Part IB Physics B 2009
Answers to Classical Dynamics and Fluids Examples 2

These worked answers to the Dynamics problems are offered to students and supervisors as a guide, and I hope they contain some generally useful hints. In some cases, I have indulged in a little research that goes beyond the stated problem.

Health Warning

These solutions must be used with caution.

- The worked answers will be useless to you unless you have already made a very serious attempt to solve the problem yourself and have discussed it with your supervisor. If you consult the solutions earlier, they will just act as 'spoilers'.
- You must remember that there is often more than one way to solve a physics problem. I have sometimes suggested alternatives, but your supervisors will know many others.
- The most difficult part of a physics problem is knowing where to start. It is therefore quite possible that I taken too much for granted at the beginning of some problems. If so, please let me know.
- I have often given the numerical answers to several more decimal places than is really justified, so that you can check your calculations (and mine) carefully. I have used values of the physical constants taken from the formula book that you will use in the Examination.

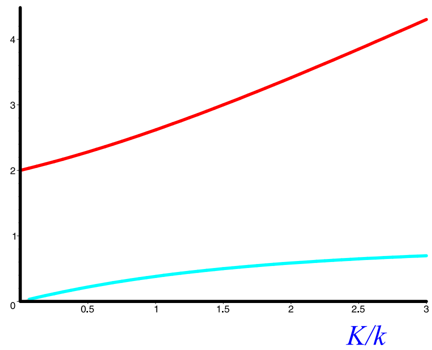
Please let me me know about any errors, typos and suggestions for improvement.

Steve Gull
`email: steve@mrao.cam.ac.uk`

`http://www.mrao.cam.ac.uk/~steve/part1bdyn`

1. The kinetic energy is $\frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$ so the mass matrix is $\underline{\underline{M}} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$. The potential energy is $\frac{1}{2}(Kx_1^2 + k(x_1 - x_2)^2)$ so $\underline{\underline{K}} = \begin{pmatrix} K+k & -k \\ -k & k \end{pmatrix}$. The determinant of $\underline{\underline{K}} - \omega^2\underline{\underline{M}}$ must vanish for non-trivial solutions

$$\det\begin{pmatrix} K+k-m\omega^2 & -k \\ -k & k-\frac{1}{3}m\omega^2 \end{pmatrix} = 0 \Rightarrow kK - (2k+K)m\omega^2 + m^2\omega^4 = 0$$



The solution is $m\omega^2 = k + \frac{1}{2}K \pm \sqrt{k^2 + \frac{1}{4}K^2}$, which is plotted as a function of K/k. For small K the modes approach free translation ($\omega^2 = 0$) and symmetric vibration at $\omega^2 = 2k/m$. For large K the two modes approach $\omega^2 = k/m$ (mass near the wall stationary) and $\omega^2 = K/m$ (outer mass stationary).

2. The centre of mass of the rod is at

$$(x, y) = (a\sin\theta_1 + a\sin\theta_2/2, a(1-\cos\theta_1) + a(1-\cos\theta_2)/2)$$

relative to its equilibrium position, so that its kinetic energy is (not approximated yet)

$$T = \tfrac{1}{2}ma^2(\dot{\theta}_1^2 + \tfrac{1}{4}\dot{\theta}_2^2 + \cos(\theta_2 - \theta_2)\dot{\theta}_1\dot{\theta}_2) + \tfrac{1}{2}I\dot{\theta}_2^2$$

Using $I = \frac{1}{12}ma^2$ and the approximations $\cos(\theta_2 - \theta_2) \approx 1$ and $1 - \cos\theta \approx \frac{1}{2}\theta^2$ we assemble the 'mass matrix' $\underline{\underline{M}} = ma^2\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$. The potential energy is mgy, and in the same approximation we get $\underline{\underline{K}} = mga\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

The determinant of $\underline{\underline{K}} - \omega^2\underline{\underline{M}}$ must vanish for non-trivial solutions

$$\det\begin{pmatrix} mga - ma^2\omega^2 & -\frac{1}{2}ma^2\omega^2 \\ -\frac{1}{2}ma^2\omega^2 & \frac{1}{2}mga - \frac{1}{3}ma^2\omega^2 \end{pmatrix} = 0 \Rightarrow \tfrac{1}{2}m^2g^2a^2 - \tfrac{5}{6}m^2ga^3\omega^2 + \tfrac{1}{12}m^2a^4\omega^4 = 0$$

The solution is $\omega^2 = (5 \pm \sqrt{19})g/a$.

3. The normal modes of the system are obvious because of the symmetry: normalised modes are

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \underset{\text{free rotation}}{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \quad \underset{\text{slow vibration}}{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \quad \underset{\text{fast vibration}}{\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$$

The rotation mode has $\omega^2 = 0$ (it is still rotating, of course, but not oscillating). The restoring couple per unit displacement for the 'fast' mode is 3 times that of the 'slow' mode, so using the general rule ω^2= (restoring force per unit displacement)/mass, the frequencies are in the ratio $\sqrt{3} : 1$, i.e. if the 'slow' mode is Ω the 'fast' mode is $\sqrt{3}\Omega$.

The problem is a model for a bird strike on the front of the jet engine, so that the angular velocity change is $\begin{pmatrix} \Delta\omega \\ 0 \\ 0 \end{pmatrix}$. This can be expressed in terms of the normal modes (it's easy here as we have normalised the modes — the coefficients are just the dot products with the velocity change) as

$$\begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{pmatrix} = \Delta\omega \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \Delta\omega \left(\frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right)$$

We have found the initial change in the angular velocities, but we are asked about the displacements $(\theta_1\, \theta_2, \theta_3)$, so we have to integrate with respect to time. The time dependence of the zero frequency mode is $\propto t$ and that of the vibrational modes $\propto \dfrac{1}{i\Omega}e^{i\Omega t}$ and $\frac{1}{i\sqrt{3}\Omega}e^{i\sqrt{3}\Omega t}$. The general solution is real form is

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{pmatrix} = \Delta\omega \left(\frac{t}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{\sin\Omega t}{2\Omega}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{\sin\sqrt{3}\Omega t}{6\sqrt{3}\Omega}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right)$$

The zero frequency mode does not contribute to the relative displacement of the first two discs, and the 'fast' mode contributes 3 times as much as the 'slow' mode, so the displacement is thus

$$\Delta\theta(t) = \frac{\Delta\omega}{2\Omega}\left(\sin\Omega t + \frac{1}{\sqrt{3}}\sin\sqrt{3}\Omega t \right)$$

and the maximum possible displacement is $\dfrac{\Delta\omega}{2\Omega}\left(1 + \dfrac{1}{\sqrt{3}}\right)$.

4. (a) The normal modes are obvious: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with $\omega^2 = k/m$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with $\omega^2 = (k+2K)/m$. The normal modes don't actually vary with K/k at all and I think I'll rephrase the question next year.
(b) The normal frequencies are found from det $\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - M\omega^2 \end{pmatrix} = 0$, i.e. $\omega^2 = \dfrac{k}{mM}\left(m + M \pm \sqrt{m^2 - mM + M^2}\right)$.
The normal mode shapes are

$$\frac{X_2}{X_1} = \frac{\left(M - m \mp \sqrt{m^2 - mM + M^2}\right)}{M}$$

As the ratio M/nm gets large the high frequency mode approaches $\omega^2 = 2k/m$, with the large mass stationary. The low frequency mode approaches $\omega^2 = 3k/2M$, with the small mass oscillating with half the amplitude of the large one.
(c) If $m = M$ the inverse of $\underline{\underline{K}} - \omega^2\underline{\underline{K}}$ is

$$\frac{1}{(k - m\omega^2)(3k - m\omega^2)}\begin{pmatrix} 2k & k \\ k & 2k \end{pmatrix}$$

5. This is a rather instructive normal modes/wave problem.

   The velocity on the first part of the string is $\sqrt{T/4\rho}$ so is half that of the second part of the string and the k-vector is twice the magnitude. The boundary conditions imply that we can write the spatial part of the solution oscillating as $\exp(-i\omega t)$ in the form

$$y = A\sin 2kx \quad (0 < x < l)\ ; \quad y = B\cos k(2l - x) \quad (l < x < 2l)$$

   where $\omega/k = \sqrt{T/\rho}$. The second mode is obvious, since it is a half wavelength in the first part of the string and a quarter wavelength in the second, clearly satisfying the boundary conditions for $k = \pi/2l$. Doing the matching properly we find

$$A\sin 2kl = B\cos kl \qquad (1); \quad 2kA\cos 2kl = kB\sin kl \qquad (2)$$

   Dividing (1) by (2) then implies $\tan 2kl = 2\cot kl$. The formula $\tan 2x = 2\tan x/(1 - \tan^2 x)$ then implies that $\tan kl = \pm 1/\sqrt{2}$. The lowest mode is $kl = 0.615$.

   You might like to ponder how the other mode comes out! Think about (1) and (2) as set of homogenous equations; we really should be solving

$$\sin 2kl\sin kl - 2\cos 2kl\cos kl = 0$$

   (vanishing determinant) and finding the eigenmodes afterwards. This equation does have $k = \pi/2$ as a solution. So, be careful when dividing equations in case they are zero on both sides!

6. Taking the position of the trolley x and the angle of the string θ as generalised coordinates, the x-position of the mass m is $x + l\theta$ for small θ. In the small angle approximation we find $T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} + l\dot{\theta})^2$ and $U = \frac{1}{2}mgl^2\theta^2$. We then have $\underline{\underline{M}} = \begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}$ and $\underline{\underline{K}} = \begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}$ (note that the dimensions of the terms in the matrix are not all the same — our generalised coordinates do not all have the same dimensions). We find $\omega^2 = 0$ or $\omega^2 = (M+m)g/ml$.

   The normal modes are translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and oscillation about centre of mass $\begin{pmatrix} 1 \\ -\frac{M}{ml} \end{pmatrix}$.

   Using Lagrangians, you get the same results...

7. The balance of forces on the sides of the tube (pressure P, radius a and thickness t) are

$$2t\tau_\phi = 2aP \Rightarrow \tau_\phi = \frac{Pa}{t}$$

   On the ends we have $\pi a^2 P = 2\pi a t\tau_z \Rightarrow \tau_z = \frac{Pa}{2t}$. The radial stress is approximately P and is negligible if $t \ll a$. Writing the elastic equation for e_z:

$$Ee_z = \tau_z - \sigma\tau_\phi = \frac{P(1-2\sigma)a}{2t} \Rightarrow B = \frac{E}{3(1-2\sigma)} = \frac{Pa}{6te_z}$$

   Using the values provided we get $B = 1.5 \times 10^{11}$ Pa. We can only determine the bulk modulus unambiguously. We would need to know σ to determine E.

8. When the column is vertical the stress $\tau_z = -\rho g(H-z)$, and $\tau_x, \tau_y \approx 0$. If the extension is $Z(x)$ we have

$$Ee_z = E\mathrm{d}Z/\mathrm{d}z = -\rho g(H-z) \Rightarrow Z(z) = -\rho g(Hz - \tfrac{1}{2}z^2)$$

so that $h_1 = \frac{1}{2}\rho g H^2/E$.

When the valley is flooded we have $\tau_x, = \tau_y = -\rho_\mathrm{w} g(H-z)$ so that

$$Ee_z = -\rho g(H-z) + 2\sigma\rho_\mathrm{w} g(H-z) \Rightarrow Z(z) = -g(Hz - \tfrac{1}{2}z^2)(\rho - 2\sigma\rho_\mathrm{w})$$

We therefore have $h_2 = \frac{1}{2}gH^2(\rho - 2\sigma\rho_\mathrm{w})/E$.

When $\rho = \rho_\mathrm{w}$, we have $e_x = e_y = e_z$ and we recover the argument given in lectures that $B = E/3(1-2\sigma)$.

9. When the weight W is placed at the end of the beam the stress along the bar is constant and equal to W. The bending moment is zero at the free end and rises linearly so $B(x) = W(L-x)$, where L is the length. Using $B = EIy''$ we get $IEy' = W(Lx - \frac{1}{2}x^2)$ (the integration constant is zero as $y'(0) = 0$) and $IEy(x) = W(\frac{1}{2}Lx^2 - \frac{1}{6}x^3)$ (constant of integration is again zero). The deflection at the end is $WL^3/3IE$, and at the mid-point we find $y = 5WL^3/48IE$.

   When the load is applied half-way along the stress and bending moment are zero for $L/2 < x < L$, and $S = W, B = W(L/2 - x)$ for $0 < x < L/2$. A similar integration yields for $0 < x < L/2$:

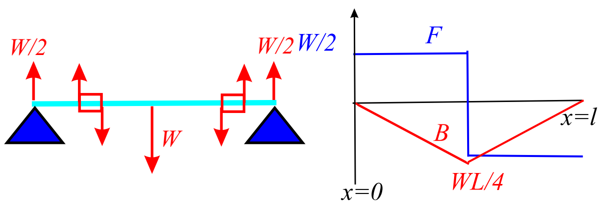
$$y'(x) = \frac{W}{2IE}\left(Lx - x^2\right) \;\; ; \quad y(x) = \frac{W}{2IE}\left(\tfrac{1}{2}Lx^2 - \tfrac{1}{3}x^3\right)$$

   so that, at $x = L/2$, $y' = WL^2/8IE$ and $y = WL^3/24IE$. The beam is straight for $L/2 < x < L$, so continues downwards by a further $WL^3/16IE$ due to the slope at $x = L/2$. The total is indeed $y(L) = 5WL^3/48IE$.

   The reciprocity theorem proved in lectures shows that this result is true for any two points, on a beam with arbitrary cross-section.

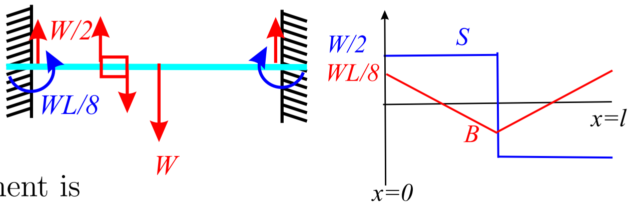
10. The moment of area is $\propto$ side^2, so is in the ratio of $4:1$ for this beam. The diagonal is vertical, so the gravitational force resolves $2:1$ in favour of the stiffer axis. In the diagram I will do shortly $\theta = \tan^{-1}(1/2)$. The deflections will be in the ratio $1:2$, i.e. the weaker axis deflects more, i.e. at $\tan^{-1} 2$ to the stiff axis. With respect to the vertical the angle is $\tan^{-1} 2 - \tan^{-1}(1/2) = 36°.87$.

11. When doing problems about bent beams, a diagram of the forces and bending moments really does help.



    For the freely-supported beam the bending moment is zero at the ends so $B = -Wx/2$ for $x < L/2$. From $EIy'' = B$ we get $y'(x) = \frac{W}{4EI}\left(\frac{1}{4}L^2 - x^2\right)$ since $y'(L/2) = 0$ by symmetry. Integrating, we have $y(x) = \frac{W}{4EI}\left(\frac{1}{4}L^2x - \frac{1}{3}x^3\right)$ and the maximum deflection is $= \dfrac{W}{48EI}$ at $x = \frac{1}{2}L$. A beam overloaded like this will break in the middle where the bending moment is greatest.

The beam cantilevered at both ends must, by symmetry, have a point of inflection at $x = \frac{1}{4}L$, so the couple required to set $y'(0) = 0$ must be $\frac{1}{8}WL$ (you can get that from $y'(\frac{1}{2}L) = 0$ as well). The bending moment is



$B = WL(\frac{1}{8} - \frac{1}{2}x)$, so $y(x) = \frac{W}{8EI}\left(\frac{1}{2}Lx^2 - \frac{1}{3}x^3\right)$. The maximum deflection is $= \dfrac{W}{96EI}$ at $x = \frac{1}{2}L$.The bending moment is equal and opposite at the ends and the middle, so there are three places an overloaded beam could break.

We've done the last part already: it's $\dfrac{5W}{48EI}$. The overloaded beam will fail at the support.

12. The angle of shear for the straight configuration is $\theta r/l$, so the element ar radius r contributes a couple $\mathrm{d}C = 2\pi r \mathrm{d}r r^2 G\theta/l$. Integrated over the radius a, the total is $C = \pi G\theta a^4/2l$. The stored energy is the integral of $C(\theta)\mathrm{d}\theta$ from $\theta = 0$ to its final value $\pi G a^4 \theta_\mathrm{m}/2l = C_\mathrm{m}$, i.e.

$$U_1 = \frac{\pi G a^4 \theta_\mathrm{m}^2}{4l} = \frac{C_\mathrm{m}^2 l}{\pi a^4 G}$$

In the coiled configuration[1], we just have a bent rod, with radius of curvature given by $R\theta = l$. The bending moment is $B = EI/R$, where the moment of area $I = \frac{1}{4}(\pi a^2)a^2$ as we saw before for the moment of inertia of a disc about an axis in its plane. The bending moment is $B = \pi E a^4 \theta/4l$ and the stored energy

$$U_2 = \frac{\pi E a^4 \theta_\mathrm{m}^2}{8l} = \frac{2B_\mathrm{m}^2 l}{\pi a^4 E}$$

For the same final torque $B_\mathrm{m} = C_\mathrm{m}$, the energies $U_1 : U_2$ are in the ratio $1/G : 2/E$. Since $G = E/2(1+\sigma)$, the coiled configuration will have the lower energy if $\sigma > 0$.

13. This is a tricky problem that requires some insight and algebraic persistence. The first thing to realise is that the strains (e_r, e_θ, e_ϕ) are given in terms of the radial distortion $r \to r + R(r)$ as

$$e_r = \frac{\mathrm{d}R}{\mathrm{d}r}\ ; \quad e_\theta = e_\phi = \frac{R}{r}$$

The relation to the stresses τ_r and $\tau_\theta = \tau_\phi$ is then

$$E\frac{\mathrm{d}R}{\mathrm{d}r} = \tau_r - 2\sigma\tau_\theta \qquad (1); \quad E\frac{R}{r} = -\sigma\tau_r + (1-\sigma)\tau_\theta \qquad (2)$$

We also need the equilibrium conditions. The downwards force due to hoop stresses on the element at $r \to r + \mathrm{d}r$ is $2\pi r \mathrm{d}r \tau_\theta$. The balancing forces upwards due to the pressure $(P = -\tau_r)$ are $\mathrm{d}(\pi r^2 \tau_r)$. Hence the final equation

$$\tau_\theta = \tau_r + \tfrac{1}{2} r \frac{\mathrm{d}\tau_r}{\mathrm{d}r} \qquad (3)$$

[1] The most difficult part of this question is the visualisation of the coiled configuration. If you've ever (foolishly) taken a coiled hosepipe with N turns off its reel without uncoiling it first, you will probably remember that it then has N full twists in it, which must be relieved before you can use it properly.

I'm getting old and I don't find the resulting substitutions as easy as I used to, but my facility with computer algebra currently more than makes up for that. Hence, multiplying (2) by r and differentiating, and setting it equal to (1), then substituting for τ_θ from (3), we get

$$r\frac{\mathrm{d}^2\tau_r}{\mathrm{d}r^2} + 4\frac{\mathrm{d}\tau_r}{\mathrm{d}r} = 0$$

as required. The solution is $\tau_r \propto r^{-3}$ (the other solution is τ_r constant, of course). We set $P = -\tau_r = P_0 a^3/r^3$

Substituting back into the equations for the strain we find $e_r = -2e_\theta = -A/r^3$, so that the strain is traceless (there is no volume strain (see below)). Similarly the stresses are traceless too ($\tau_r = -2\tau_\theta$), with the medium compressed in the radial direction and stretched in the other two directions (i.e. we have a pure shear). Substituting into (2) we find $ER/r = -\frac{1}{2}\tau_r(1+\sigma) \Rightarrow R = P_0 a^3/4Gr^2$. The displacement at $r = a$ is $R(a) = P_0 a/4G$, so that the volume increase is approximately $4\pi a^2 R(a) = \pi a^3 P_0/G = 3P_0 V/G$.

Now the quick way. If there was any curl present in the distortion field $\boldsymbol{R}(\boldsymbol{r})$, it would propagate to infinity as a shear wave. If there was any divergence, it would propagate to infinity as a compressional wave. The distortion field can be written as $\boldsymbol{R} = \nabla\Phi(r)$, with $\nabla^2\Phi = 0$. The solution is $\Phi \propto r^{-1}$, $R(r) \propto r^{-2}$ and $\tau_r(r) \propto r^{-3}$. We still need equations (1) and (2), but the difficult substitution is short-circuited.

14. The jet flow obeys incompressible Bernoulli, so $P + \frac{1}{2}\rho v^2 = P_0 + \frac{1}{2}\rho v_0^2$, where v_0 is the jet speed and P_0 the external pressure. where the jets meet in the middle there is a stagnation point, where the pressure is highest $P = P_0 + \frac{1}{2}\rho v_0^2$. The region of higher pressure forces the jets to emerge as a disc. At a couple of jet radii the pressure has again fallen to P_0, so the flow velocity in the disc must be v_0.

    The volume flow rate in the disc at radius r and thickness d is $Q = 2\pi r d v_0$. Conservation of volume then means that the quantity Q is constant with radius and equal to the flow rate coming in from the jets, i.e. $Q = 2 \times \pi a^2 v_0$. Hence $d = a^2/r$.

    The second part is similar: we appeal to Bernoulli to show that the final velocities on each sides is v_0. Considerations of symmetry tell us that there must be two jets in the plane $z = 0$, travelling in the $\pm x$-directions. Since they have the same speed, their thicknesses must be different: conservation of volume means that the sum of the thicknesses is $2a$. The overall momentum flow rate $2a\sin\alpha$ must also be conserved, hence the thicknesses are $a(1 \pm \sin\alpha)$.

15. The pressure is the same at the outflow and at the top of the bath. Applying Bernoulli to the stream line gets the outflow velocity $v = \sqrt{2gh}$ and volume rate $Q = vA_\mathrm{p}\epsilon = -\dot{h}A_\mathrm{b}$, where ϵ is the efflux coefficient, which is in the range 0.5 to 1. Integrating, we get

$$t = \sqrt{\frac{2h_0}{g}}\frac{A_\mathrm{b}}{\epsilon A_\mathrm{p}}$$

    With an efflux coefficient of 0.62, it takes a bit over 10 minutes to drain the bath.

16. Use an image source at $z = -d$ to get the correct boundary conditions at the plate. The velocity potential is

$$\Phi = -\frac{Q}{4\pi}\left(\frac{1}{(x^2+y^2+(z-d)^2)^{1/2}} + \frac{1}{(x^2+y^2+(z+d)^2)^{1/2}}\right)$$

The velocity at the plane $z = 0$ is $\dfrac{Q}{2\pi}\dfrac{(x, y, 0)}{(x^2 + y^2 + d^2)^{3/2}}$, i.e. in the plane and radially outwards. Setting $r = \sqrt{x^2 + y^2}$, we get from Bernoulli $P(r) = P_0 - \dfrac{\rho Q^2 r^2}{8\pi^2(r^2 + d^2)^{3/2}}$. Integrating the pressure defect over the plane $z = 0$ gives the force

$$F = \frac{\rho Q^2}{8\pi^2}\int_0^\infty \mathrm{d}r\ 2\pi r\ \frac{r^2}{(r^2 + d^2)^3}$$

The integral can be done either with the substitution $y = r^2$ or $r = d\tan\theta$ or Maple. In either case $\int_0^\infty \mathrm{d}r\ r^3((r^2 + d^2))^{-3} = 1/4d^2$ and $F = \frac{\rho Q^2}{16\pi d^2}$. There is a pressure defect at the plane $z = 0$, and no momentum is being transferred across the surface, so this is the attractive force between sources or sinks[2]

The case of a source + sink is more interesting. There is still a pressure defect at the surface $z = 0$, of the same amount, but this time there is a momentum transfer per unit time equal to twice the attractive force. The net effect is a repulsion $F = \frac{\rho Q^2}{4\pi d^2}$. This case also appeals to generalised Bernoulli to say $P + \frac{1}{2}\rho v^2$ is constant in all space.

[It might seem to be cheating to do the momentum sums on the $z = 0$ plane. Of course, one can extend the surface to be a large closed surface without changing the answer, but one still worries that we might not have calculated the actual force felt at the source or sink. However, all is well — consider the momentum change in volume V due to pressure forces and momentum transport

$$\int \mathrm{d}V\ \rho\dot{\boldsymbol{v}} = -\oint \mathrm{d}\boldsymbol{S}P - \oint \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{v}\ \rho\boldsymbol{v}$$

Conversion of the RHS to a volume integral and using $\nabla\cdot\boldsymbol{v} = 0$ shows that we can convert the surface to one immediately enclosing the source or sink.]

17. The flow velocity on the surface of the sphere is $\frac{3}{2}v_0\sin\theta$. Bernoulli gives

$$P_0 + \tfrac{1}{2}\rho v_0^2 = P(\theta) + \tfrac{9}{8}\rho v_0^2\sin^2\theta$$

The pressure at the stagnation points $\theta = 0, \pi$ is higher than that at infinity $P(0, \pi) = P_0 + \frac{1}{2}\rho v_0^2$, whereas the pressure at $\theta = \pi/2$ is lower: $P(\pi/2) = P_0 - \frac{5}{8}\rho v_0^2$. Cavitation might occur if the minimum pressure around the sphere is negative ($\frac{5}{8}\rho v_0^2 > P_0$), which evaluates to $v_0 = 12.6$ m s^{-1}.

18. The solutions of $\nabla^2\Phi = 0$ with the correct symmetry are $r\cos\phi$ and $r^{-1}\cos\phi$. For $r < a$ we set $\Phi = vr\cos\theta$, and for $r > a$ we also have a dipole term:

$$\Phi = v_0 r\cos\phi + \frac{A\cos\phi}{r}$$

The boundary condition at $r = a$ is that the **normal flux** is continuous on both sides, so that the flow must satisfy $v_r^- = 2v_r^+$, since the depth on the sandbank is half that

[2] As a way of deriving the force between sources and sinks, this is an interesting, but overelaborate method. If a source Q_2 is at distance d from a source Q_1, the velocity at the source Q_1 due to Q_2 is $v_{12} = Q_2/4\pi d^2$. The source Q_1 has to provide the momentum of its outflow at a rate $\rho Q_1 v_{12} = \rho Q_1 Q_2/4\pi d^2$ which, by Newton's third law, is the force felt by the source Q_1.

outside. The velocity potential also has to be continuous, which is also the condition that the transverse velocity is continuous:

$$v_0 a + \frac{A}{a} = va \ ; \quad v_0 - \frac{A}{a^2} = \frac{v}{2}$$

Adding theses equations we find $v = \frac{4}{3}v_0$.

19. **Method 1**: balance viscous forces on element between r and $r + \mathrm{d}r$ with total force on unit length of pipe $2\pi r \mathrm{d}r \Delta P/d$. The stress is $\tau_{rz} = \eta \mathrm{d}v_z/\mathrm{d}r$ and the force $2\pi r\eta \mathrm{d}v_x/\mathrm{d}r$. The net force is $\mathrm{d}(\eta \mathrm{d}v_z/\mathrm{d}r)$. Dividing by $\mathrm{d}r$ thus get

$$\eta \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v_z}{\mathrm{d}r}\right) = -\frac{r\Delta P}{d}$$

Method 2: Use the Navier-Stokes equation $0 = -\nabla P + \eta\nabla^2 \boldsymbol{v}$. This is easy for this case because the flow is in the same direction everywhere. We just have to remember the expression for ∇^2 in cylindrical polars (it is in the book of mathematical formulae you get in the Exam). Clearly, we get the same equation.

So, in either case, we integrate

$$r\frac{\mathrm{d}v_z}{\mathrm{d}r} = -\frac{r^2\Delta P}{2\eta d} + A$$

The constant of integration A must be zero, since the stress at $r = 0$ is not infinite, and is actually zero. Dividing by r and integrating again we get the parabolic Poiseuille profile

$$v_z = -\frac{r^2\Delta P}{4\eta d} + B$$

The velocity has to vanish at the wall, so $v_z = \Delta P(a^2 - r^2)/(4\eta d)$. The flow rate is $Q = \int_0^a \mathrm{d}r 2\pi r v_z = \pi\Delta P a^4/(8\eta d)$.

The stress at the wall is $\tau_{rz} = -\eta\Delta P a/(2\eta d)$. The total force on the inside of the pipe is $\pi\Delta P a^2$, equal to the applied force on the ends.

If the flow becomes turbulent, this profile only applies in the boundary layer near $r = a$. The flow in the centre will be slower, with an essentially flat velocity profile. The velocity approaches $\mathrm{d}P/\mathrm{d}z = f\rho v^2/2a$, with $f \approx 0.03$, depending on the roughness of the pipe. We will see this demonstrated in a film.

[You can get into lots of trouble with $\nabla^2\boldsymbol{v}$ in polar coordinates — the problem is not ∇^2, which is a nice, scalar differential operator. The problem is with $\boldsymbol{v}$ — it is a vector. You must think of it as $\boldsymbol{v} = v_r\hat{\boldsymbol{u}}_r + v_\theta\hat{\boldsymbol{u}}_\theta + v_z\hat{\boldsymbol{u}}_z$, not just as components (v_r, v_θ, v_z). The ∇^2 operator has to act on the unit vectors $\hat{\boldsymbol{u}}_r, \hat{\boldsymbol{u}}_\theta$, which vary in space, as well as on the components. Here it doesn't matter, because the only component of the velocity is in the direction of $\hat{\boldsymbol{u}}_z$, which does not vary. Some people (I am not one of them) think it's easier to use $\nabla^2\boldsymbol{v} = \nabla\times(\nabla\times\boldsymbol{v}) - \nabla\nabla\cdot\boldsymbol{v}$, for which there are formulae in the book. (In fact $\nabla^2\hat{\boldsymbol{u}}_r = -\hat{\boldsymbol{u}}_r/r^2, \nabla^2\hat{\boldsymbol{u}}_\theta = -\hat{\boldsymbol{u}}_\theta/r^2$).]

20. This is a very important and interesting problem best done in an Excel spreadsheet. The drag force on a sphere is $F = \rho d^2v^2 f(R) = \pi d^3(\rho_\mathrm{S} - \rho_\mathrm{L})g/6$ where the Reynolds number is $R = \rho dv/\eta$. Given the data, we can add some columns of R and $f(R)$.

Indeed, the data are entirely consistent with a single $f(R)$ that is essentially constant for high Reynolds number and $\propto 1/R$ when viscosity is important.

To find the velocity of the hailstone we can pretend we know the velocity, and calculate the drag coefficient and Reynolds number as above. This is not the same curve, of course, but it crosses the experimental one somewhere (at about $R = 700$), that's the self-consistent value to use for the hailstone. We find $v \approx 6$ m s^{-1}. The point is also quite close to the value where the drag coefficient levels out to about $f = 0.36$, so could we use that instead.