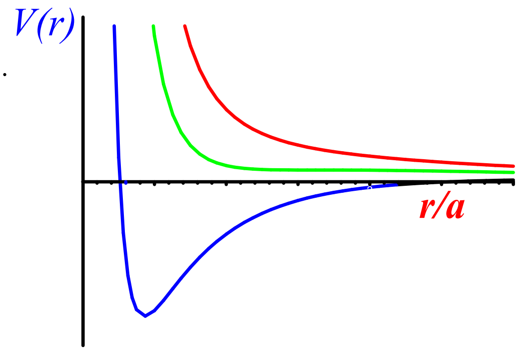
$$\rho\left(\ddot{r} - \frac{v^2}{r}\right) = -\frac{\mathrm{d}P}{\mathrm{d}r} + 2\rho v\Omega\sin\lambda$$

   Setting $\ddot{r} = 0$ we get a quadratic $v^2 + 2v\Omega r\sin\lambda - \frac{r}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = 0$. The data given imply a density for air of 1.16 kg m^{-3} (using a molecular mass of 29 g) and the resulting wind speed is 27 km hr^{-1}.

   You can also formulate the problem in the accelerated frame moving with the air, in which there is a centrifugal term $\rho v^2/r$, confirming the above result.

   It is interesting to note that, if you tried to do the same problem for an **anticyclone**, there are some sign changes and the resulting quadratic has no real solutions. The reason is that cyclones tend to wind themselves up and increase the pressure gradient, whereas anticyclones tend to disperse and decrease the pressure gradient. Hence, no anticyclone can have a pressure gradient as high as that given in the question.

8. The effective potential is $V(r) = -\frac{k}{r}\exp(-r/a) + \frac{J^2}{2mr^2}$.



This is illustrated in the figure opposite for the cases where r/a is less than, equal to and greater than the critical value $r/a = \frac{1}{2}(1+\sqrt{5})$.

The exponential fall-off eventually causes the potential to be insufficiently attractive to permit bound states.

If there is a minimum in the potential it occurs at $r = r_0$ where

$$\left.\frac{\mathrm{d}V}{\mathrm{d}r}\right|_{r_0} = \frac{k}{r^2}\exp(-r/a)\left(1+\frac{r}{a}\right) - \frac{J^2}{mr^3} = 0$$

We now form $\mathrm{d}^2V/\mathrm{d}r^2$ to see if there is a minimum there

$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -\frac{k}{r^3}\exp(-r/a)\left(2+\frac{2r}{a}+\frac{r^2}{a^2}\right) + \frac{3J^2}{mr^4}$$

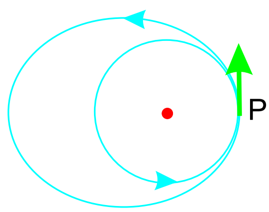
Using the value of J^2 at r_0 above we get

$$\left.\frac{\mathrm{d}^2V}{\mathrm{d}r^2}\right|_{r_0} = \frac{k}{r^2}\exp(-r/a)\left(1+\frac{r}{a}-\frac{r^2}{a^2}\right)$$

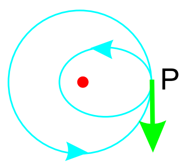
which confirms that the orbit is stable below the critical value of $r/a = \frac{1}{2}(1+\sqrt{5})$.

9. This is a very instructive short question.

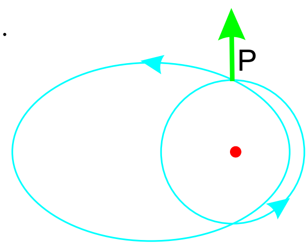
   (a) If the impulse is in the direction of motion, then the velocity is still perpendicular to the radius vector, but the energy is increased. Assuming that the orbit is still bound, the satellite moves to an elliptical orbit with a larger semi-major axis. The only points on the orbit where the velocity is perpendicular to the radius is at the points of furthest and closest approach. Here it is clearly the point of closest approach.



   (b) Again, assuming that the velocity is not reversed, the point of impact must be at the point of furthest approach.



   (c) The impulse is directed away from the central mass, so the angular momentum and hence the semi-latus rectum are unchanged. The only two points at which the orbit crosses the distance of the semi-latus rectum are at $\phi = \pm\pi/2$ as illustrated.



10. This question refers to pulsar formation, and explains their very wide distribution about the Galactic plane compared to the normal stellar population.

    Denote the masses m and αm and the angular velocity of the original circular orbit as ω. Consider the balance of gravitational and centrifugal forces on either star to get the formula

$$\omega^2 = \frac{Gm(1+\alpha)}{r^3}\ . \qquad (0.1)$$

After the explosion, the initial velocities do not change, but the centre of mass of the remaining stars is different, so that they each have a speed $\omega r/2$ *relative to the new centre of mass.* (This is the trickiest step in the problem.) To see if the new orbit is bound, we simply evaluate the total energy

$$E = -\frac{Gm^2}{r} + 2 \times \tfrac{1}{2} m(\omega r/2)^2 = \frac{Gm^2}{4r}(\alpha - 3) \ . \tag{0.2}$$

The orbit is unbound if $\alpha > 3$.

11. The radius and velocity are correct, hence the energy and orbital period are correct too, and the distance a given in the question is indeed the semi-major axis of the orbit. From the polar equation of the ellipse we conclude that $a(1-e^2) = r_0$ and that the maximum and minimum distances are $r_{\substack{\max \\ \min}} = r_0/(1 \mp e) = a(1 \pm e)$. However, the angular momentum is incorrect $J^2 = GMm^2 a \cos^2\theta = GMm^2 r_0 = GMm^2 a(1-e^2)$. Hence we conclude that $e = \sin\theta$ as required.

    The new orbit has a semi-major axis $a' = a(1+e)$, so by Kepler's third law $T' = T(1+e)^{3/2} \approx 1 \text{ day} + 37.6 \text{ s}$.

    The only purpose of putting a satellite in a geostationary orbit is that is has a period of 1 day. Since it now doesn't have this period, it was a big mistake to make the correction...

    Alternative
    Most of these orbit problems can also be done by using the effective potential. Here the effective potential of the orbit is

$$U_{\text{eff}} = -\frac{GMm}{r} + \frac{J^2}{2mr^2} \ . \tag{0.3}$$

    Differentiate this to find the radius of the circular orbit a, and evaluating the angular momentum J_0 and the energy E_0

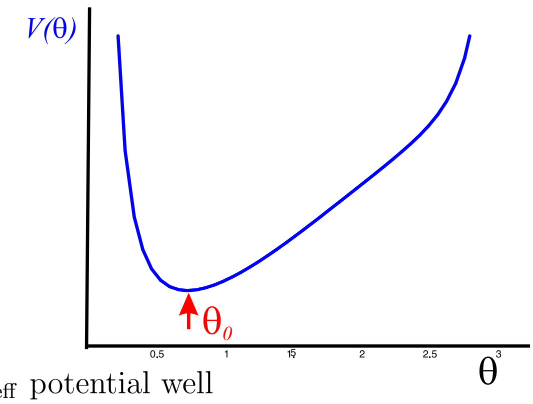
$$J_0^2 = GMm^2 a \ , \quad E_0 = -\frac{GMm}{2a} \ .$$

    Now suppose that, as stated, the velocity v is misaligned by θ, so that the energy is unchanged, but the new angular momentum is $J = J_0 \cos\theta$. We can now solve the equation $U_{\text{eff}} = E_0$ to find the new turning points:

$$E_0 = -\frac{GMm}{r^2} + \frac{J_0^2 \cos^2\theta}{2mr^2} \ .$$

    Rearranging, we get $r^2 - 2ra + a^2\cos^2\theta$, so that, as required, $r = a\,(1 \pm \sin\theta)$.

12. The potential energy V and kinetic energy T are $V(\theta) = mgl(1 - \cos\theta)$, $T = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)$ The angular momentum about the polar axis is $J = ml^2\sin^2\theta\dot{\phi}$ and is conserved.



We can therefore substitute for $\dot{\phi}$ to get the energy equation as required: $E = U_{\text{eff}} + \frac{1}{2}ml^2\dot{\theta}^2$, where $U_{\text{eff}} = mgl(1 - \cos\theta) + \dfrac{J^2}{2ml^2\sin^2\theta}$.

For the circular orbit, we locate the bottom of the U_{eff} potential well

$$\frac{dU_{\text{eff}}}{d\theta} = 0 = mgl\sin\theta - \frac{J^2\cos\theta}{ml^2\sin^3\theta} . \tag{0.4}$$

Solving this for $J^2 = m^2l^3g\sin^4\theta/\cos\theta$ and comparing with our earlier form for J yields the steady angular velocity $\Omega = \dot{\phi}$

$$\Omega^2 = \frac{g}{l\cos\theta} . \tag{0.5}$$

Differentiating again, we find

$$\frac{d^2U_{\text{eff}}}{d\theta^2} = mgl\cos\theta + \frac{J^2}{ml^2\sin^4\theta}\left(\sin^2\theta + 3\cos^2\theta\right) . \tag{0.6}$$

and substituting for J^2 as above yields

$$\frac{d^2U_{\text{eff}}}{d\theta^2} = \frac{mgl}{\cos\theta}\left(1 + 3\cos^2\theta\right) = \Omega^2ml^2\left(1 + 3\cos^2\theta\right) . \tag{0.7}$$

This is the required result, but many students are unsure how to finish it off rigorously. We write a quadratic approximation for the potential

$$U_{\text{eff}}(\theta) = V(\theta_0) + \delta\theta V'(\theta_0) + \tfrac{1}{2}(\delta\theta)^2V''(\theta_0) . \tag{0.8}$$

Now take the energy equation

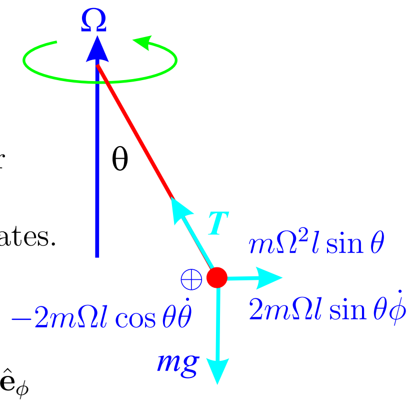
$$E = U_{\text{eff}}(\theta) + \tfrac{1}{2}ml^2\dot{\theta}^2 = \text{constant} \tag{0.9}$$

and differentiate with respect to time

$$0 = \dot{\theta}\left(V''\delta\theta + ml^2\ddot{\theta}\right) , \tag{0.10}$$

which is SHM at the required frequency $\omega^2 = \Omega^2\left(1 + 3\cos^2\theta\right)$.

Optional part



A proper solution of this using Newtonian methods (whether or not you are in the rotating frame) is rather difficult, as it requires the use of the formulae for the velocity and acceleration in spherical polar coordinates. For this case the particle is constrained to the sphere $r = l$, so it simplifies somewhat:

$$\text{velocity}: \quad l\dot\theta\ \hat{\mathbf{e}}_\theta + l\sin\theta\dot\phi\ \hat{\mathbf{e}}_\phi$$

$$\text{acceleration}: \quad -l(\dot\theta^2 + \sin^2\theta\dot\phi^2)\ \hat{\mathbf{e}}_r + l(\ddot\theta - \cos\theta\sin\theta\dot\theta^2)\ \hat{\mathbf{e}}_\theta + l(\sin\theta\ddot\phi + 2\cos\theta\dot\theta\dot\phi)\ \hat{\mathbf{e}}_\phi$$

where we recognise, for example, the first term as the centripetal acceleration.

In the frame rotating at the steady state $\Omega^2 = g/l\cos\theta_0$ we have a centrifugal force $m\Omega^2 l\sin\theta\ \hat{\mathbf{e}}_\rho$, and components of the Coriolis force $2m\Omega l(\sin\theta\dot\phi\ \hat{\mathbf{e}}_\rho - \cos\theta\dot\theta\ \hat{\mathbf{e}}_\phi)$.

The full equations of motion are then

$$\begin{aligned}
-ml(\dot\theta^2 + \sin^2\theta\dot\phi^2) &= -T + mg\cos\theta + ml\Omega^2\sin^2\theta + 2ml\Omega\sin^2\theta\dot\phi \\
ml(\ddot\theta - \cos\theta\sin\theta\dot\theta^2) &= -mg\sin\theta + ml\Omega^2\sin\theta\cos\theta + 2ml\Omega\sin\theta\cos\theta\dot\phi \\
ml(\sin\theta\ddot\phi + 2\cos\theta\dot\theta\dot\phi) &= -2ml\Omega\cos\theta\dot\theta
\end{aligned}$$

Phew! Since we're going to expand to first order about θ_0 we didn't really need to worry about the terms that are second-order in the velocities, and we don't actually need the first equation at all, since it simply tells us what the tension in the string is. . .

We now set $\sin\theta \approx \sin\theta_0 + \cos\theta_0\delta\theta$ and $\cos\theta \approx \cos\theta_0 - \sin\theta_0\delta\theta$ and use $mg = ml\Omega^2\cos\theta_0$ to get the much simplified set of equations

$$\begin{aligned}
ml\ddot\theta &= -ml\Omega^2\sin_0^2\theta_0\delta\theta + 2ml\Omega\sin\theta_0\cos\theta_0\dot\phi \\
ml\sin\theta_0\ddot\phi &= -2ml\Omega\cos\theta_0\dot\theta
\end{aligned}$$

Integrating the second equation to get $\sin\theta_0\dot\phi = -2\Omega\cos\theta_0\delta\theta$[1], and substituting in the first, we recover

$$\ddot\theta = -\Omega^2(1 + 3\cos^2\theta_0)$$

13. The satellite starts off in a circular orbit with $a = \alpha r$ potential energy $-GMm/\alpha r$ and kinetic energy $GMm/2\alpha r$. After the burn it has kinetic energy $GMm/8\alpha r$, making a total of $-7GMm/8\alpha r$, so that the new semi-major axis is $4\alpha r/7$. we want this to be equal to $(1 + \alpha)r/2$, so that the satellite grazes the surface. Hence $\alpha = 7$ and $e = \frac{3}{4}$.

    The alternative burn actually increases the kinetic energy to $5GMm/8\alpha r$, but doesn't affect the angular momentum or the semi-latus rectum, which remains $r_0 = \alpha r$. The new total energy is $-3GMm/8\alpha r$ and the semi-major axis has increased to $4\alpha r/3$, so that $(1 - e^2)a = r_0 \Rightarrow 1 - e^2 = 3/4$, $e = \frac{1}{2}$, $r_{\min} = 2\alpha r/3, r_{\max} = 2\alpha r$ and the satellite does not hit the surface.

[1] We have set a constant of integration to zero here. If it were non-zero, the resulting small oscillations in θ would be offset from θ_0 in the appropriate way.

14. The polar equation of the ellipse $r_0 = r(1 + e\cos\phi)$ is truly marvellous. For $e > 1$ we have hyperbolic orbits and the major axis is now negative, but is still related to the semi-latus rectum $r_0 = a(1 - e^2)$ and the energy ($E = -A/2a$, now positive) in the same way. When you set $\phi > \phi_\infty$ in the formula, r becomes negative, the orbit switches to the repulsive branch, but all the physical results remain valid. At $\phi = \pi$, which is the distance of closest approach c on the repulsive branch, we still get $c = r_0/(1 - e) = a(1 + e)$.

We get e from $\sec\phi_\infty = -e$ and relate this to the scattering $\Phi = 2\phi_\infty - \pi$ (in lectures I defined the scattering angle $\chi = -\Phi = \pi - 2\phi_\infty$ since it is a negative angle, but here Φ is clearly intended to be positive), so $e = \mathrm{cosec}\Phi/2$.

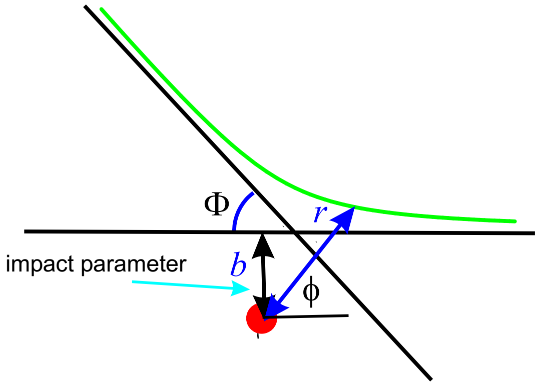
The semi-major axis a is related to the energy in the usual way (see above), so

$$c = \frac{zZe^2}{8\pi\epsilon_0 E}(1 + \mathrm{cosec}\Phi/2)$$

where $Z = 82$ is the charge on the nucleus and $z = 2$ is the charge of the alpha-particle. The distance of closest approach for $\Phi = 60^\circ$ is 1.4×10^{-14} m.

Alternative

Since the use of negative r has a slightly 'Alice through the looking-glass' quality to it, I offer an alternative. This *ab initio* derivation is known as the the 'linear momentum trick'.



The revised geometry is shown in the diagram. Take the initial velocity as v_∞ along the x-direction and the impact parameter as b, so that the angular momentum is $J = mbv_\infty$ and energy is $\frac{1}{2}mv_\infty^2$.

Note that the angular momentum at angle ϕ and radius r is $J = mr^2\dot{\phi}$ The change in x-momentum scattering to angle Φ is

$$mv_\infty(1 - \cos\Phi) = \frac{zZe^2}{4\pi\epsilon_0}\int_0^\Phi \mathrm{d}t\ \frac{\cos\phi}{r^2} = \frac{zZe^2 m}{4\pi\epsilon_0 J}\int_0^\Phi \mathrm{d}\phi\ \cos\phi = \frac{zZe^2 m}{4\pi\epsilon_0 J}\sin\Phi\ .$$

Using $2\sin^2\Phi = (1 - \cos 2\Phi)$ and $2\sin\Phi\cos\Phi = \sin 2\Phi$, we can quickly rearrange this to get

$$b = \frac{zZe^2}{8\pi\epsilon_0 E}\cot\Phi/2\ .$$

The next trick is to note that the velocity v at the radius c of closest approach satisfies

$$cv = bv_\infty\ , \quad \tfrac{1}{2}mv_\infty^2 = \tfrac{1}{2}mv^2 + \frac{zZ}{4\pi\epsilon_0 c}\ .$$

This neatly unwraps to give a quadratic for c

$$c^2 - 2c\frac{zZ}{8\pi\epsilon_0 E} - b^2 = 0 = c^2 - 2cb\tan\Phi/2 - b^2\ ,$$

so that $c/b = \tan\Phi/2 + \sec\Phi/2$ (positive root) and

$$c = \frac{zZ}{8\pi\epsilon_0 E}\,(1 + \mathrm{cosec}\Phi/2)\ .$$

15. The disc has moments of inertia $I_1 = I_2 = \frac{1}{4}ma^2 = 10^{-3}$ kg m^2 and $I_3 = \frac{1}{2}ma^2 = 2 \times 10^{-3}$ kg m^2. It is rotating at 45° to its axis, so let $\boldsymbol{\omega} = (1/\sqrt{2}, 0, 1/\sqrt{2})$ *wrt* the body axes. Using $\boldsymbol{J} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$ we get $\boldsymbol{J} = (1/\sqrt{2}, 0, \sqrt{2}) \times 10^{-3}$ kg m^2 s^{-1} and $T = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{J} = 0.75 \times 10^{-3}$ J.

16. The moments of inertia (*wrt* the CoM) about the x- and y-axes are $I_x = \frac{1}{3}mb^2$ and $I_y = \frac{1}{3}ma^2$ respectively. Suppose that the impulse of the collision is P, the resulting velocity of the centre of mass is v and the final angular velocities are ω_x and ω_y.

    Whilst this problem can be done in a variety of ways, I **strongly advise** that everything is considered *wrt* the CoM, i.e. as an impulse P to the CoM and as angular impulses Pb and Pa about the x- and y-axes.

    The equations for the linear momentum, angular momentum and energy conservation then become

$$\begin{aligned} P &= m(v+u) && (1)\\ Pb &= I_x\omega_x && (2)\\ Pa &= I_y\omega_y && (3)\\ \tfrac{1}{2}mu^2 &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 && (4) \end{aligned}$$

    Eliminating P between (1) and (2) we find $I_x\omega_x = mb(u+v)$ and similarly ω_y is found from (1) and (3): $I_y\omega_y = ma(u+v)$. Substituting into (4) we get

$$m(u^2 - v^2) = \frac{m^2b^2}{I_x}(u+v)^2 + \frac{m^2a^2}{I_y}(u+v)^2 = 6m(u+v)^2$$

    where we have used the formulae above for the moments of inertia.

    Although this is a quadratic, we already know that $v = -u$ is a solution (i.e. $P = 0$) so, cancelling the factor of $u + v$ and rearranging, we see that the other solution is $v = -\frac{5}{7}u$. We also find $\omega_x = \frac{6}{7}u/b$ and $\omega_y = \frac{6}{7}u/a$, so that the instantaneous velocity of the point of impact is $v + b\omega_x + a\omega_y = u$, i.e. it has reversed.

17. The vertical motion is completely decoupled from the rotation and the horizontal motion so that, on bouncing elastically, $u_2 = -u_1$.

    The floor gives a horizontal impulse P (P negative as defined) so that the momentum and angular momentum equations are

$$P = m(v_2 - v_1)\ ; \quad Pa = I(\Omega_2 - \Omega_1) \Rightarrow I(\Omega_2 - \Omega_1) = ma(v_2 - v_1)$$

    The energy equation can be written $\frac{1}{2}I(\Omega_2^2 - \Omega_1^2) = -\frac{1}{2}m(v_2^2 - v_1^2)$ so that $v_2 + a\Omega_2 = -(v_1 + a\Omega_1)$, i.e. the velocity of the point of impact reverses as usual.

    Substituting for Ω_2 we find $v_2(Ma^2/I + 1) = -2\Omega_1 + (ma^2/I - 1)v_1$ and, using $I = \frac{2}{5}ma^2$, we get $v_2 = (3v_1 - 4a\Omega_1)/7, \Omega_2 = -(10v_1/a + 3\Omega_1)/7$.

18. The collision is relatively straightforward again, with $I = \frac{1}{4}ma^2$ and impulse P, velocities u' and v for the mass and the disc afterwards, and angular velocity ω, we get:

$$\begin{aligned} P &= m(u-u') && (1)\\ P &= mv && (2)\\ Pa/2 &= I\omega && (3)\\ \tfrac{1}{2}m(u^2-(u')^2) &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 && (4) \end{aligned}$$

We find that $\omega = mva/2I \Rightarrow I\omega^2 = mv^2$ for this case. Substituting u' from (1) into (4) we find (of course) $v = 0$ or, more interestingly, $v = \frac{2}{3}u, u' = \frac{1}{3}u, \omega = \frac{4}{3}u/a$.

It is now supposed that the disc was rotating at $\frac{2}{3}u/a$. This implies that $J_3 = \frac{1}{3}mau = J_1$, since $I_3 = \frac{1}{2}ma^2$. The total $J = \frac{1}{3}\sqrt{2}mau$, and the space precession frequency is $J/I_1 = \frac{4}{3}\sqrt{2}u/a$. The disc returns to its original spatial orientation in a time $2\pi/\Omega_s$ or in a distance $\pi a/\sqrt{2}$.

19. I give a much more complete answer to this question in the Appendix. The coin has radius a, mass m and moment of inertia $I = \frac{1}{4}ma^2$. Suppose the impulse is P and that the velocities before and after are $-u$ and v as in the previous question. If the angular velocity afterwards is ω, we can write the equations of linear momentum, angular momentum and energy conservation as

$$\begin{aligned} P &= m(v+u) && (1) \\ Pa\cos\theta &= I\omega && (2) \\ \tfrac{1}{2}mu^2 &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 && (3) \end{aligned}$$

where θ is the angle the coin makes to the horizontal Eliminating P between (1) and (2) we find $I\omega = ma(u+v)\cos\theta$, and substituting into (3) we get

$$m(u^2 - v^2) = \frac{m^2a^2\cos^2\theta}{I}(u+v)^2 = 4m(u+v)^2$$

This again has $u + v$ as a factor. For small θ we can set $\cos\theta = 1$ implying $v = -\frac{3}{5}u$ and $\omega = \frac{8}{5}u/a$. This means that the velocity of the point of impact is $v + \omega a = u$ (i.e. it reverses) and that the velocity of the other edge is $v + -\omega a = -\frac{11}{5}u$.

For small angle θ, the other edge is approximately $2a\theta$ above the ground. By the time the edge hits the ground the original point of impact has travelled $\frac{5}{11} \times 2a$ upwards, so that the angle the coin makes is $\frac{5}{11}\theta$.

20. The model for the equatorial bulge has a ring of mass m_r, so that the two moments of inertia are $2mr^2/5 + m_r^2/2$ and $2mr^2/5 + m_r r^2$. The data given then imply that $m_r \approx m/375$.

We now find the torque on the ring. Let an element of mass $m_r d\phi/2\pi$ be at $\boldsymbol{r}$ and the Sun be at $\boldsymbol{R}$, where

$$\boldsymbol{r} = (r\cos\phi, r\sin\phi, 0) \ , \quad \boldsymbol{R} = (R\cos\theta, 0, R\sin\theta) \ . \tag{0.11}$$

The gravitational force on the element is

$$\mathrm{d}\boldsymbol{F} = \frac{GMm_r\mathrm{d}\phi}{2\pi}\ \frac{(R\cos\theta - r\cos\phi, -r\sin\phi, R\sin\theta)}{(R^2 + r^2 - 2Rr\cos\phi\cos\theta)^{3/2}} \ . \tag{0.12}$$

The couple is $\mathrm{d}\boldsymbol{G} = \boldsymbol{r}\times\mathrm{d}\boldsymbol{F}$:

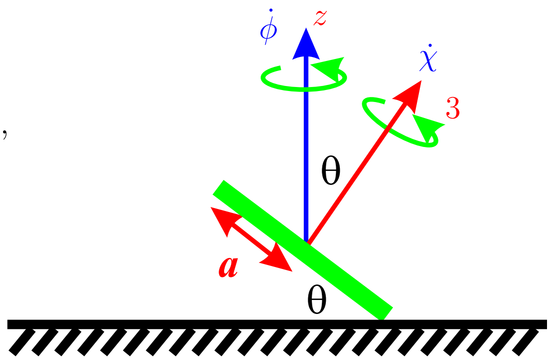
$$\mathrm{d}\boldsymbol{G} = \frac{GMm_r d\phi}{2\pi}\ \frac{(-Rr\sin\theta\sin\phi, -Rr\sin\theta\cos\phi, Rr\cos\theta\sin\phi)}{(R^2 + r^2 - 2Rr\cos\phi\cos\theta)^{3/2}} \ . \tag{0.13}$$

Integrating around the ring, the G_x and G_z components are zero, and the G_y we could probably find in a book if we wanted to. However, here we expand in $r/R \ll 1$

$$
\begin{aligned}
G_y &= -\frac{GMm_\mathrm{r} r \sin\theta}{2\pi R^2}\int_0^{2\pi} \mathrm{d}\phi\ \cos\phi \left(1 - 2r\cos\phi\cos\theta/R + r^2/R^2\right)^{-3/2} \\
&\approx -\frac{GMm_\mathrm{r} r \sin\theta}{2\pi R^2}\int_0^{2\pi} \mathrm{d}\phi\ \cos\phi (1 + 3r\cos\theta\cos\phi/R + \ldots) \qquad (0.14)\\
&= \frac{3GMm_\mathrm{r} r^2}{4R^3}\sin 2\theta = \frac{1}{500}\frac{GMmr^2}{R^3}\sin 2\theta\ .
\end{aligned}
$$

To find the precession rate look down from the pole of ecliptic plane and see the projected angular momentum $J_p = (2mr^2/5 + m_\mathrm{r} r^2)\Omega\sin\theta$. As always with gyroscope problems this gives the precession rate as $\dot\phi = G_y/J_p = 1.6 \times 10^{-4}$ rads yr^{-1}.

21. The diagram shows that the angular velocities along the body axes are $(\omega_1, \omega_2, \omega_3) = (0, \dot\phi\sin\theta, \dot\chi + \dot\phi\cos\theta)$. The condition of zero slip at the base is $a(\dot\chi + \dot\phi\cos\theta) = 0$, so we must have $\omega_3 = 0$.



As usual, look down from above, and see that that the horizontal component of the angular momentum is $J_x = J_2\cos\theta = I_1\dot\phi\sin\theta\cos\theta$ and that it precesses at $\dot\phi$ due to the couple $G = mga\cos\theta$. The precession rate is $\Omega = \dot\phi$ and satisfies the gyroscope formula $J_x\Omega = G$ so that, with $I_1 = \frac{1}{4}ma^2$ we get $\Omega^2 = 4g/a\sin\theta$.

The coin is rolling around a circle of radius $a\cos\theta$ so does not make a complete revolution is every precession cycle, The difference being $1 - \cos\theta \approx \frac{1}{2}\theta^2$ for small angle θ. Using $\sin\theta \approx \theta$, we see that the head of the coin rotates at approximately $\sqrt{g\theta^3/a}$.

Appendix: more on problem 19

Problem 19 is easy enough, but invites the question as to what happens next. The coin is left spinning, but still approaching the table, so another impact is inevitable, even without gravity. Here I give the answer for the number of impacts it makes in the absence of gravity, and also determine its final linear and angular velocities as a function of the initial angle θ.

Coin has mass m and moment of inertia $I = \frac{1}{4}ma^2$. For any one impact let the initial and final velocities and angular velocities be u_1, Ω_1 and u_2, Ω_2, and have vertical impulse P when the coin hits the table. Momentum equation:

$$P = m\left(u_2 - u_1\right)\ .$$

Angular momentum equation:

$$Pa\cos\theta = I\left(\Omega_2 - \Omega_1\right)\ .$$

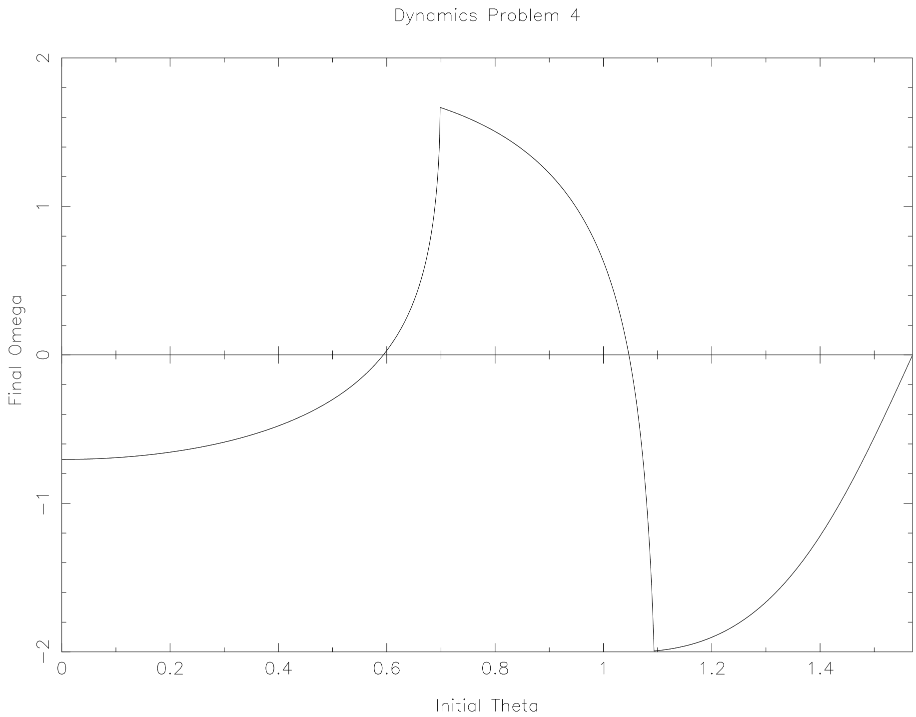


Figure 1: Final Ω of the coin, plotted as a function of the initial θ. There are regions with 1, 2 and 3 impacts Yes, I know it says Problem 4....

Eliminating P gives

$$ma\cos\theta\,(u_2 - u_1) = I\,(\Omega_2 - \Omega_1)\ .$$

The conservation of energy equation can be written in the form

$$m\left(u_2^2 - u_1^2\right) = I\left(\Omega_2^2 - \Omega_1^2\right)\ ,$$

which can be divided by () and rearranged to give

$$u_2 + a\cos\theta\Omega_2 = -\left(u_1 + a\cos\theta\Omega_1\right)\ ,$$

i.e. the vertical velocity of the point of contact reverses, as expected.

Solving for u_2, Ω_2 gives answer for the general case:

$$u_2 = \frac{u_1(ma^2\cos^2\theta - I) + \Omega_1 2aI\cos\theta}{ma^2\cos^2\theta + I};$$

$$\Omega_2 = \frac{u_1 2ma\cos\theta - \Omega_1(ma^2\cos^2\theta - I)}{ma^2\cos^2\theta + I}\ .$$

At this point I resorted to solving the problem numerically, in the absence of gravity, and for arbitrary moment of inertia. The solution of the impact (given above) is easy, it's not hard to work out whether another impact will happen, but finding the time of the next impact (and hence the next θ) does not have a closed form solution. I used a binary chop, of course. The plot shows the final Ω as a function of the initial θ.

There are regions with 1 , 2 and 3 impacts, and it's particularly interesting to see how fast it spins at the transition between 1 and 2 impacts. The general pattern is as follows: for θ

near $\pi/2$, the coin will only make 1 impact, but as the angle is reduced the linear velocity after the impact decreases and the angular velocity increases. Eventually, the other side of the coin will make another impact with the table. For $I = \frac{1}{4}ma^2$ the limiting case $\theta = 0$ has 3 impacts, and the coin is left moving away from the table with a non-zero angular velocity.