CLASSICAL DYNAMICS

20 Lectures Prof. S.F. Gull

- HANDOUT — comprehensive set of notes containing all relevant derivations.
  Please report all errors and typos.
- NOTES — Provisional hardcopy available in advance.
  Definitive copies of overheads available on web.
  Please report all errors and typos.
- SUMMARY SHEETS — 1 page summary of each lecture.
- EXAMPLES — 2 example sheets — 2 examples per lecture.
- WORKED EXAMPLES — Will be available on the web later.
- WEB PAGE — For feedback, additional pictures, movies etc.
  http://www.mrao.cam.ac.uk/∼steve/part1bdyn/
  There is a link to it from the Cavendish teaching pages.

NEWTONIAN MECHANICS

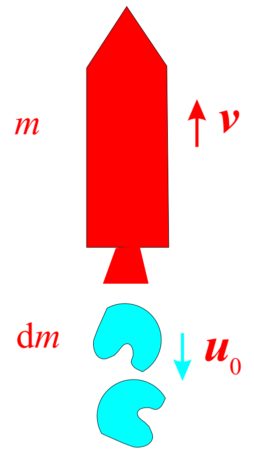
- **Newtonian Mechanics** is:
  - **Non-relativistic** i.e. velocities $v \ll c$ (speed of light=3×10^8 m s^{-1}.)
  - **Classical** i.e. $Et \gg \hbar$ (Planck's constant= 1.05×10^{-34} J s.)
- **Assumptions**:
  - mass independent of velocity, time or frame of reference;
  - measurements of length and time are independent of the frame of reference;
  - all parameters can be known precisely.
- **Mechanics**:
  = Statics (absence of motion);
  \+ Kinematics (description of motion, using vectors for position and velocity);
  \+ Dynamics (prediction of motion, and involves forces and/or energy).

Basic Principles of Newtonian Dynamics

- This course contains many applications of Newton's Second Law.
- Masses accelerate if a force is applied...
- The rate of change of momentum (mass × velocity) is equal to the applied force.
- Vectorially ($\boldsymbol{p} = m\boldsymbol{v}$):

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\boldsymbol{v})}{\mathrm{d}t} = \boldsymbol{F}$$

- Usually m is a constant so $m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{F}$
- General case $m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}\boldsymbol{v} = \boldsymbol{F}$ enables you to do **rocket science**.
- Rocket of mass $m(t)$ moving with velocity $v(t)$ expels a mass $\mathrm{d}m$ of exhaust gases backwards at velocity $-u_0$ relative to the rocket.
- In the absence of gravity or other external forces $\mathrm{d}m u_0 + m\mathrm{d}v = 0$.
- Integrating, we find $v = u_0 \log(m_\mathrm{i}/m_\mathrm{f})$, where $m_{\mathrm{i,f}}$ are the initial and final masses.
- For a rocket accelerating upwards against gravity $m\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}u_0 + mg = 0$.



Simple Harmonic Oscillator

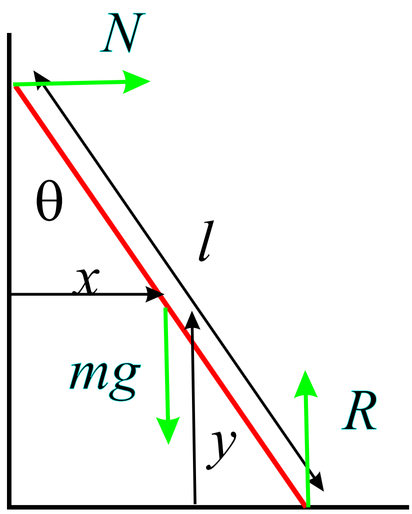
- The simple harmonic oscillator (SHO) occurs many times during the course.
- Mass m moving in one dimension with coordinate x on a spring with restoring force $F = -kx$. The constant k is known as the **spring constant**.
- Newtonian equation of motion: $m\ddot{x} = -kx$, where $\dot{x}$ denotes $\frac{\mathrm{d}x}{\mathrm{d}t}$ etc.
- General solution: $x = A\cos\omega t + B\sin\omega t$ where $\omega^2 = k/m$.
- Can also write solution as $x = \Re(Ae^{i\omega t})$, where A is complex.
- We can integrate the equation of motion to get a conserved quantity — the **energy**.
- Multiplying the equation of motion by $\dot{x}$ (a good general trick) we get

$$m\dot{x}\ddot{x} + k\dot{x}x = 0 \Rightarrow \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = E = \text{ constant}$$

- Here the quantity $T \equiv \frac{1}{2}m\dot{x}^2$ is the **kinetic energy** of the mass and $V \equiv \frac{1}{2}kx^2$ is the **potential energy** stored in the spring.
- For many dynamical systems (such as the SHO) the time t does not appear explicitly in the equations of motion and the **total energy** $E = T + V$ is conserved. This conserved quantity is also known as the **Hamiltonian**.

- If we know from physical grounds that the energy is conserved, we can always derive the equations of motion of systems that only have one degree of freedom (such as the Simple Harmonic Oscillator): $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E \Rightarrow \dot{x}(m\ddot{x} + kx) = 0 \Rightarrow m\ddot{x} = -kx$.
- This works because $\dot{x}$ is not always zero. We will call this the **energy method**.
- We can sometimes derive the equations of motion of much more complicated systems with n degrees of freedom is a similar way. It's certainly not rigorous, but works for most of the systems studied in this course. The theoretically more advanced methods of Lagrangian and Hamiltonian mechanics derive the equations of motion from a variational principle. They **are** rigorous, but still use the kinetic energy T and potential energy V (actually in the combination $\mathcal{L} = T - V$).
- We will see later in the course the use of the energy method to derive the equation of motion of a particle at radius r in a central force: $\frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r) = E$.

  Differentiating with respect to time we get: $\dot{r}\left(m\ddot{r} + \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}\right) = 0$
- We strike out the $\dot{r}$ to obtain the equation of motion.



- Ladder leaning against a smooth wall, resting on a smooth floor (not recommended practice).
- Newtonian method needs reaction forces N and R.

  Take moments to get the angular acceleration. Try it. . .
- The energy method is easier:

  (1) Potential energy. This is easy: $V = \frac{1}{2}mgl\cos\theta$.

  (2) Kinetic energy. Write this as the sum of the kinetic energy of the centre of mass plus the energy of rotation about the centre of mass: $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2$,

  where $I = \frac{1}{12}ml^2$ is the moment of inertia of a uniform rod about its centre.

  Coordinates of centre of mass: $x = \frac{1}{2}l\sin\theta\ ;\ \ y = \frac{1}{2}l\cos\theta$

  Work out velocities: $\dot{x} = \frac{1}{2}l\cos\theta\,\dot{\theta}\ ;\ \ \dot{y} = -\frac{1}{2}l\sin\theta\,\dot{\theta}$

  $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{8}ml^2\dot{\theta}^2 + \frac{1}{24}ml^2\dot{\theta}^2 = \frac{1}{6}ml^2\dot{\theta}^2$

  (3) Energy method: $\frac{\mathrm{d}(T+V)}{\mathrm{d}t} = 0 \Rightarrow \dot{\theta}\left(\frac{1}{3}\ddot{\theta}ml^2 - \frac{1}{2}mgl\sin\theta\right) = 0$.

  (4) Equation of motion: $\ddot{\theta} = \frac{3g}{2l}\sin\theta$.

(See Section 1.2 of Handout)

- In dynamics we use **vectors** to describe the positions, velocities and accelerations of particles and other bodies, as well as the forces and couples that act on them
- We need to revise vectors, vector functions, vector identities and integral theorems.
- Revise **scalar product** $\boldsymbol{a}\cdot\boldsymbol{b}$ and **vector product** $\boldsymbol{a}\times\boldsymbol{b}$ (preferable to $\boldsymbol{a}\wedge\boldsymbol{b}$).
- There are only a couple of vector identities, but you MUST LEARN THEM.

(1) Scalar triple product

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} \quad \text{(Interchange of dot and cross)}$$

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot\boldsymbol{c}\times\boldsymbol{a} = -\boldsymbol{b}\cdot\boldsymbol{a}\times\boldsymbol{c} \quad \text{(Permutations change sign)}$$

(2) Vector triple product

This is the most important identity:

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{c}\,\boldsymbol{b} - \boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{c}$$

Rule: Vector outside bracket appears in both scalar products.

Rule: Outer pair takes the plus sign.

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{a}\cdot\boldsymbol{c}\,\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{c}\,\boldsymbol{a}$$

(See Section 1.2 of Handout)

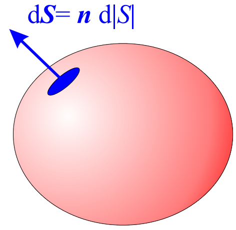
Gradient operator $\boldsymbol{\nabla}$ (Vector differential operator).

- grad(Φ); $\boldsymbol{\nabla}\Phi$ — **Vector** gradient of a **scalar** field Φ.
- div$(\boldsymbol{E})$; $\boldsymbol{\nabla}\cdot\boldsymbol{E}$ — **Scalar divergence** of a **vector** field E.
- curl$(\boldsymbol{E})$; $\boldsymbol{\nabla}\times\boldsymbol{E}$. — **Vector curl** of a **vector** field E.

Divergence Theorem (Gauss' Theorem)

- Relates integral of flux vector $\boldsymbol{E}$ through closed surface $\boldsymbol{S}$ ($\boldsymbol{n}$ outwards) to volume integral of $\boldsymbol{\nabla}\cdot\boldsymbol{E}$.

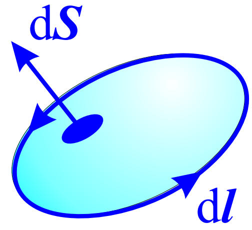
$$\oint \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{E} = \int \mathrm{d}V\,\boldsymbol{\nabla}\cdot\boldsymbol{E}$$



Stokes' Theorem

- Relates line integral vector $\boldsymbol{E}$ around closed loop $\boldsymbol{l}$ to surface integral of $\boldsymbol{\nabla}\times\boldsymbol{E}$.

$$\oint \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{E} = \int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{\nabla}\times\boldsymbol{E}$$



Further identities:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\Phi) = 0; \quad \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) = 0; \quad \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E}) - \nabla^2\boldsymbol{E}$$

MECHANICS — REVIEW OF PART IA

Revision of dynamics of many-particle system.(Detailed derivations in Section 3 of Handout.)

- System of N particles. The ath particle of mass m_a is at position $\boldsymbol{r}_a$ and velocity $\boldsymbol{v}_a$. Acted on by external force $\boldsymbol{F}_{a0}$ and internal forces $\boldsymbol{F}_{ab}$ from other particles.
- **Important definition**: Centre of mass $\boldsymbol{R}$. Define $M \equiv \sum_a m_a$ and $M\boldsymbol{R} \equiv \sum_a m_a \boldsymbol{r}_a$.
- **Other concepts:** Total momentum $\boldsymbol{P}$. Total angular momentum $\boldsymbol{J}$. Total external force $\boldsymbol{F}_0$ and couple $\boldsymbol{G}_0$. Kinetic energy T, potential energy U, total energy E.
- **Total momentum** acts as if it were acted upon by the **total external force**.
- **Total angular momentum** acts as if it were acted upon by the **total external couple**.
- **Intrinsic angular momentum:** $\boldsymbol{J}'$ in the frame S' in which $\boldsymbol{P}' = 0$ (zero-momentum, or Centre of Mass (CoM) frame). The intrinsic angular momentum is independent of origin.
- The **Centre of Mass frame** is thus special, and should be used wherever possible.
- Galilean transformation from CoM frame S' to frame S moving at velocity $\boldsymbol{V}$ (CoM at $\boldsymbol{R}' = \boldsymbol{V}t$). **Momentum**: $\boldsymbol{P} = \boldsymbol{P}' + M\boldsymbol{V}$
  Angular Momentum: $\boldsymbol{J} = \boldsymbol{J}' + M\boldsymbol{R}' \times \boldsymbol{V}$. **Kinetic energy**: $T = T' + \frac{1}{2}M\boldsymbol{V}^2$

OVERALL MOTION

(See Section 3.1 of Handout)

- Development from Newton's Laws: $m_a \ddot{\boldsymbol{r}}_a = \boldsymbol{F}_a$,
  where $a = 1, N$ for the ath of N particles. A rigid body is a special case.
- **Overall motion** $$\sum_a m_a \ddot{\boldsymbol{r}}_a = \sum_a \boldsymbol{F}_a = \sum_a \boldsymbol{F}_{a0} + \sum_a \sum_b \boldsymbol{F}_{ab},$$
  where $\boldsymbol{F}_{a0}$ is the **external** force on particle a and $\boldsymbol{F}_{ab}$ is the force on a due to b.
  Since $\boldsymbol{F}_{ab} = -\boldsymbol{F}_{ba}$ by Newton's 3rd Law, the $\sum_a \sum_b$ term above sums to zero.
- Define $M \equiv \sum_a m_a$ and $M\boldsymbol{R} \equiv \sum_a m_a \boldsymbol{r}_a$. $\boldsymbol{R}$ is the position of the Centre of Mass.
- With these definitions $$M\ddot{\boldsymbol{R}} = \sum_a \boldsymbol{F}_{a0} \equiv \boldsymbol{F}_0$$
- The Centre of Mass moves as if it were a particle of mass M acted upon by the total external force $\boldsymbol{F}_0$.
- In terms of momentum $\dot{\boldsymbol{p}}_a = \boldsymbol{F}_a$; $\dot{\boldsymbol{P}} = \boldsymbol{F}_0$, where $\boldsymbol{P}$ is the total momentum.

(See Section 3.2 of Handout)

- **Couple, torque**: $\boldsymbol{G} \equiv \boldsymbol{r}\times\boldsymbol{F}$. **Angular momentum:** $\boldsymbol{J} \equiv \boldsymbol{r}\times\boldsymbol{p}$.
- Since $\dot{\boldsymbol{p}}_a = \boldsymbol{F}_a$, we have $\sum_a \boldsymbol{r}_a\times\dot{\boldsymbol{p}}_a = \sum_a \boldsymbol{r}_a\times\boldsymbol{F}_a$
- Expand RHS:

$$\text{RHS} = \sum_a \boldsymbol{r}_a\times\boldsymbol{F}_{a0} + \underbrace{\sum_a\sum_b \boldsymbol{r}_a\times\boldsymbol{F}_{ab}}_{\sum_b\sum_{a<b}\underbrace{(\boldsymbol{r}_a-\boldsymbol{r}_b)\times\boldsymbol{F}_{ab}}_{=0}}$$

- The latter term is zero since $\boldsymbol{F}_{ab}$ is assumed to be along the line between a and b. We have again used Newton's Third Law.
- The LHS for one particle is $\dot{\boldsymbol{J}}_a = \frac{d}{dt}(\boldsymbol{r}_a\times\boldsymbol{p}_a) = \underbrace{\dot{\boldsymbol{r}}_a\times\boldsymbol{p}_a}_{\text{zero, since } m\dot{\boldsymbol{r}} = \boldsymbol{p}} + \boldsymbol{r}_a\times\dot{\boldsymbol{p}}_a$
- For the system of particles

$$\dot{\boldsymbol{J}} \equiv \sum_a \dot{\boldsymbol{J}}_a = \sum_a \boldsymbol{r}_a\times\dot{\boldsymbol{p}}_a = \text{RHS} = \sum_a \boldsymbol{r}_a\times\boldsymbol{F}_{a0} \equiv \boldsymbol{G}_0$$

- $\boldsymbol{G}_0$ is the resultant couple $\boldsymbol{G}$ from all **external** forces.

(See Section 3.3 of Handout)

- Suppose we move the origin by a constant $\boldsymbol{a}$, giving new coordinates $\boldsymbol{r}'$ with $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{a}$.
- Then $\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}'$ and the overall motion is unaffected.
- What about the angular momentum $\boldsymbol{J}$? For one particle $\boldsymbol{J}_a = \boldsymbol{J}'_a + \boldsymbol{a}\times\boldsymbol{p}_a$, or for the system

$$\boldsymbol{J} = \boldsymbol{J}' + \sum_a \boldsymbol{a}\times\boldsymbol{p}_a = \boldsymbol{J}' + \boldsymbol{a}\times\boldsymbol{P}$$

i.e. $\boldsymbol{J}$ depends on the choice of origin *unless* $\boldsymbol{P} = 0$.

- **Intrinsic angular momentum:** $\boldsymbol{J}$ in the frame in which $\boldsymbol{P} = 0$ (zero-momentum, or Centre of Mass frame). The Intrinsic angular momentum is independent of origin.
- The Centre of Mass frame is thus special, and should be used wherever possible.
- Similarly $\boldsymbol{G} = \boldsymbol{G}' + \boldsymbol{a}\times\boldsymbol{F}$.

KINETIC ENERGY

(See Section 3.4 of Handout)

- **Work done:** force $\times$ distance moved $\|$ force $=$ change in **energy**.
- For a single particle $$\boldsymbol{F}\cdot\mathbf{d}r = m\ddot{\boldsymbol{r}}\cdot\mathbf{d}r = m\,\underbrace{\ddot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}}_{\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}})}\,\mathrm{d}t$$

  or $$\boldsymbol{F}\cdot\mathbf{d}r = \mathrm{d}(\tfrac{1}{2}mv^2)$$
- **Kinetic energy:** $T \equiv \frac{1}{2}mv^2$.
- Work done on particle $=$ change in kinetic energy.
- For a system of particles

$$\begin{aligned}\mathrm{d}T &= \sum_a \mathrm{d}T_a = \sum_a \boldsymbol{F}_a\cdot\mathbf{d}r_a \\ &= \sum_a \boldsymbol{F}_{a0}\cdot\boldsymbol{d}r_a + \sum\sum_{a<b} \boldsymbol{F}_{ab}\cdot(\mathbf{d}r_a - \mathbf{d}r_b)\end{aligned}$$

where we have used $\boldsymbol{F}_{ab} = -\boldsymbol{F}_{ba}$. We can write the ab-term as $-\mathcal{F}_{ab}\,\mathrm{d}|\boldsymbol{r}_a - \boldsymbol{r}_b|$, where $\mathcal{F}_{ab}$ has magnitude $= |F_{ab}|$ and is positive if force is attractive, negative if repulsive.

POTENTIAL AND TOTAL ENERGY

(See Section 3.4 of Handout)

- **Potential energy:** U is defined as $$\mathrm{d}U = \sum\sum_{a<b} \mathcal{F}_{ab}\,\mathrm{d}|\boldsymbol{r}_a - \boldsymbol{r}_b|$$
- Note the zero of $U = \int \mathrm{d}U$ is undefined. It is often taken with $U = 0$ with particles at infinite separation, giving negative U for a system of particles with attractive forces.
- For a rigid body $\mathrm{d}U = 0$ since $|\boldsymbol{r}_a - \boldsymbol{r}_b|$ is fixed.
- **Total Energy:** $E = T + U$.
- As defined above $$\mathrm{d}E = \mathrm{d}T + \mathrm{d}U = \sum_a \boldsymbol{F}_{a0}\cdot\boldsymbol{d}r_a$$
- The RHS term is the work done by external forces; it can be incorporated into U if desired.

Galilean Transformation

(See Section 3.5 of Handout)

- Go from frame S' to S with $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{V}t$; $\boldsymbol{V}$ steady; $t = t'$.
- **Momentum** $\quad \boldsymbol{p} = \boldsymbol{p}' + m\boldsymbol{V}; \quad \boldsymbol{P} = \boldsymbol{P}' + M\boldsymbol{V}$
  i.e. $\boldsymbol{P}$ in S and $\boldsymbol{P}'$ in S' change together (or remain steady together if there if no external force). If $\boldsymbol{P}' = 0$, then S' is the **zero-momentum** or **Centre of Mass** frame.
- **Angular momentum** $\quad \boldsymbol{J} = \sum_a (\boldsymbol{r}'_a + \boldsymbol{V}t)\times(\boldsymbol{p}'_a + m_a\boldsymbol{V})$
- There are 4 terms. The 4th is $\boldsymbol{V}\times\boldsymbol{V} = 0$. The others give

$$\boldsymbol{J} = \boldsymbol{J}' + \boldsymbol{V}t\times\boldsymbol{P}' + \underbrace{\sum_a \boldsymbol{r}'_a\times m_a\boldsymbol{V}}$$

$$\sum_a (m_a\boldsymbol{r}'_a)\times\boldsymbol{V} = M\boldsymbol{R}'\times\boldsymbol{V}$$

- Thus if S' is the zero-momentum frame, $\boldsymbol{P}' = \boldsymbol{0}$ and

$$\underset{\text{in } S}{\boldsymbol{J}} = \underset{\text{intrinsic}}{\boldsymbol{J}'} + \underset{\text{motion of C of M in } S}{\underbrace{M\boldsymbol{R}'\times\boldsymbol{V}}}.$$

Galilean Transformation — Energy

(See Section 3.5.3 of Handout)

- **Energy**

$$\begin{aligned} T &= \sum_a \tfrac{1}{2}m_a v_a^2 = \tfrac{1}{2}m_a(\boldsymbol{v}'_a + \boldsymbol{V})\cdot(\boldsymbol{v}'_a + \boldsymbol{V}) \\ &= T' + \underbrace{\sum_a m_a\boldsymbol{v}'_a\cdot\boldsymbol{V}} + \tfrac{1}{2}MV^2. \end{aligned}$$

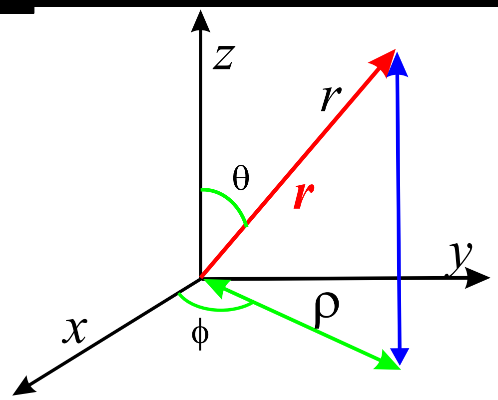
$$(= 0, \text{ if } S' = \text{zero-momentum frame})$$

or

$$T = \text{KE in zero-momentum frame} + \tfrac{1}{2}MV^2$$

Coordinate Systems

(See Section 1.1 of Handout)



- Position vector $\boldsymbol{r}$ has Cartesian coordinates (x, y, z), cylindrical polar coordinates (ρ, ϕ, z) and spherical polar coordinates (r, θ, ϕ).
- Relation between coordinate systems:

$$\begin{aligned} x &= \rho\cos\phi &= r\sin\theta\cos\phi \\ y &= \rho\sin\phi &= r\sin\theta\sin\phi \\ z &= z &= r\cos\theta \end{aligned}$$

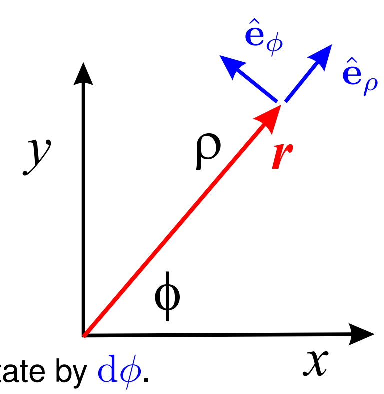
$$\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ r &= \sqrt{x^2+y^2+z^2} \end{aligned}$$

- We define unit vectors $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$ along x, y, z axes.
- Similarly we define unit vectors $(\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\theta)$ along directions of increasing (ρ, ϕ, θ) (unambiguous because these are orthogonal coordinate systems).
- Position vector $\boldsymbol{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z = \rho\hat{\mathbf{e}}_\rho + z\hat{\mathbf{e}}_z = r\hat{\mathbf{e}}_r$

Dynamics in Cylindrical Polars

(See Section 2.1 of Handout)

- In Cartesians, the equation of motion of a particle is $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$; or $m\ddot{x} = F_x$ etc, where we denote time derivatives $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \equiv \dot{\boldsymbol{r}}$, $\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \equiv \ddot{\boldsymbol{r}}$ etc.



- Consider cylindrical polars; ignore z-motion for the moment $\boldsymbol{r} = \rho\hat{\mathbf{e}}_\rho$ where $\hat{\mathbf{e}}_\rho$, $\hat{\mathbf{e}}_\phi$ and $\hat{\mathbf{e}}_z$ are unit vectors in the directions of increasing ρ, ϕ, z.
- Note that the direction of the vectors $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ change as the particle moves: $\dot{\boldsymbol{r}} = \dot{\rho}\hat{\mathbf{e}}_\rho + \rho\dot{\hat{\mathbf{e}}}_\rho$
- As the particle moves from say P to P' in $\mathrm{d}t$, $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ rotate by $\mathrm{d}\phi$.
- Elementary geometry gives $\mathbf{d}\hat{\mathbf{e}}_\rho = \mathrm{d}\phi\,\hat{\mathbf{e}}_\phi$ or $\dot{\hat{\mathbf{e}}}_\rho = \dot{\phi}\hat{\mathbf{e}}_\phi$ and similarly $\dot{\hat{\mathbf{e}}}_\phi = -\dot{\phi}\hat{\mathbf{e}}_\rho$

giving

$$\dot{\boldsymbol{r}} = \underbrace{\dot{\rho}\hat{\mathbf{e}}_\rho}_{\text{radial}} + \underbrace{\rho\dot{\phi}\hat{\mathbf{e}}_\phi}_{\text{transverse}}$$

- In cylindrical polar coordinates the radial velocity is $\dot{\rho}$ and the transverse velocity is $\rho\dot{\phi}$.

ACCELERATION IN POLAR COORDINATES

(See Section 2.1 of Handout)

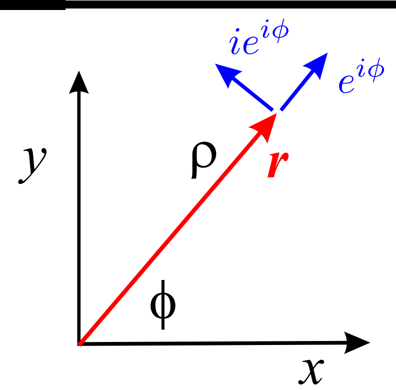
- Similarly, we can work out the rate of change of velocity:

$$\begin{aligned}\ddot{\boldsymbol{r}} &= \ddot{\rho}\hat{\mathbf{e}}_\rho + \dot{\rho}\underbrace{\dot{\hat{\mathbf{e}}}_\rho}_{\dot{\phi}\hat{\mathbf{e}}_\phi} + \dot{\rho}\dot{\phi}\hat{\mathbf{e}}_\phi + \rho\ddot{\phi}\hat{\mathbf{e}}_\phi + \rho\dot{\phi}\underbrace{\dot{\hat{\mathbf{e}}}_\phi}_{-\dot{\phi}\hat{\mathbf{e}}_\rho} \\ &= \underbrace{(\ddot{\rho} - \rho\dot{\phi}^2)}_{\text{radial}}\hat{\mathbf{e}}_\rho + \underbrace{(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})}_{\text{transverse}}\hat{\mathbf{e}}_\phi\end{aligned}$$

- The z-motion is independent: $(\ddot{\boldsymbol{r}})_z$ is just $\ddot{z}\hat{\mathbf{e}}_z$ since $\dot{\hat{\mathbf{e}}}_z = 0$.
- The radial acceleration is $\ddot{\rho} - \rho\dot{\phi}^2$, the second term being the **centripetal acceleration** required to keep a particle in an orbit of constant radius.
- The transverse acceleration is $2\dot{\rho}\dot{\phi} + \rho\ddot{\phi} = \dfrac{1}{\rho}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\rho^2\dot{\phi}\right)$ and shows that it is related to the angular momentum per unit mass $\rho^2\dot{\phi}$.
- Spherical polars can be treated by putting $\boldsymbol{r} = r\hat{\mathbf{e}}_r$, and expanding $\dot{\boldsymbol{r}}$ etc. with $\dot{\hat{\mathbf{e}}}_r$ expressed in terms of $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$. We shall not need this here, as it's slightly complicated, but if you have computer algebra available it's very useful. . .

POLARS AND THE ARGAND DIAGRAM

(See Handout Section 2.1)



- The complex plane $z \equiv x + iy = \rho e^{i\phi}$ has the same structure as the two-dimensional plane

$$\boldsymbol{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y = \rho\hat{\mathbf{e}}_\rho.$$

- The unit vectors correspond to complex numbers:

$$\begin{aligned}\hat{\mathbf{e}}_\rho &\leftrightarrow e^{i\phi} \\ \hat{\mathbf{e}}_\phi &\leftrightarrow ie^{i\phi}\end{aligned}$$

- We can therefore derive the radial and transverse components easily using the Argand diagram.
  - **Velocity**: $\dfrac{\mathrm{d}}{\mathrm{d}t}(\rho e^{i\phi}) = \dot{\rho}e^{i\phi} + \rho\dot{\phi}\, ie^{i\phi}$.
  - **Acceleration**:

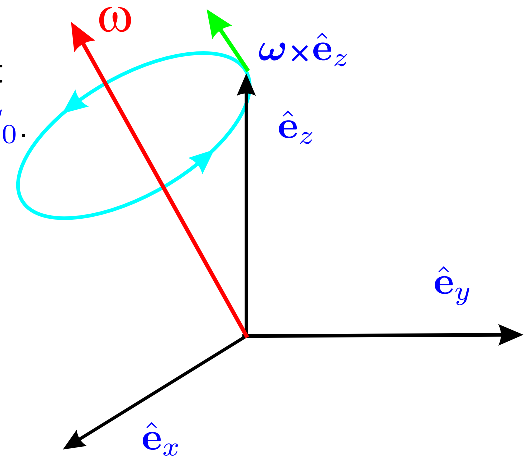
$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho e^{i\phi}) &= \ddot{\rho}e^{i\phi} + 2\dot{\rho}\dot{\phi}\, ie^{i\phi} + \rho\ddot{\phi}\, ie^{i\phi} - \rho\dot{\phi}^2 e^{i\phi} \\ &= \underbrace{(\ddot{\rho} - \rho\dot{\phi}^2)}_{\text{radial}}\, e^{i\phi} + \underbrace{(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})}_{\text{transverse}}\, ie^{i\phi}\end{aligned}$$

(See Handout Section 2.2)

- Suppose we have a frame S_0 in which $m\ddot{\boldsymbol{r}}_0 = \boldsymbol{F}$, with $\boldsymbol{F}$ ascribed to known physical causes. What is the apparent equation of motion in a moving frame S?
- **Case 1**: Suppose $\boldsymbol{r} = \boldsymbol{r}_0 - \boldsymbol{R}(t)$. Suppose the axes in S_0 and S remain parallel and $t = t_0$ (as always in classical physics): $\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_0 - \ddot{\boldsymbol{R}}$
- For the special case $\ddot{\boldsymbol{R}} = 0$ (i.e. steady motion between frames), $m\ddot{\boldsymbol{r}} = m\ddot{\boldsymbol{r}}_0 = \boldsymbol{F}$, i.e. the **same** equation of motion (Galilean transformation).
- For general $\boldsymbol{R}(t)$ $\quad m\ddot{\boldsymbol{r}} = m\ddot{\boldsymbol{r}}_0 - m\ddot{\boldsymbol{R}} = \boldsymbol{F} - m\ddot{\boldsymbol{R}}$
- The **apparent force** in S includes both the actual force $m\ddot{\boldsymbol{r}}_0$ and a **fictitious force** $-m\ddot{\boldsymbol{R}}$.
- Fictitious forces are: (a) associated with accelerated frames; (b) proportional to mass.
- Question: Is gravity a fictitious force?
- Answer: (according to general relativity). Yes! Gravity is equivalent to acceleration.

(See Section 2.3 of Handout)

- **Case 2**: Frame S rotates with angular velocity $\boldsymbol{\omega}$, so that the unit vectors rotate with respect to the inertial frame S_0.



- The rate of change is given by $\dot{\hat{\mathbf{e}}}_z = \boldsymbol{\omega}\times\hat{\mathbf{e}}_z$ etc.

  Let the frames coincide at $t = 0$:

$$\begin{aligned} \boldsymbol{r}_0 &= x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z = \boldsymbol{r} \\ \dot{\boldsymbol{r}}_0 &= \dot{x}\hat{\mathbf{e}}_x + x\dot{\hat{\mathbf{e}}}_x + y \text{ and } z \text{ terms} \;, \\ &= \boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{r} \end{aligned}$$

  $\boldsymbol{v} \equiv \dot{x}\hat{\mathbf{e}}_x + \dot{y}\hat{\mathbf{e}}_y + \dot{z}\hat{\mathbf{e}}_z$ is the **apparent velocity** in S.

- The acceleration in S_0 is

$$\begin{aligned} \ddot{\boldsymbol{r}}_0 &= \ddot{x}\hat{\mathbf{e}}_x + 2\dot{x}\dot{\hat{\mathbf{e}}}_x + x\ddot{\hat{\mathbf{e}}}_x + y \text{ and } z \text{ terms} \\ &= \ddot{x}\hat{\mathbf{e}}_x + 2(\boldsymbol{\omega}\times\hat{\mathbf{e}}_x)\dot{x} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\hat{\mathbf{e}}_x)x + y \text{ and } z \text{ terms} \\ &= \boldsymbol{a} + 2\boldsymbol{\omega}\times\boldsymbol{v} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) \end{aligned}$$

  where $\boldsymbol{a} \equiv \ddot{x}\hat{\mathbf{e}}_x + \ddot{y}\hat{\mathbf{e}}_y + \ddot{z}\hat{\mathbf{e}}_z$ is the **apparent acceleration** in S. We rewrite the momentum equation $m\ddot{\boldsymbol{r}}_0 = \boldsymbol{F}$ in terms of the apparent quantities $\boldsymbol{r}$, $\boldsymbol{v}$ and $\boldsymbol{a}$:

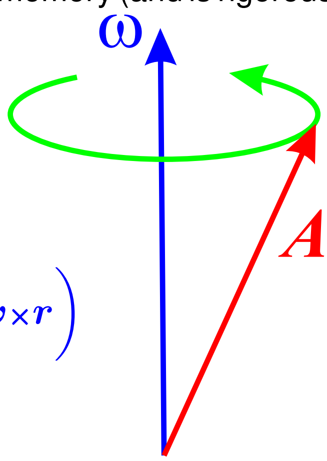
$$\boxed{m\boldsymbol{a} = \boldsymbol{F} - 2m(\boldsymbol{\omega}\times\boldsymbol{v}) - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})}$$

- The observer in S adds Coriolis and Centrifugal forces (**inertial** or **fictitious** forces).

ROTATING FRAMES

(See Section 2.3 of Handout)

- There is an operator approach to rotating frames that is a good aid to memory (and is rigorous).
- For any vector $\mathbf{A}$ the rates of change in frame S_0 and in frame S are related by $\left[\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right]_{S_0} = \left[\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right]_S + \boldsymbol{\omega}\times\boldsymbol{A}$



- Apply this operator relation twice to $\boldsymbol{r}$ ($\boldsymbol{r}=\boldsymbol{r}_0$ at $t=0$):

$$\left[\frac{\mathrm{d}^2\boldsymbol{r}_0}{\mathrm{d}t^2}\right]_{S_0} = \left(\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]_S + \boldsymbol{\omega}\times\right)\left(\left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]_S + \boldsymbol{\omega}\times\boldsymbol{r}\right)$$

- Expanding and setting $\left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]_S = \boldsymbol{v}$ and $\left[\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right]_S = \boldsymbol{a}$ we recover

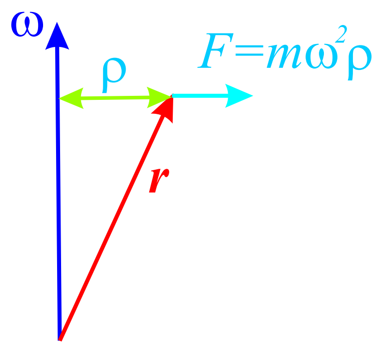
$$m\ddot{\boldsymbol{r}}_0 = \boldsymbol{F} = m\boldsymbol{a} + 2m(\boldsymbol{\omega}\times\boldsymbol{v}) + m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$$

- **General case**: Observer moves on a path $R(t)$ and uses a frame rotating at angular velocity $\boldsymbol{\omega}(t)$ which is also changing. From previous results and, because the time derivative now operates on $\boldsymbol{\omega}$ we get the general formula:

$$m\boldsymbol{a} = \boldsymbol{F} - 2m(\boldsymbol{\omega}\times\boldsymbol{v}) - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) - m\ddot{\boldsymbol{R}} - m\dot{\boldsymbol{\omega}}\times\boldsymbol{r}$$

- The $m\dot{\boldsymbol{\omega}}\times\boldsymbol{r}$ term is called the Euler force.

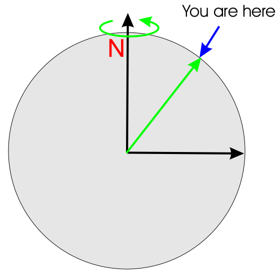
CENTRIFUGAL AND CORIOLIS FORCES



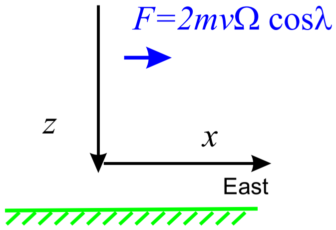
- **Centrifugal Force:** $-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$.
  $-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) = m(\boldsymbol{\omega}^2\boldsymbol{r} - \boldsymbol{r}\cdot\boldsymbol{\omega}\;\boldsymbol{\omega})$
- Centrifugal Force $m\omega^2\rho$ outwards.
- **Coriolis Force:** $-2m(\boldsymbol{\omega}\times\boldsymbol{v})$ appears if a body is moving with respect to a rotating frame.
- Coriolis Force is a sideways force, perpendicular both to the rotation axis and to the velocity.
- Problems involving Coriolis Force can often be done by considering angular momentum.
- **Advice:** Do not meddle with the signs or the ordering of the terms. The minus sign reminds us that these terms came from the other side of the equation, and $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ construction reminds us of the operator relation $\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]_{S_0} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\right]_S + \boldsymbol{\omega}\times$.

FICTITIOUS FORCES — APPLICATIONS

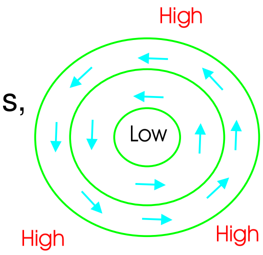
- Centrifugal force gives rise to the Earth's equatorial bulge: $\sim \dfrac{\Omega^2 R}{g} \approx \dfrac{1}{300}$.
- **Coriolis force due to motion on Earth's surface:** $F = 2m\Omega v \sin\lambda$
  Direction is sideways and to the right in the Northern Hemisphere.
  Independent of direction of travel [NSEW].

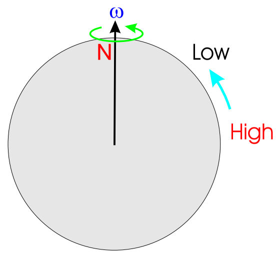


- **Coriolis force on a falling body.**
  Starts from rest at time $t = 0$. $v = gt$
  $m\ddot{x} = 2mgt\Omega\cos\lambda \Rightarrow x = \frac{1}{3}g\Omega t^3\cos\lambda$



- Foucault pendulum. Precesses at $\Omega\sin\lambda$.
  Which way?
- Roundabouts and other fairground rides. Rollercoasters.
- Weather patterns, trade winds, jet streams, tornados, bathtubs (??).

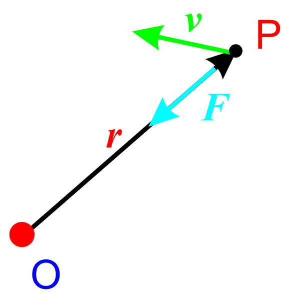




ORBITS — CENTRAL FORCE FIELD

(See Section 4.1 of Handout)

- Particle moving in central force field. Potential $U(r)$ yields radial force $\quad \boldsymbol{F} = -\boldsymbol{\nabla} U = -\dfrac{\mathrm{d}U}{\mathrm{d}r}\hat{\mathbf{e}}_r$.
- Motion remains in the plane defined by position vector $\boldsymbol{r}$ and velocity $\boldsymbol{v}$.



- No couple from central force $\Rightarrow$ angular momentum is conserved:

$$mr^2\dot{\phi} = J = \text{ constant} \quad \text{(Kepler II)}$$

- Total energy is conserved:

$$E = U(r) + \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) = \tfrac{1}{2}m\dot{r}^2 + U(r) + \frac{J^2}{2mr^2}$$

- The **effective potential** $U_{\text{eff}}(r)$ has a contribution from the angular velocity.

$$U_{\text{eff}}(r) \equiv U(r) + \frac{J^2}{2mr^2}$$

- The effective potential has a centrifugal repulsive term $\propto \dfrac{1}{r^2}$.

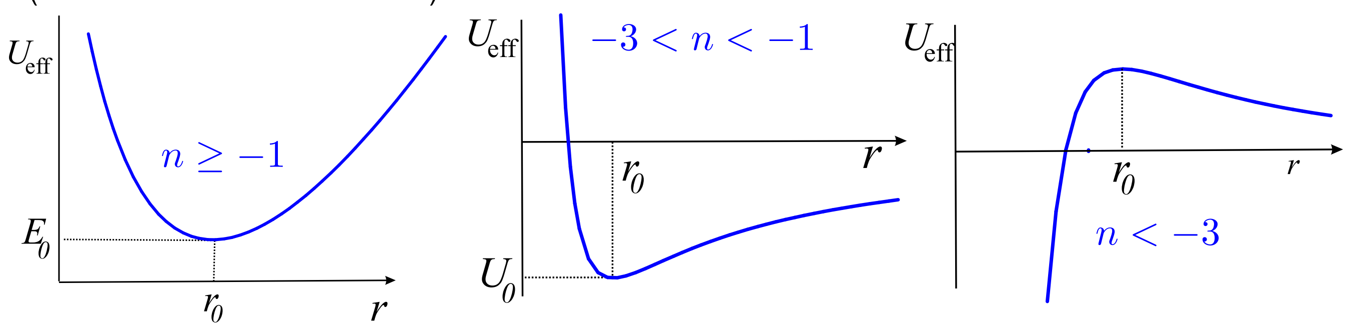
ORBITS IN POWER-LAW FORCE

(See Section 4.2 of Handout)

- We can gain a lot of insight into orbits by studying the force law $F = -Ar^n$ with A positive, so force is attractive.
- The effective potential is then $U_{\text{eff}}(r) = \dfrac{Ar^{n+1}}{n+1} + \dfrac{J^2}{2mr^2}$ the only exception being $n = -1$ (U_{eff} then contains a $\log r$ term).
- The centrifugal potential is repulsive and $\propto r^{-2}$. A plot of $U_{\text{eff}}(r)$ shows which values of the index n lead to bound or unbound orbits, and which lead to stable or unstable orbits.
- For $n \geq -1$ (including the $\log r$ potential), the potential increases as $r \to \infty$ and the orbits are bound and stable.
- For $-3 < n < -1$ the potential goes to zero at $r = \infty$ and the orbits can either be bound or unbound.
- For $n < -3$ the attraction at $r \to 0$ overcomes the centrifugal repulsion and the orbits are not stable (this is the case for the central region of black holes in GR).

ORBITS IN POWER-LAW FORCE

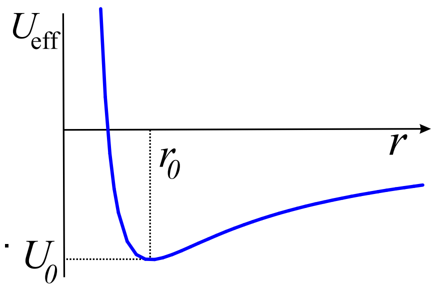
(See Section 4.2 of Handout)



- The potential is qualitatively different for different values of n:

$n \geq -1$:	Orbit at r_0 stable.	All orbits bound.
$-3 < n < -1$:	Orbit at r_0 stable.	Unbound orbits for $E > 0$.
$n < -3$:	Orbit at r_0 unstable.	Will go to $r = 0$ or $r = \infty$

Nearly Circular Orbits in Power-law Force



- Let $F = -Ar^n$, $n =$ index, with common cases $n = +1$ (2D SHM) and $n = -2$ (gravity, electrostatics).

$$U_{\rm eff} = \frac{Ar^{n+1}}{n+1} + \frac{J^2}{2mr^2}$$

- **Nearly circular orbits** are oscillations/perturbations about r_0. Taylor expansion of $U_{\rm eff}$ gives

$$U_{\rm eff} = U_0 + (r - r_0)\left.\frac{dU_{\rm eff}}{dr}\right|_{r_0} + \tfrac{1}{2}(r - r_0)^2 \left.\frac{d^2U_{\rm eff}}{dr^2}\right|_{r_0} + \cdots$$

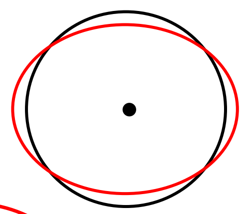
- At $r = r_0$ $\frac{dU_{\rm eff}}{dr}$ is zero, giving $\frac{dU_{\rm eff}}{dr} = Ar^n - \frac{J^2}{mr^3} = 0$ at r_0.
- The second derivative of $U_{\rm eff}$ is $\frac{d^2U_{\rm eff}}{dr^2} = nAr^{n-1} + \frac{3J^2}{mr^4} = \frac{(n+3)J^2}{mr_0^4}$ at r_0.
- Using the energy method $\frac{d}{dt}\left(\tfrac{1}{2}m\dot{r}^2 + U_{\rm eff}\right) = \dot{r}\left(m\ddot{r} + \frac{dU_{\rm eff}}{dr}\right) = 0$

  we get the SHM equation $m\ddot{r} + \frac{(n+3)J^2}{mr_0^4}(r - r_0) = 0$,

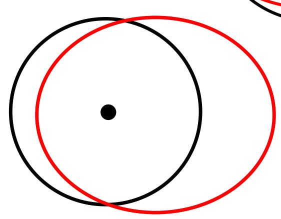
  i.e. simple harmonic motion about r_0 with angular frequency $\omega_p = \sqrt{n+3}\,\frac{J}{mr_0^2}$.

Nearly Circular Orbits II

- How does ω_p of the perturbation compare with ω_c of the circular orbit at r_0?
  $\omega_c = \dot{\phi} = J/mr_0^2$. Therefore $\omega_p = \sqrt{n+3}\,\omega_c$.
- The simple cases are

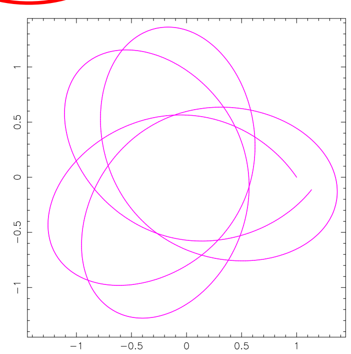


1. $n = 1$. Force proportional to r, i.e. simple harmonic motion. $\omega_p = 2\omega_c$, giving a central ellipse (Lissajous figure).



2. $n = -2$. Inverse square force. $\omega_p = \omega_c$, giving an ellipse with a focus at $r = 0$ (planetary orbit).

- General n gives non-commensurate ω_p and ω_c, with non-repeating orbits (e.g. galactic orbits). Case illustrated is $n = -1$.



Kepler's Laws

- 150AD Ptolemy - Earth at centre of solar system, with Sun rotating around it, and the planetary orbits described by a combination of circles (epicycles).
- Tycho Brahe (1546-1601) made observations of planetary and stellar positions to an accuracy of 10 arcsec (the resolution of the eye is 1 arcmin).
- Johannes Kepler (1571 -1630) spent 5 years fitting circles to the Brahe's data for Mars's orbit and found differences of the order of 8 arcmin. Rejected model because of known accuracy of Brahe's measurements.
- Eventually Kepler concluded that the orbits were ellipses.
- **Kepler's Laws**
  - **First Law:** Planetary orbits are ellipses with the Sun at one focus.
  - **Second Law:** The line joining the planets to the Sun sweeps out equal areas in equal times. (Implies conservation of angular momentum.)
  - **Third Law:** The square of the period of a planet is proportional to the cube of its mean distance to the Sun (it is proportional to the cube of the length of the orbit's major axis).

Orbits in Inverse Square Law Force

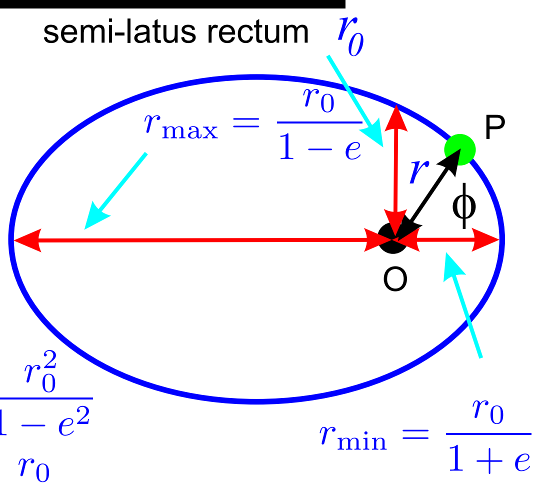
- Inverse square law force: $F = -\dfrac{A}{r^2}$.
- **Angular momentum**: $J = mr^2\dot{\phi}$ **Energy**: $\frac{1}{2}m\dot{r}^2 + \dfrac{J^2}{2mr^2} - \dfrac{A}{r} = E$.
- Slightly easier to work with $u = 1/r$: $\dot{r} = \dfrac{\mathrm{d}r}{\mathrm{d}\phi}\dot{\phi} = -\dot{\phi}r^2\dfrac{\mathrm{d}u}{\mathrm{d}\phi} = -\dfrac{J}{m}\dfrac{\mathrm{d}u}{\mathrm{d}\phi}$.
- Substitute into the energy equation $\left(\dfrac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2 - \dfrac{2m}{J^2}(E + Au) = 0$.

  Complete square: $\left(\dfrac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \dfrac{e^2}{r_0^2} - \left(u - \dfrac{1}{r_0}\right)^2$, where $\dfrac{e^2}{r_0^2} \equiv \dfrac{2mE}{J^2} + \dfrac{1}{r_0^2}$

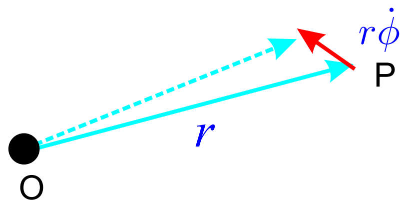
  and we have defined $r_0 = \dfrac{J^2}{mA}$, the radius of the circular orbit with the same J.

  Standard integral: $\dfrac{\mathrm{d}u}{\sqrt{\dfrac{e^2}{r_0^2} - \left(u - \dfrac{1}{r_0}\right)^2}} = \mathrm{d}\phi \Rightarrow u = \dfrac{1}{r_0}(1 + e\cos(\phi - \phi_0))$
- Equation of conic section: $r_0 = r(1 + e\cos\phi)$
- For a repulsive potential $r_0 = r(e\cos\phi - 1)$

- Ellipse of eccentricity e ($0 < e < 1$).
  Centre of attraction at one focus.
- Polar equation: $r_0 = r(1 + e\cos\phi)$
  r_0 is called the semi-latus rectum.



- Cartesian equation: $r = r_0 - ex$
  $y^2 + x^2(1 - e^2) + 2er_0x = r_0^2$.
  Set $x' = x + \frac{r_0 e}{1 - e^2} \Rightarrow y^2 + (x')^2(1 - e^2) = \frac{r_0^2}{1 - e^2}$
  Ellipse: $\frac{(x')^2}{a^2} + \frac{y^2}{b^2} = 1\ ;\ a = \frac{r_0}{1 - e^2}\ ;\ b = \frac{r_0}{\sqrt{1 - e^2}}$
- Area of an ellipse is $\pi ab = \frac{\pi r_0^2}{(1 - e^2)^{3/2}}$



- Period T is $\frac{\text{Area}}{\text{Rate of sweeping out area}}$.
- Rate of sweeping out area: $\frac{1}{2}r^2\dot\phi = \frac{J}{2m}$,
  hence period $T = \frac{2\pi r_0^2 m}{J(1 - e^2)^{3/2}} = 2\pi\sqrt{\frac{ma^3}{A}}$ (Kepler's 3rd Law).

(See Section 4.3.1)

- **Shape of orbit**. Since the vectors $\boldsymbol{J}$, $\dot{\boldsymbol{v}}$ and $\dot{\hat{\mathbf{e}}}_r$ have magnitudes $mr^2\dot\phi$, A/mr^2 and $\dot\phi$ respectively and are mutually perpendicular, we may write
  $$\boldsymbol{J}\times\dot{\boldsymbol{v}} = -A\dot{\hat{\mathbf{e}}}_r \qquad \text{(the sign is obtained by inspection).}$$
- Since $\boldsymbol{J}$ is constant, the equation may be integrated to give $\boldsymbol{J}\times\boldsymbol{v} + A(\hat{\mathbf{e}}_r + \boldsymbol{e}) = 0$, where $\boldsymbol{e}$ is a vector integration constant.
- Taking the dot-product of this equation with $\boldsymbol{r}$ gives
  $$\underbrace{\boldsymbol{J}\times\boldsymbol{v}\cdot\boldsymbol{r}}_{= \boldsymbol{J}\cdot\boldsymbol{v}\times\boldsymbol{r} = -J^2/m} + A(r + \boldsymbol{e}\cdot\boldsymbol{r}) = 0$$
- Therefore $r(1 + \boldsymbol{e}\cdot\hat{\mathbf{e}}_r) = r(1 + e\cos\phi) = \frac{J^2}{mA} = r_0$,
  which is the polar equation of a conic with **focus** at $r = 0$ (Kepler's 1st Law).
- The major axis is in the direction of $\boldsymbol{e}$; e is the eccentricity: a circle ($e = 0$); ellipse ($e < 1$), parabola; ($e = 1$) or hyperbola ($e > 1$).

ALTERNATIVE — ENERGY OF THE ORBIT

- To get the energy take the scalar product of $Ae = -(\boldsymbol{J}\times\boldsymbol{v} + A\hat{\mathbf{e}}_r)$ with itself (note that $\boldsymbol{J}$ and $\boldsymbol{v}$ are perpendicular).

$$A^2e^2 = J^2v^2 + 2\underbrace{\boldsymbol{J}\times\boldsymbol{v}\cdot\hat{\mathbf{e}}_r}\,A + A^2$$

$$= \boldsymbol{J}\cdot\boldsymbol{v}\times\hat{\mathbf{e}}_r = -J^2/mr$$

Therefore

$$A^2(e^2-1) = J^2\left(v^2 - \frac{2A}{mr}\right) = \frac{2EJ^2}{m} = 2AEr_0$$

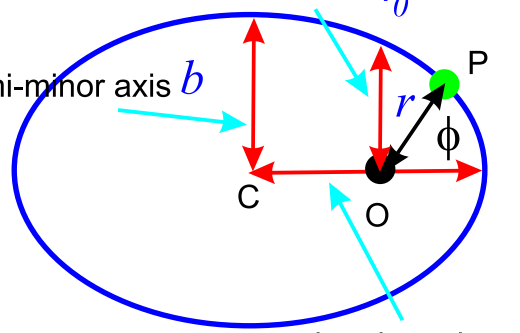
where E is the total energy. The major axis of the orbit is given by

$$2a = r_0\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{2r_0}{1-e^2} = -\frac{A}{E}$$

i.e. $E = -A/2a$, independent of eccentricity.

- The distance of closest approach occurs at $\phi = 0$: $r_{\rm min} = \dfrac{r_0}{1+e}$ and the distance of furthest approach occurs at $\phi = \pi$, at $r_{\rm max} = \dfrac{r_0}{1-e}$.
- The semi-major axis a satisfies $2a = r_{\rm min} + r_{\rm max}$, so that $a = \dfrac{r_0}{1-e^2}$.
- Substituting, we find the useful relations $r_{\rm max} = a(1+e)$ and $r_{\rm min} = a(1-e)$.

ELLIPTICAL ORBITS — IMPORTANT THINGS TO REMEMBER



- The equation of an ellipse in polar coordinates

$$r_0 = r(1 + e\cos\phi)$$

- The distances of closest and furthest approach follow from this: $r_{\rm min} = \dfrac{r_0}{1+e}$ and $r_{\rm max} = \dfrac{r_0}{1-e}$
- The semi-major axis a satisfies $2a = r_{\rm min} + r_{\rm max}$ $\Rightarrow$ $a = \dfrac{r_0}{1-e^2}$ so that $r_{\rm max,min} = a(1\pm e)$. Also $b = \dfrac{r_0}{\sqrt{1-e^2}}$
- The semi-major axis a determines the **energy** and the **period** of the orbit

$$E = -\frac{A}{2a}\;;\quad T = \frac{2\pi}{\omega}\;;\quad \omega^2 = \frac{A}{ma^3}$$

- The semi-latus rectum r_0 determines the **angular momentum** of the orbit: $J^2 = Amr_0$
- If you need to derive any of these formulae in a hurry **consider a silly case**. Use the simple balance of forces argument for circular motion: $\dfrac{A}{r^2} = m\omega^2 r$.

Time in Orbit — (Details Non-Examinable)

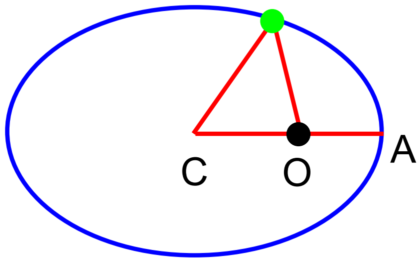
- We haven't so far got a formula for the coordinates (r, ϕ) as a function of time. There is an easy way of doing this, which is coming a bit later, but we can in fact get a formula for $t(\phi)$, rather than $\phi(t)$. It's just not very pretty...
- From the equation of the ellipse and $h \equiv \frac{J}{m} = r^2\dot{\phi}$ we get

$$\frac{d\phi}{(1 + e\cos\phi)^2} = \frac{h dt}{r_0^2}$$

- The integral is found as formula 2.551 of Gradshteyn & Ryzhik:

$$t = \frac{r_0^2}{h(1-e^2)}\left(-\frac{e\sin\phi}{1+e\cos\phi} + \frac{2}{\sqrt{1-e^2}}\tan^{-1}\left(\frac{\tan(\phi+\frac{1}{4}\pi)+e}{\sqrt{1-e^2}}\right)\right)$$

- You can also get a formula for it by subtracting the area of the triangle CPO from the sector CPA.

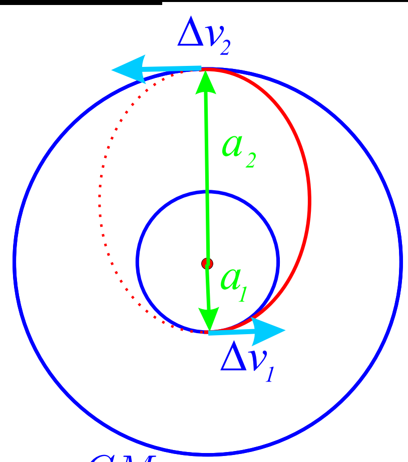


Elliptical Orbits — Another (Non-Examinable) Way

- Return to the energy equation: $\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{A}{r} = E$.
- Change the **independent variable**, defining a radially-scaled "time" $r ds = dt$ so that $\frac{dr}{dt} = \frac{1}{r}\frac{dr}{ds}$ We then get $\left(\frac{dr}{ds}\right)^2 - \frac{2E}{m}r^2 - \frac{2A}{m}r + J^2 = 0$
- Defining $\Omega^2 = \frac{-2E}{m}$ (remember E is negative) we get the same form of equation as before, but for r instead of u. The distances of furthest and closest approach are $a(1 \pm e)$, so the solution is just $r = a(1 - e\cos\Omega s)$.
- We can now integrate $r = \frac{dt}{ds}$ to find a nice parametric form for the time t

  $t = as - \frac{ae}{\Omega}\sin\Omega s$. The period is $T = \frac{2\pi a}{\Omega} = 2\pi\sqrt{\frac{ma^3}{A}}$.
- There is also a simple closed form for $\phi(s)$: $\tan\left(\sqrt{1-e^2}\frac{\phi}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\frac{\Omega s}{2}$.
- This is related to the "square root" of the Kepler problem, which transforms the orbit to a central ellipse (2-D SHM). Many essential features of this solution were known to Newton, but the procedure is called the "Kustaanheimo-Stiefel" transformation.

Example: The Hohmann Transfer Orbit

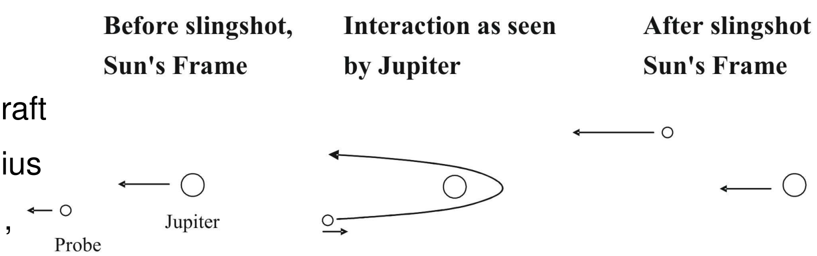


- The Hohmann transfer orbit is one half of an elliptic orbit that touches both the initial orbit and the desired orbit.
- For gravitational case put $A = GMm$.
- In circular orbits $T = -E = -\frac{1}{2}U$, for elliptical orbits $\langle T \rangle = -\langle E \rangle = -\frac{1}{2}\langle U \rangle$ (Virial Theorem).
- The initial energy (for a spacecraft of unit mass) is $E_1 = -\frac{GM}{2a_1}$, and the velocity has to be increased until the spacecraft has the energy of the transfer orbit $E_t = -\frac{GM}{a_1 + a_2}$.
- The important thing to know is Δv_1, since that determines the amount of fuel used, but we can work all that out from the energies:

$$E_t = -\frac{GM}{a_1 + a_2} = -\frac{GM}{a_1} + \frac{1}{2}v_{t_1}^2 \Rightarrow \frac{1}{2}v_{t_1}^2 = \frac{GMa_2}{a_1(a_2 + a_1)}$$

- The rest of relations are easy enough, but not particularly informative...
- The Hohmann transfer is the most fuel efficient orbit, unless there are other massive bodies in the vicinity, in which case you can use the gravitational slingshot.

Gravitational Slingshot



- The escape velocity of a spacecraft from the Solar system at the radius of the Earth's orbit is 42 km s^{-1}, which should be compared to its orbital velocity of 30 km s^{-1}.
- We can use a gravitational 'slingshot' around planets to increase kinetic energy and/or change direction in order to visit other bodies in the Solar system.
- Voyager 2 made a 'grand tour'.

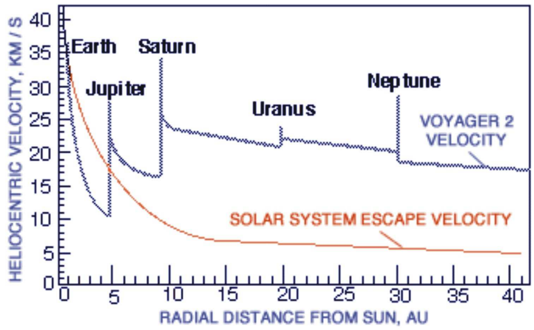
Consider the Earth/Jupiter transfer orbit:

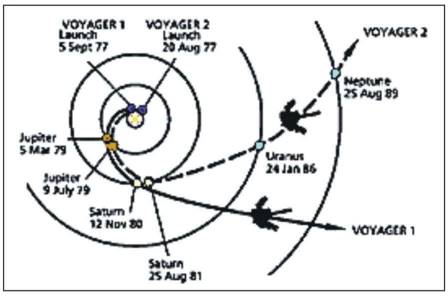
Ratio of Jovian/Earth's Orbits	5.2
Orbital velocity of Earth (w.r.t. Sun)	29.8km/s
Solar escape velocity at Earth	42.1km/s
Transfer orbital velocity at Earth	38.6km/s
Transfer orbital velocity at Jupiter	7.4km/s
Orbital velocity of Jupiter	13.1km/s
Closing speed Jupiter/probe	13.1-7.4=5.7km/s
Max slingshot speed of probe post Jupiter	13.1+5.7=18.8km/s
Solar escape velocity at Jupiter	18.5km/s

(these figures ignore the eccentricity of the orbits)

Gravitational Slingshot — Voyager 2

Velocity of Voyager 2 as a function of distance from the sun





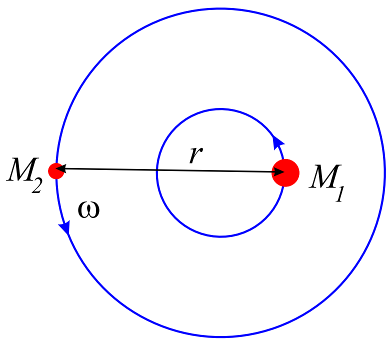
Voyager 2 left Earth at about 36 km/s. Gravitational slingshot at Jupiter, increases its speed above solar system escape velocity. The same is seen at Saturn and Uranus. The Neptune flyby design put Voyager close by Neptune's moon Triton rather than attain more speed.

(source http://www.jpl.nasa.gov/basics/)

Classical Dynamics

The Two-Body Problem and Reduced Mass

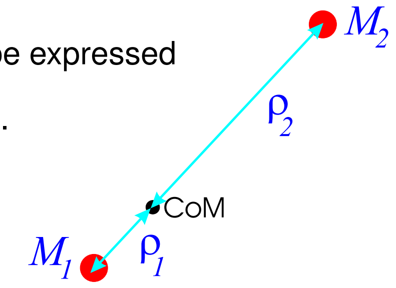
- The two masses M_1 and M_2 orbit the centre of mass.
- Each orbit is an ellipse in a common plane with the centre of mass at one focus. The ellipses have the same eccentricity and phase.
- The important case is circular motion. Mass M_1 is distance $\dfrac{M_2 r}{M_1 + M_2}$ from the CoM.



- Balance of forces for M_1: $\dfrac{GM_1M_2}{r^2} = M_1\omega^2 \dfrac{M_2 r}{M_1 + M_2}$

  $\Rightarrow \omega^2 = \dfrac{G(M_1 + M_2)}{r^3}$
- This is the really important result and is usually the best starting point (e.g. in problem Q12).
- You get the same result by considering the balance of forces for M_2.
- We can rearrange the result as $\mu r\omega^2 = \dfrac{GM_1M_2}{r^2}$, in which we have the true separation r and the actual force $\dfrac{GM_1M_2}{r^2}$, but a modified **reduced mass** term $\mu \equiv \dfrac{M_1M_2}{M_1 + M_2}$.
- I do not advocate the use of reduced mass, despite its widespread use in the textbooks. . .

The Two-Body Problem



- Two particles of masses M_1 and M_2 orbiting each other — positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.
- The energy, angular momentum and equations of motion can be expressed in terms of the **reduced mass** $\mu \equiv \dfrac{M_1 M_2}{M_1 + M_2}$ and $\boldsymbol{r}_1 - \boldsymbol{r}_2$.
- The centre of mass is at $R_0 = \dfrac{M_1 \boldsymbol{r}_1 + M_2 \boldsymbol{r}_2}{M_1 + M_2}$.
- Define

$$\boldsymbol{\rho}_1 \equiv \boldsymbol{r}_1 - R_0 = \frac{M_2}{M_1 + M_2}(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$
$$\boldsymbol{\rho}_2 \equiv \boldsymbol{r}_2 - R_0 = \frac{M_1}{M_1 + M_2}(\boldsymbol{r}_2 - \boldsymbol{r}_1)$$

- Kinetic energy in the centre of mass frame:

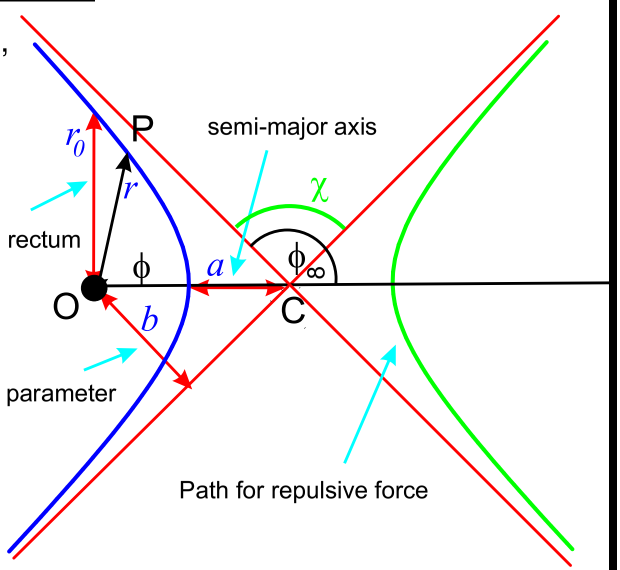
$$T = \tfrac{1}{2}M_1\dot{\boldsymbol{\rho}}_1^2 + \tfrac{1}{2}M_2\dot{\boldsymbol{\rho}}_2^2 = \frac{1}{2}\left(\frac{M_1 M_2^2}{(M_1+M_2)^2} + \frac{M_1^2 M_2}{(M_1+M_2)^2}\right)(\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2)^2 = \frac{\mu}{2}(\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2)^2$$

- Angular momentum: $\boldsymbol{J} = M_1 \boldsymbol{\rho}_1 \times \dot{\boldsymbol{\rho}}_1 + M_2 \boldsymbol{\rho}_2 \times \dot{\boldsymbol{\rho}}_2 = \mu(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times (\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2)$.
- Equations of motion: $M_1 \ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_{12}\ ;\ \ M_2 \ddot{\boldsymbol{r}}_2 = \boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$

$$\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2 = \left(\frac{1}{M1} + \frac{1}{M_2}\right)\boldsymbol{F}_{12} = \frac{1}{\mu}\boldsymbol{F}_{12}\ ;\ \ \ddot{\boldsymbol{R}}_0 = \boldsymbol{0}$$

- Reduced to the one-body problem in the centre of mass frame.

Hyperbolic Orbits



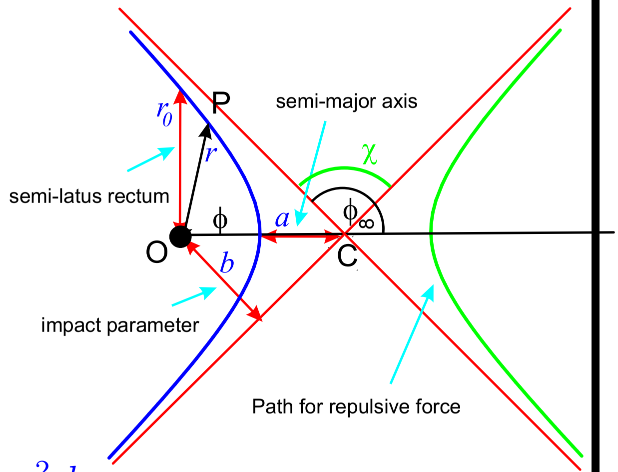
- **Attractive potential**: all previous formulae still valid, but $e > 1$ so $a < 0$ and energy $E = -\dfrac{A}{2a} = \dfrac{(e^2-1)A}{2r_0} > 0$.
- Impact parameter b and velocity at infinity v_∞ determine angular momentum $J = mbv_\infty$ and energy $E = \tfrac{1}{2}mv_\infty^2$.
- Most problems (e.g. Rutherford scattering Q14) require the total angle of deflection $\chi = 2\phi_\infty - \pi$ (χ positive).
- Asymptotes are at $\pm\phi_\infty$; from the equation of the conic section (valid for $e < 1$ or $e > 1$) we have

$$\cos\phi_\infty = -1/e \Rightarrow \sec\phi_\infty = -e \Rightarrow \tan^2\phi_\infty = e^2 - 1$$

- Note that $\pi/2 < \phi_\infty < \pi$.

HYPERBOLIC ORBITS II

(See Section 4.4 of Handout)



Attractive potential

- We need to find the eccentricity from the physical parameters E and J.
- From the definition of $r_0 = \dfrac{J^2}{mA}$ we can write
  $(e^2 - 1) = \dfrac{2r_0E}{A} = \dfrac{2J^2E}{mA^2}$.
- In terms of b and v_∞ this means
  $\tan^2\phi_\infty = e^2 - 1 = \dfrac{m^2v_\infty^4b^2}{A^2} \Rightarrow \tan\phi_\infty = \dfrac{mv_\infty^2 b}{A}$.

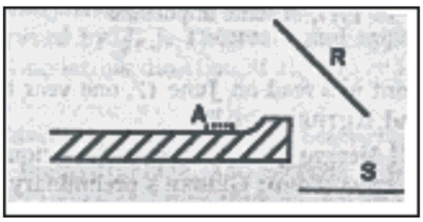
Repulsive potential:

- Change $\dfrac{A}{r^2} \to -\dfrac{B}{r^2}$, define $J^2 = Bmr_0$ and use other branch $r_0 = r(e\cos\phi - 1)$.
- a is positive again and the total angle of deflection is now $\chi = \pi - 2\phi_\infty$ (χ negative).
- The asymptotes are still related to the physical parameters by $\tan\phi_\infty = \dfrac{mv_\infty^2 b}{B}$.
- The distance of closest approach is $a(1+e)$.

RUTHERFORD SCATTERING

Geiger and Marsden 1909 – first evidence of backscattering

Alpha source (A) shielded by lead (shaded) from scintillation screen (S). Presence of foil (R) causes backscattering (scintillation)

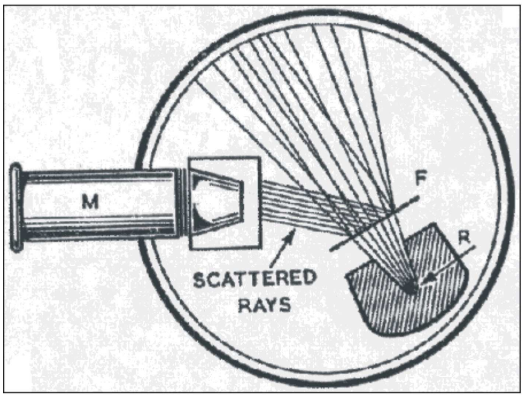


Count rate drops with decreasing atomic weight of foil

1. Metal	2. Atomic weight, A	3. Number of scintillations per minute, Z.	4. A/Z.
Lead	207	62	30
Gold	197	67	34
Platinum	195	63	33
Tin	119	34	28
Silver	108	27	25
Copper	64	14·5	23
Iron	56	10·2	18·5
Aluminium.	27	3·4	12·5

Rutherford Scattering II

Angular Deflection of Alpha Particles From Gold Foil (1911)



Alpha Particle:
Energy: 8MeV
Velocity $2\text{x}10^{7}$m/s

Gold foil thickness
$8.6\text{x}10^{-8}$m (glass supported) about 350 atom spacings

Chance of deflection > 90 degrees: 1/20000
– forces required for large deflection only possible with very close approach to something massive compared to alpha particle – impossible with 'plum pudding' model.

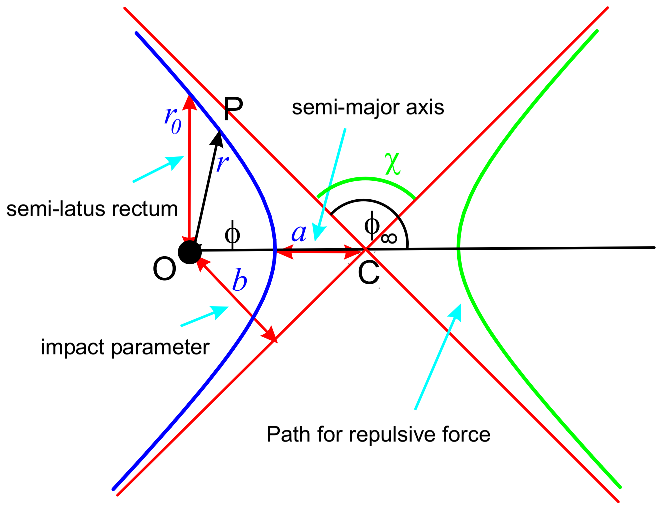
Distance of closest approach: $2.8\text{x}10^{-14}$m

λ of alpha particle: $5\text{x}10^{-15}$m classical scattering ok (ish)

Rutherford: 'As if you fired a 15-inch naval shell at a piece of tissue paper and the shell came right back and hit you'

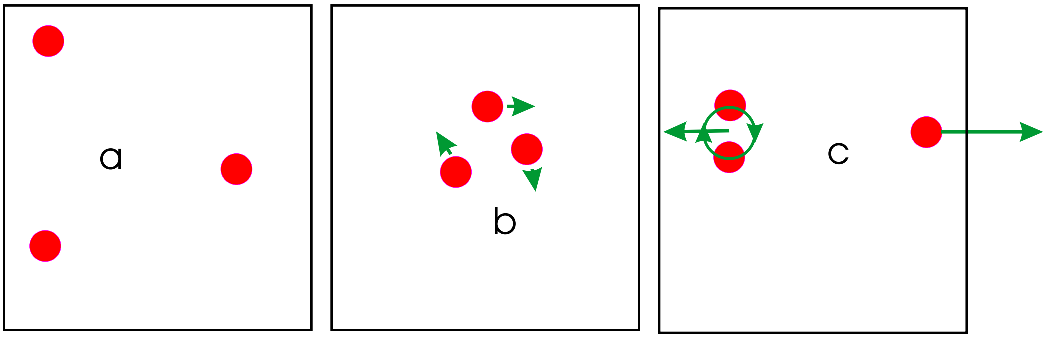
Hyperbolic Orbits in Repulsive Potential — Another Way

- We can simply use the results for the attractive potential case ($r_0 = r(1 + e\cos\phi)$) and let ϕ exceed ϕ_∞.
- The radius r is then **negative** and the particle traces out the repulsive branch, getting closest to O at $\phi = \pi$, so that $r(\pi) = a(1 + e)$, where a is now negative.
- This works because the potential energy $-\dfrac{A}{r}$ is positive when $r < 0$ and so represents a repulsive potential.
- This approach has the considerable advantage that no sign changes are needed, but it has something of a “Alice through the looking glass” quality to it.



THE THREE-BODY PROBLEM

- Some hierarchical systems can be stable indefinitely e.g. Sun, Earth and the Moon.
- A general 3-body encounter can be very complicated, but a general feature emerges.



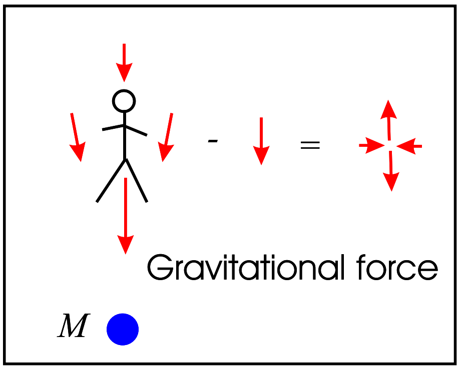
- If 3 bodies are allowed to attract each other from a distance (a), they will speed up and interact strongly (b). Eventually the interaction is likely to form a close binary (negative gravitational binding energy) releasing kinetic energy, which may be enough for the bodies to escape to infinity (c).
- This mechanism is responsible for "evaporation" of stars from star clusters. (maybe also invalidates the virial theorem?)
- The planet Pluto has a close companion Charon, and has an eccentric orbit which takes it inside Neptune's orbit. A 3-body collision amongst Neptune's moons is the most likely cause.

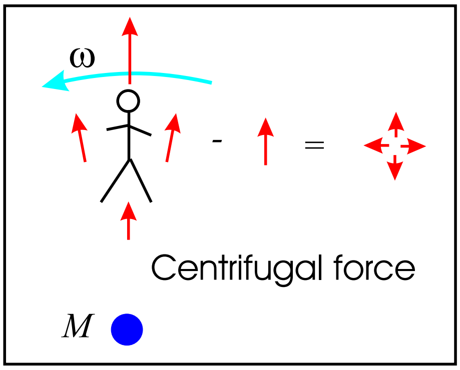
GRAVITATIONAL POTENTIAL, FIELD AND TIDAL FORCES

There three important aspects to gravitation (Newtonian or GR).

- **Gravitational potential** $\phi(\boldsymbol{r})$ This determines energies and redshifts; velocities of objects and temperature of gases. Always relative — can't have an absolute value. For point mass $\phi = -\frac{GM}{R}$. [Newtonian potential is one part of the metric of GR.]
- **Gravitational field** $\boldsymbol{g}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi$ This determines accelerations and orbits. The field is also relative (perhaps surprisingly). e.g. we (and the Local Group of galaxies) could all be accelerating to a "Great Attractor" and nothing would change here. For point mass $|\boldsymbol{g}| = \frac{GM}{R^2}$. [Gravitational field is one part of the "connection" in GR].
- **Gravitational tidal field** $\boldsymbol{R}(\boldsymbol{a}) = \boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{g}$ This is all one can feel and measure locally — it describes how the gravitational field varies in space. In components $[\boldsymbol{R}(\boldsymbol{a})]_i = R_{ij}a_j$ where $R_{ij} \equiv \frac{\partial g_i}{\partial x_j}$. The gravitational tidal field varies as $1/R^3$.
  For a point mass it is $R(\hat{\boldsymbol{u}}_r) = \frac{2GM}{R^3}\hat{\boldsymbol{u}}_r;\ R(\hat{\boldsymbol{u}}_\theta) = -\frac{GM}{R^3}\hat{\boldsymbol{u}}_\theta;\ R(\hat{\boldsymbol{u}}_\phi) = -\frac{GM}{R^3}\hat{\boldsymbol{u}}_\phi;$
  There is a radial stretching, and a squashing by half as much in the transverse directions. The tidal acceleration tensor R_{ij} gives the coefficients of the quadratic term in the Taylor expansion of the gravitational potential. [Tidal field is part of the Riemann tensor of GR].

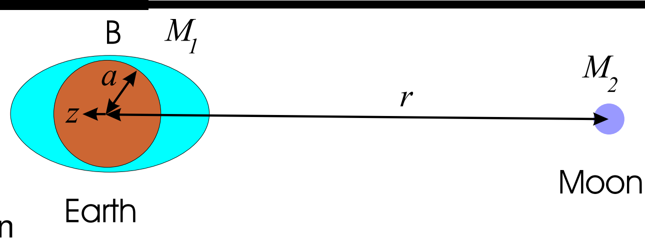
More about tidal forces





- The gravitational field near a point mass is directed radially and is proportional to $1/r^2$. The tidal forces consist of a radial stretching $2GM/r^3$ and a sideways compression $-GM/r^3$.
- For the two-body potential we must also add the contribution from centrifugal force — this is a stretching in two directions in the plane of rotation, and no contribution in the direction of the rotation axis. This assumes our stick man is corotating with the orbit (i.e. keeping the same relative orientation with respect to the mass M).
- The sum is a stretch of $3GM/R^3$ along the radial direction, no contribution in the orbital plane and $-GM/R^3$ perpendicular to the plane.
- Tidal forces are weak on Earth and not very strong in the solar system (exception: Jupiter and Io), but tidal forces can be colossal near compact objects such as neutron stars.

Origin of the tides



- Looking down on the orbital plane, we see the Earth rotating under a tidal bulge of water, making two tides a day ($\approx$ 1 hr later per day).
- The tidal acceleration in the Earth-Moon direction is $\dfrac{3GM_2z}{r^3}$, where z is the distance from the centre of the Earth.
- Integrating to get the tidal potential we find $\phi_{\rm tide} = -\dfrac{3GM_2a^2}{2r^3}$ at the surface at point A.
- There is no tidal potential at point B, so the height h of the tide is given by $\phi_{\rm tide} = -gh$, where the gravity $g = \dfrac{GM_1}{a^2}$.
- Eliminating g, the height of tides is $h = \dfrac{3M_2a^4}{2M_1r^3} = 0.5$ m.
- Can be less due to limited flow (Mediterranean), or much more in estuaries (e.g. St. Malo and Bristol Channel). Can be complicated due to amplification by resonances (e.g. Solent).
- Tide from the Sun about half that from the Moon — explains “spring” and “neap” tides.
- The Moon now keeps the same face towards us — its initial additional rotation was dissipated against Earth tides of 16 m. It was once much closer to us, and it is still receding. (The rotation of the Earth is also slowing down.)