Part IB Physics B

Classical Dynamics — Examples 2 — 2010

Problem grading:

(A) Problems that can be answered directly by quoting the lectured material or by straightforward calculation.

(B) Problems that require some algebraic formulation and manipulation as well as calculation.

(C) Problems which are either harder or longer than (B) problems. You should feel a sense of achievement in completing these.

Normal Modes

1. (B) When a diatomic molecule is adsorbed onto the surface of a metal the frequency of its internal vibrational mode is changed. If we consider a horizontal surface with the axis of the molecule vertical, a simple model which might describe this phenomenon is as follows. The molecule consists of two point masses, m, separated by a light spring of spring constant k. The mass nearer to the metal is attached to a fixed point (the metal surface) with a light spring of constant K. The two springs are collinear and the motion of the masses is regarded as confined to the line of the springs. Find the normal modes and how their frequencies vary with the ratio of K to k. Sketch the results and comment on their physical significance for large and small K.

   [ANS: $m\omega^2/k = 1 + \frac{1}{2}K/k \pm \sqrt{1 + \frac{1}{4}K^2/k^2}$.]

2. (B) A uniform rod of length a hangs vertically on the end of an inelastic string of length a, the string being attached to the upper end of the rod. What are the frequencies of the normal modes of oscillation in a vertical plane?

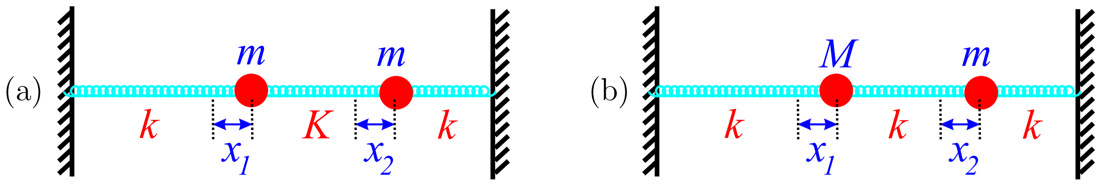
   [ANS: $\omega^2 = (5 \pm \sqrt{19})g/a$.]

3. (C) A simple model of a jet engine comprises three identical thin rigid discs mounted equidistant on a uniform light shaft. Describe the normal modes of oscillation of the system.

   A small object entering the engine produces an abrupt change $\varDelta\varOmega$ in the angular velocity ω of the first disk. Obtain an expression for the maximum resultant angle of twist of the shaft between the discs, given that the angular frequency of the lowest vibrational normal mode is $\varOmega$.

   [ANS: $\leq (1 + 1/\sqrt{3})\varDelta\varOmega/2\varOmega$ - an upper bound from the difference between two non-harmonically related sine waves.]

4. (B)



   (a) Find the frequencies of the normal modes of the two-mass system shown in (a) above, and sketch how they vary with K/k, the ratio of the spring constant of the centre spring to the outer ones. Describe how the normal modes vary with K/k.

   (b) Find the frequencies of the normal modes of the two-mass system shown in (b) above, and sketch how they vary with M/m (show the ratio M/m varying from 1 to ∞). Annotate your sketch with pictures showing the normal modes associated with each eigenvalue at $M \approx m$ and $M \gg m$.

   (c) The two-mass system of part (b), with $M = m$, is driven by a force $F \sin(\omega t)$ applied to the first mass, where ω is not equal to either of the normal mode frequencies. As with a simple harmonic oscillator, the response can be represented as the sum of a steady-state response of frequency ω and a free response. Find the steady-state response of the system. Sketch the amplitude of the responses of x_1 and x_2 as a function of ω.

5. (B) One end of a string of length l and mass per unit length 4ρ is joined to a firm support. The other is joined end to end with a string of length l and mass per unit length ρ, which in turn is joined to a very long, very light string under constant tension T. Calculate the two lowest frequencies at which transverse vibrations can occur in this composite system and sketch the corresponding standing wave patterns.

6. (B) A block of mass M can move along a smooth horizontal track. Hanging from the block is a mass m on a light rod of length l that is free to move in a vertical plane that includes the line of motion of the block. Find the frequency and displacement patterns of the normal modes of oscillation of the system firstly by 'spotting' the normal modes of the system and then secondly by writing the Lagrangian $\mathcal{L} = T - U$ for the system and solving the Euler-Lagrange equations.

Elasticity

7. (A) A straight tube 1 m long, of radius 10 mm and wall thickness 100 μm, is closed at both ends. It is found that when the pressure inside is increased from 10^5 Pa to 10^6 Pa the tube lengthens by 100 μm. What is the bulk modulus of the material of the tube?

8. (B) A pillar has length H when lying horizontally; it is of uniform isotropic material and has uniform diameter $\ll H$. When it was set up vertically on a valley floor its height became $H - h_1$. At a later date the valley was flooded; when the water reached the top of the pillar, its height was $H - h_2$. Find h_1 and h_2 in terms of Young's modulus E, Poisson's ratio σ and the density ρ of the pillar, the density $\rho_{\rm w}$ of water, and the acceleration of gravity.

   By considering the case $\rho = \rho_w$, find the relation between E, σ and the bulk modulus.

9. (B) A horizontal cantilever of uniform cross section is such that, when a weight of 1 kg is hung from its free end, a point half way along the cantilever is displaced downwards by 1 cm. Show by direct calculation that, if the same weight is hung from the point half way along, the displacement of the free end is also 1 cm.

   Re-examine the question, but for any two points on a cantilever whose cross section varies arbitrarily along its length. Show that the deflection at A due to unit load at B is always equal to the deflection at B due to unit load at A.

   [Use an energy argument to show that the stored energy when loads are applied at A and B must be the same whether the load at B is applied before or after that at A.]

10. (B) A horizontal beam of rectangular section 2×1 has one end built into a wall, a diagonal of the rectangle being vertical. When a weight is hung on the free end, in what direction will the beam deflect?

11. (B) A light uniform horizontal beam is loaded at its mid-point. Determine the relative maximum deflections for the cases when the beam is

    (a) freely supported at both ends;
    (b) rigidly clamped at both ends;
    (c) rigidly clamped at one end, with the other end unsupported.

    State clearly the difference in boundary conditions in the three cases. Where, in each case, is the beam most likely to break if it is overloaded?

12. (B) A long straight cylindrical wire is fitted with universal couplings at its two ends. Through these couplings a torque is applied, and their design is such that the wire is free to choose between remaining straight but twisted or coiling up into a flat spring. Show that the energy stored in these two configurations for a given torque are in the ratio $1/G : 2/E$, and hence that the wire will coil up if $\sigma > 0$.

13. (C) An isotropic elastic medium of infinite extent contains a spherical hole of volume V and gas is pumped into the hole until a pressure P_0 has been built up. Show that as a function of radius r the radial stress within the medium obeys the differential equation $4P' + rP'' = 0$ and hence that $P \propto 1/r^3$ in equilibrium. Show that the gas dilates the hole by an amount $\Delta V = 3P_0V/4G$.

Fluid Dynamics

14. (B) (a) Two cylindrical jets of incompressible fluid, which have the same radius a and velocity components $(0, 0, v)$ and $(0, 0, -v)$ respectively, meet head-on at the origin and spread out to form a sheet in the $z = 0$ plane. Show that the thickness d of this sheet at distance r from the origin is a^2/r.

    (b) Two jets of incompressible fluid have a rectangular cross-section, with thickness a in the x direction and width b ($b \gg a$) in the y direction. Now suppose them to be tilted through angles $\pm\alpha$, so that the velocity components with which they meet become $(v \sin\alpha, 0, \pm v \cos\alpha)$. Show that the resultant sheet has thickness $a(1 + \sin\alpha)$ for $x > 0$ and $a(1 - \sin\alpha)$ for $x < 0$.

15. (B) A bath of cross-sectional area 2 m^2 is filled to a depth of 20 cm. Estimate the time taken for the water to drain away when the plug is pulled out, if the area of the plug hole is 10 cm^2.

16. (B) A small bubble is expanding at a constant volume rate Q a distance d away from a solid plane in a fluid of density ρ. Show that the pressure distribution on the plane is given by

$$P = P_0 - \frac{Q^2\rho}{8\pi^2}\frac{r^2}{(d^2+r^2)^3}$$

where r is the radial distance from the point of symmetry on the plane, and P_0 is the pressure at a large distance from the bubble. Hence, or otherwise, show that the force on the bubble is $\rho Q^2/16\pi d^2$.

Is the force repulsive or attractive? Explain why.

17. (B) Recall the potential for steady flow of incompressible fluid, with uniform speed v_0 at infinity, past a stationary sphere. Find an expression for the velocity at any point and sketch the streamlines. Where on the surface of the sphere is the pressure highest and the lowest and what are these values compared with the pressure at infinity?

Estimate how rapidly a sphere must travel in still water to risk cavitation.

[Take the pressure at infinity as 10^5 Pa.]

18. (B) A river of frictionless incompressible water flows steadily over a flat circular sandbank where the water is half as deep as it is elsewhere. The depth of the river is very small compared with the diameter of the sandbank. By solving Laplace's equation in two dimensions for the flow potential, subject to appropriate boundary conditions, show that above the sandbank the current is 4/3 times as fast as it is at large distances.

19. (B) An incompressible fluid of density ρ and viscosity η flows along a pipe of length l. Assuming the flow remains laminar, determine the volume flow rate Q if the pressure difference between the ends of the pipe is Δp.

Calculate the total viscous force on the walls of the pipe.

What qualitative differences would you expect if instead the flow was turbulent?

20. (C) In a number of experiments on the terminal velocity of spheres falling through viscous fluids the following results were obtained:

(a) Aluminium spheres (density = 2.7×10^3 kg m^{-3}) in propyl alcohol (density = 0.8×10^3 kg m^{-3}, viscosity =4.50×10^{-3} Pa s):

Diameter of sphere (mm)	1.5	3.0	6.0	12.0
Terminal velocity (m s^{-1})	0.167	0.33	0.58	0.88

(b) Steel spheres (density = 7.83×10^3 kg m^{-3}) in olive oil (density = 0.93×10^3 kg m^{-3}, viscosity = 99×10^{-3} Pa s)

Diameter of sphere (mm)	10.0	17.5	30.0	52.5
Terminal velocity (m s^{-1})	0.89	1.50	2.65	3.30

Examine these results critically in the light of dimensional analysis, and deduce the terminal velocity of a spherical hailstone (density =0.9×10^3 kg m^{-3}) of diameter 2 mm in air (density = 1.3 kg m^{-3}, viscosity = 17×10^{-6} Pa s).

Quickies

1. The ringing note produced by a tea-cup when it is tapped on the rim with a spoon is liable to vary in pitch depending on whereabouts in relation to the handle the cup is tapped. Predict this variation.
2. What is the virtue of the I cross-section used for steel joists?
3. A sheet of thickness 1 mm is corrugated to a depth of 50 mm. Estimate the factor by which its sti?ness to bending about an axis perpendicular to the corrugations exceeds that of the uncorrugated sheet. [Ans: about 2500.]
4. Why is it easier to blow out a candle than suck it out?

Numerical Answers
Q5. $\omega = 0.615\sqrt{T/\rho l^2}$, $1.57\sqrt{T/\rho l^2}$.
Q7 1.5×10^{11} Pa.
Q10. 37° off vertical.
Q15. About 10 min.
Q17. 12.6 m s^{-1}.
Q20. About 6 m s^{-2}.