

- A **rigid body** is a special case of the many-particle system we have already studied, in which all the interparticle distances are fixed.
- The **location** of all the particles in the body can therefore be described by 6 coordinates: the position \mathbf{R} of the centre of mass, and 3 angles (θ, ϕ, χ) (the Euler angles), which describe the **attitude** of the body with respect to the spatial (x, y, z) -axes. We will define these angles later.

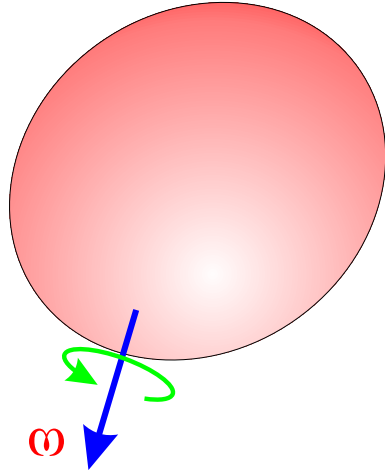
- More importantly, the **velocity** of any particle in the body is determined by the velocity \mathbf{v} of the CoM and a single angular velocity $\boldsymbol{\omega}$.

- The **dynamics** of the rigid body is then determined by its total mass and the **inertia tensor** that relates the angular momentum \mathbf{J} to the angular velocity $\boldsymbol{\omega}$.

- This inertia tensor is the generalisation of the **moment of inertia** of a body rotating about a fixed axis.

- For a body spinning about a fixed axis (say $\hat{\mathbf{e}}_z$), the moment of inertia $I = \sum m(x^2 + y^2)$ relates the angular momentum J to the angular velocity ω via $J = I\omega$.

- The kinetic energy is also related to the angular velocity via $T = \frac{1}{2}I\omega^2 = \frac{1}{2}J\omega$.



Basic equations: (See Section 5.1 of Handout.)

1) $M\ddot{R} = F_0$; centre of mass moves like particle of mass M under action of the resultant external force. Here we will only consider motion in the zero-momentum (CoM) frame.

2) $\dot{J} = G_0$; the rate of change of angular momentum is equal to the total external couple.

- The angular velocity ω determines the velocity of a

- particle at position r in the CoM frame: $v = \omega \times r$.

- The angular momentum J is therefore:

$$J = \sum r \times p = \sum r \times m(\omega \times r) = \sum m r^2 \omega - \sum m r(\omega \cdot r)$$

Therefore J is proportional to ω but not necessarily parallel to ω ,

we call this a **tensor** relationship $J = I \cdot \omega$ or $J_i = I_{ij} \omega_j$.

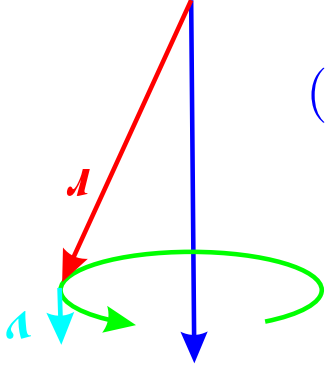
- Using matrix notation, we can write this in detail:

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \sum m(y^2 + z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2 + z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2 + y^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

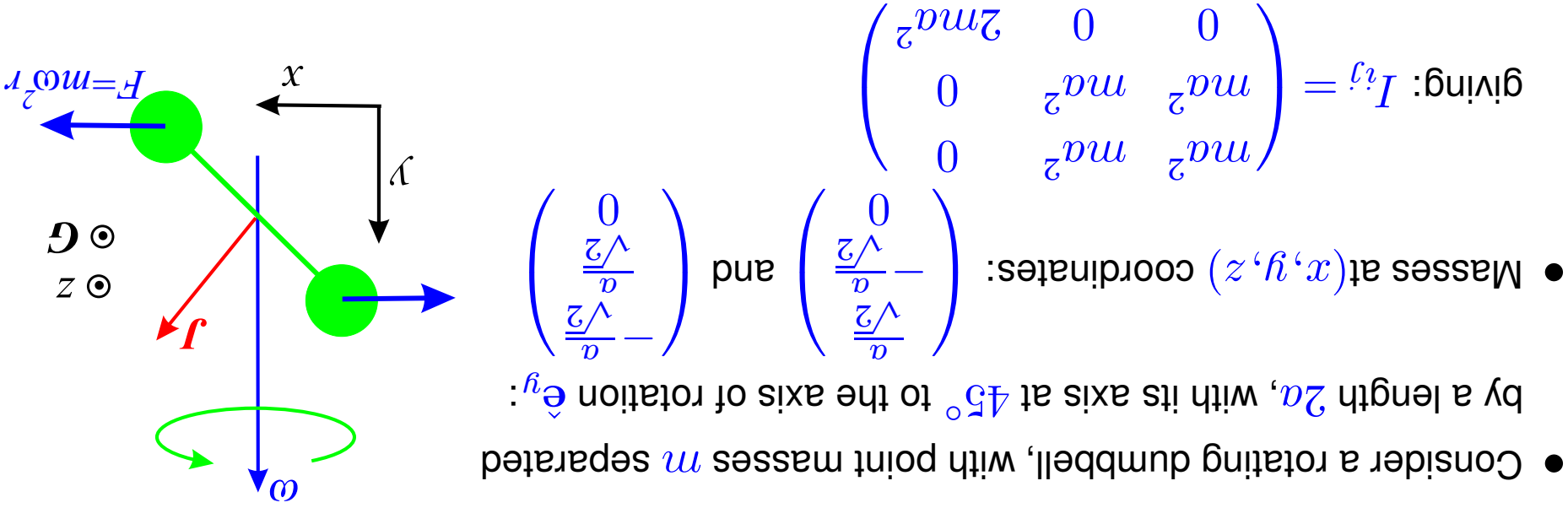
- I is the **Moment of Inertia tensor** — here written as a matrix.

- The energy can also be written in terms of the inertia tensor

$$T = \sum \frac{1}{2} m (\omega \times r) \cdot (\omega \times r) = \sum \frac{1}{2} m \omega \cdot r \times (\omega \times r) = \frac{1}{2} \omega \cdot J$$



- Consider a rotating dumbbell, with point masses m separated by a length $2a$, with its axis at 45° to the axis of rotation \hat{e}_y :



- The angular momentum is then given by

$$\mathbf{J} = \mathbf{I} \cdot \boldsymbol{\omega} = \begin{pmatrix} ma^2 & 0 & 0 \\ ma^2 & ma^2 & 0 \\ 0 & 0 & 2ma^2 \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ ma^2\omega \\ ma^2\omega \end{pmatrix}$$

which is not parallel to $\boldsymbol{\omega}$.

As the dumbbell rotates, \mathbf{J} precesses around with it, which means a couple must be applied to the dumbbell for it to rotate about the y -axis: $\mathbf{G} = \mathbf{J} \times \boldsymbol{\omega} = -ma^2\omega^2 \hat{e}_z$

- The same couple may be derived by viewing the dumbbell in a frame rotating at $\boldsymbol{\omega}$ with respect to the inertial centre of mass frame and calculating the effect of centrifugal forces.
- This couple precesses with angular velocity $\boldsymbol{\omega}$, just as \mathbf{J} — giving a time-varying couple on the bearing that supports the dumbbell, causing ‘juddering’.

(See Section 5.1 of Handout.)

- The Moment of Inertia tensor is symmetric. There are therefore 3 real eigenvalues $\{I_1, I_2, I_3\}$ and 3 mutually perpendicular eigenvectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$.

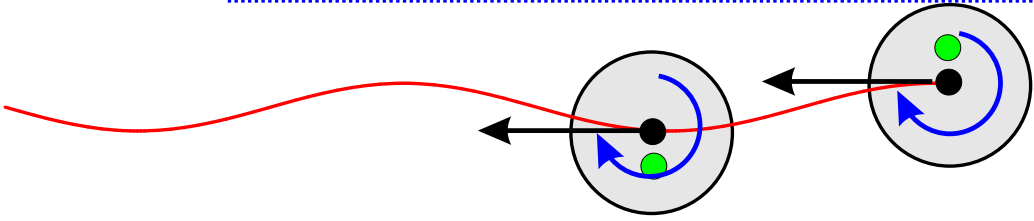
- With respect to these eigenvector axes, $I' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$ or in matrix form $J = \begin{pmatrix} I_1\omega_1 & & \\ & I_2\omega_2 & \\ & & I_3\omega_3 \end{pmatrix}$
- We can write $J = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3$, or in matrix form $J = \begin{pmatrix} I_1\omega_1 & & \\ & I_2\omega_2 & \\ & & I_3\omega_3 \end{pmatrix}$
- The kinetic energy is $T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$.

- $\{I_1, I_2, I_3\}$ are called the **Principal Moments of Inertia** and are the moments of inertia in the ordinary sense about the eigenvector axes; the eigenvector axes \hat{e}_1, \hat{e}_2 and \hat{e}_3 are called the **Principal Axes**.

- In ω -space, a surface of constant $T = T(\omega)$ is an ellipsoid, which is fixed to the body—the **Inertia Ellipsoid**; the axes of the ellipsoid have length $\propto I_i^{-1/2}$ **smallest** I_i corresponds to **longest axis**).

- In ω -space, the gradient $\Delta_\omega T = \left(\frac{\partial T}{\partial \omega_1}, \frac{\partial T}{\partial \omega_2}, \frac{\partial T}{\partial \omega_3} \right) = (I_1\omega_1, I_2\omega_2, I_3\omega_3) \equiv J$, i.e. J is perpendicular to the surface of constant T at ω .

- For an object to rotate smoothly about an axis, it must not only be 'statically' balanced (axis passes through the CoM), but also 'dynamically' balanced.
- This requires that it is set up so that ω is parallel to J .
- This occurs when the axis lies along a 'principal axis'.

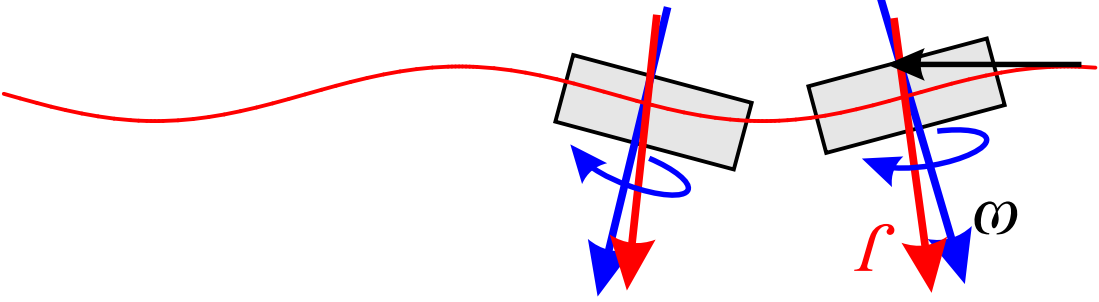


- **Case 1.** Static imbalance.

Centre of gravity is not on the axle — Causes **bouncing** of the wheel.

- **Case 2.** Dynamic imbalance.

The axle is not a principal axis.
Causes **wobbling** of the wheel.



- Words and diagrams courtesy of ATS Ltd. circa 1986. . . .

(see Sections 5.1.1 and 5.1.2 of Handout.)

- The principal axes are fixed in the body and must be perpendicular to each other, **whatever the shape of the body.**

- As far as their rotational properties are concerned, bodies come in three types:

1. Spherical Tops; all I_i 's equal, e.g. sphere, cube. For a spherical top $\mathbf{J} = I\boldsymbol{\omega}$ with a **scalar I** , i.e. $\mathbf{J} \parallel \boldsymbol{\omega}$ in this case.

Rotationally, the body is isotropic, with the same I about any axis.

2. Symmetrical Tops; $I_1 = I_2 \neq I_3$, e.g. many simple molecules. Subgroups are oblate (lens or disc shaped) and prolate (cigar shaped). $\hat{\mathbf{e}}_3$ axis unique; $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ anywhere in plane perpendicular to $\hat{\mathbf{e}}_3$.

3. Asymmetrical Tops; all three I 's different. Principal axes unique.

- No one I_i can be larger than the sum of the other two. Thus, with respect to x, y, z along principal axes) $I_1 + I_2 = \sum m(y^2 + z^2 + x^2 + z^2) = I_3 + 2 \sum m z^2 \geq I_3$

- A special case is a flat lamina, $z = 0$, for which $I_1 + I_2 = I_3$, which is known as the

Theorem of Perpendicular Axes.

- Example of a lamina: Disc, mass M , radius a :

$$I_3 = \sum m(x_2^2 + y_2^2) = \int_0^a 2\pi r dr \frac{M}{\pi a^2} r^2 = \frac{1}{2} M a^2.$$

$$I_1 = I_2 = \text{moment of inertia about a diameter} = \frac{1}{4} M a^2.$$

- Theorem of Parallel axes. For I about an axis **not** through centre of mass, say \mathbf{a} away and parallel to a principal axis.

$$I = \sum m(\mathbf{r} + \mathbf{a}) \cdot (\mathbf{r} + \mathbf{a}) = I_0 + M a^2 + 2 \left(\sum m \mathbf{r} \right) \cdot \mathbf{a} = I_0 + M a^2 \text{ since } \sum m \mathbf{r} \text{ is zero when } \mathbf{r} \text{ is with respect to CoM.}$$

[Vectors \mathbf{r} etc here are all taken as 2-D projections in plane perpendicular to I axis.]

- Other useful moments of inertia:

$$\text{- Sphere, mass } M, \text{ radius } a: I = \frac{5}{2} M a^2$$

$$\text{- Thin rod, mass } M, \text{ length } l: I = \frac{1}{12} M l^2$$

$$\text{- Rod, mass } M, \text{ length } l, \text{ radius } a: I_1 = I_2 = M \left(\frac{1}{12} a^2 + \frac{1}{12} l^2 \right), I_3 = \frac{1}{2} M a^2$$

(See Section 5.2 of Handout.)

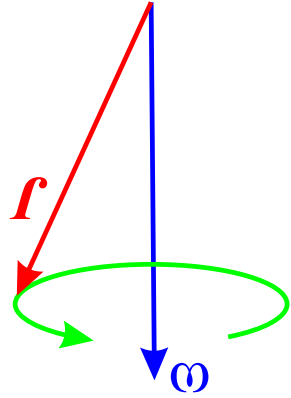
- Suppose $F = 0$, $G = 0$, i.e. an isolated body is spinning freely. The angular momentum J is constant. What about ω ? The answer is easy if J and ω are along a principal axis.
- If J and ω do not lie along a principal axis the motion is **much** more complicated: the direction of ω varies both in space and with respect to the body; for asymmetrical tops, the magnitude of ω varies too. These variations are called **free precession**, to distinguish them from the **forced** precession produced by an external couple.
- The problem of handling free precession may be treated in three quite different ways:
 1. **Euler's equations**, which are the equations of motion in a coordinate frame moving with the body. This is the easiest approach to get some useful results, but it is slightly awkward to see what is going on in space axes.
 2. **Lagrange's approach**, which gives the equations of motion with respect to fixed axes. This is straightforward once the Euler angles are defined.
 3. **Poinso's geometrical approach**, which is either deeply insightful or mind-numbing, depending on the beholder.

- The first two of these generalise readily to include forced motion, i.e. with external couples.
- We will start with Euler's equations, then briefly illustrate Poinso's construction and finally examine the Lagrange approach (probably the most systematic).

(See section 5.4 of Handout.)

Euler's equations result from consideration of the change of angular momentum in the **body frame** S . The equations of motion are relatively simple in this frame, but the body axes are rotating with respect to the inertial frame S_0 . Suppose there is an external couple G .

- In the inertial frame S_0 we have $\left[\frac{dJ}{dt} \right]_{S_0} = G$. The rates of change of J in frames S and S_0 are related^a by $\left[\frac{dJ}{dt} \right]_{S_0} = \left[\frac{dJ}{dt} \right]_S + \omega \times J$



- We therefore have the vector equation $G = \left[\frac{dJ}{dt} \right]_S + \omega \times J$

- ω in frame S is $(\omega_1, \omega_2, \omega_3)$ and J is $(I_1\omega_1, I_2\omega_2, I_3\omega_3)$, hence

$$G_1 = I_1\dot{\omega}_1 + \omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_2 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2 \omega_3 \text{ and } G_2, G_3 \text{ similarly..}$$

- **Euler's equations** for the motion of a rigid body written out in full are

$$G_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2 \omega_3$$

$$G_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3 \omega_1$$

$$G_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1 \omega_2$$

^aTo see this easily note that: (1) if J is not changing in S , then in S_0 the J vector just rotates around ω ; (2) if the body is not rotating, the rates of change of J in S and S_0 are the same. The two terms are linear in J , hence result.

(See section 5.4.1 of Handout.)

- We first study a simple case — the symmetric top with $I_1 = I_2 \neq I_3$.

- Euler's equations are $I_1 \dot{\omega}_1 = (I_1 - I_3) \omega_2 \omega_3$; $I_1 \dot{\omega}_2 = (I_3 - I_1) \omega_1 \omega_3$; $I_3 \dot{\omega}_3 = 0$.

- This implies that ω_3 is constant. Defining the **body frequency** $\Omega_b \equiv \frac{I_1 - I_3}{I_1} \omega_3$, we see

$$\text{that } \dot{\omega}_1 = \Omega_b \omega_2 \text{ and } \dot{\omega}_2 = -\Omega_b \omega_1 \Rightarrow \ddot{\omega}_1 + \Omega_b^2 \omega_1 = 0$$

- The general solution of these two coupled ODEs is

$$\omega_1 = A \sin(\Omega_b t + \phi_0), \omega_2 = A \cos(\Omega_b t + \phi_0) \text{ so that,}$$

in the body frame, ω precesses around the 3-axis with

- angular velocity Ω_b , which is known as the **body frequency**.

- The body frequency Ω_b can either be the same sign as ω_3

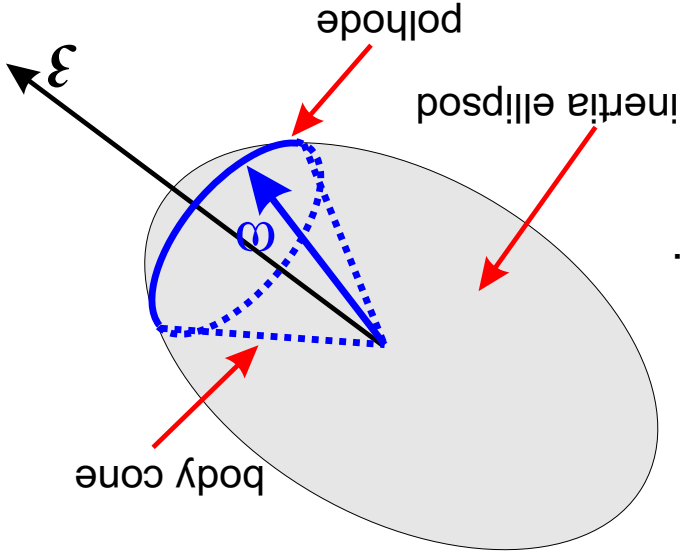
(prolate inertia ellipsoids ($I_1 > I_3$)), or have the opposite

sign ($I_3 > I_1$).

- The surface traced out by the angular velocity vector is known as the **body cone**.

- The curve traced out on the inertia ellipsoid by the angular velocity vector is known

as the **polhode**.



- (Section 5.4.1.) N.B. the case illustrated is **oblate** (i.e. $I_3 > I_1$).^a

- Consider axes fixed in the body — ω precesses about the 3-axis at the body frequency $\Omega_b = \frac{I_1}{I_1 - I_3} \omega_3$.

- f is not parallel to ω but still precesses about the 3-axis at the body frequency.

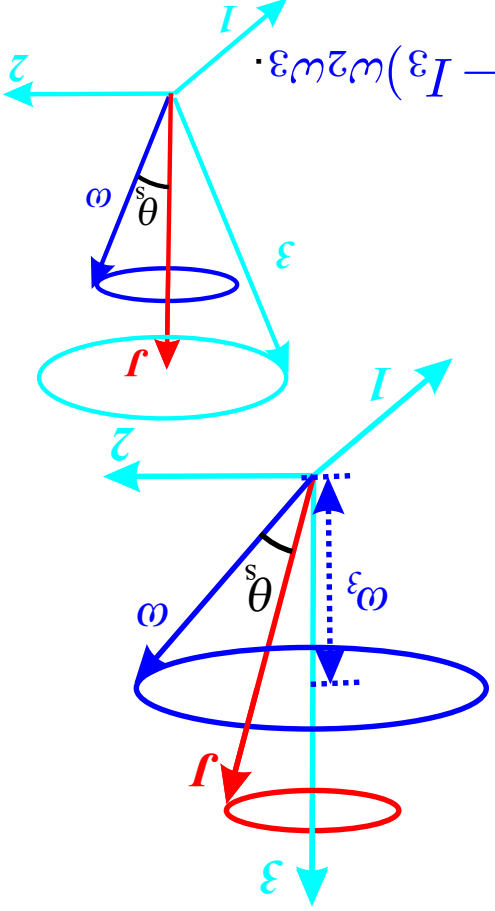
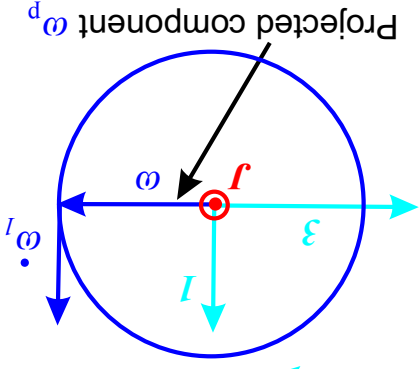
- Meanwhile, back in the space axes f is fixed and everything is rotating about ω .

- To find the space frequency, look down the f vector when $\omega_1 = 0$. The projected component of ω shown is $\omega_p \equiv |\omega| \sin \theta_s = \frac{|f \times \omega|}{|f|}$.

- At this moment $f \times \omega$ is in the 1-direction and has magnitude $= (I_1 - I_3) \omega_2 \omega_3$.

- Rate of precession (space frequency) is given by $\Omega_s = \frac{\omega_p}{\omega_1}$.

- But $I_1 \omega_1 = (I_1 - I_3) \omega_2 \omega_3$, so the space frequency is $\Omega_s = \frac{f}{I_1}$



^aRemember that the small axis of the inertia ellipsoid corresponds to the **largest** principal moment of inertia.

(See section 5.3.1 of Handout.)

- Two frequencies are important:

1. The 'Space Frequency' $\Omega_s \equiv$ the rotation speed of the plane of ω and \hat{e}_3 about J . It is

the 'free precession' speed as seen by an inertial observer. We see from the above

argument that it has the value $\Omega_s = J/I_1$ so that it is always in the sense of J and is

of the same order of magnitude as the rotation speed.

2. The 'Body Frequency' $\Omega_b \equiv$ the rate at which ω describes its cone about the 3-axis.

From the Euler equations we see that $\Omega_b = \frac{I_1}{I_1 - I_3} \omega_3$ This can take either sign wrt

to Ω_s and can be very small if the body is nearly spherical (in terms of its principal

moments of inertia).

- Two angles are important:

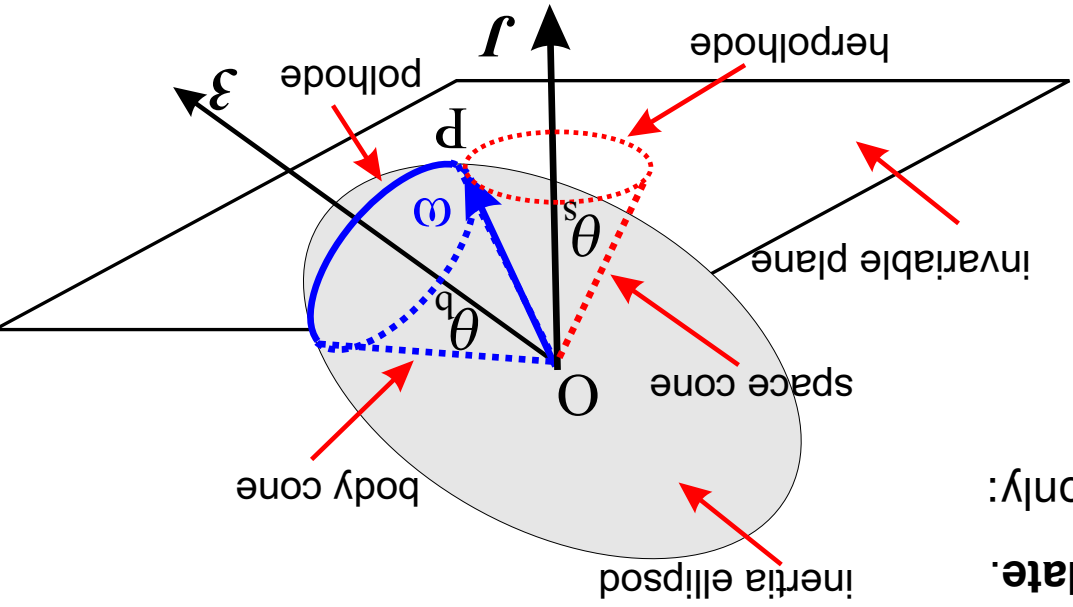
1. The 'Space Cone' of half-angle $\theta_s \equiv$ the cone swept out by ω as it precesses about J . It

is the angle of 'free precession' seen by an inertial observer.

2. The 'Body Cone' $\theta_b \equiv$ the cone swept out by ω as it precesses about \hat{e}_3 .

- We can illustrate the relation between these angles and frequencies further using **Poinsot's**

construction.



- (Section 5.3.) N.B. case illustrated is **prolate**.
- For free precession (no external couple) only: constant J and $T = \frac{1}{2}\omega \cdot J$ implies that the component of ω in the J -direction remains constant.
- As the ω -vector varies, it keeps its tip P in a plane perpendicular to J ; the plane is called the **invariant plane**.
- Since J is perpendicular to the surface of the ellipsoid at P , it follows that the ellipsoid is **tangential** to the invariant plane at P .
- The crucial step in Poinson's argument is to say that, since the instantaneous motion is a rotation about OP , P is instantaneously at rest, and the ellipsoid therefore **rolls**, rather than slides, on the invariant plane.

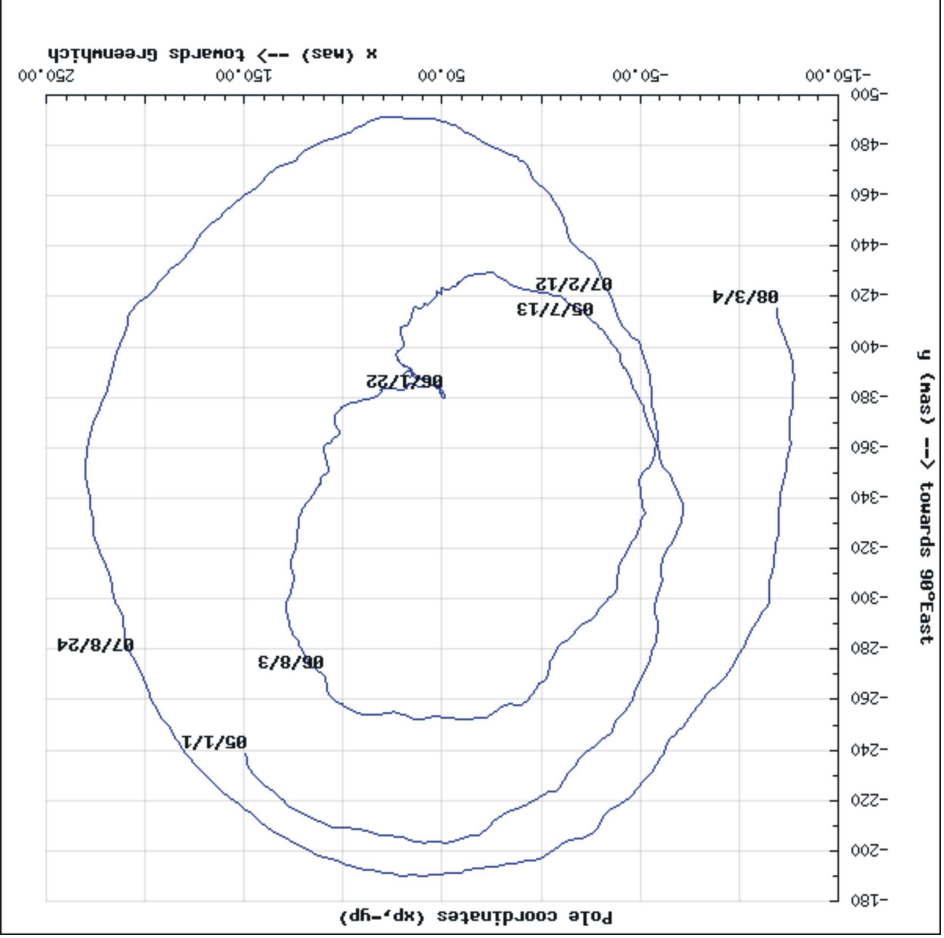
- The locus of the tip of the ω vector on the plane is called the **herpolhode**.
- You can calculate the space and body frequencies from geometric considerations, e.g. $\Omega_b \sin \theta_b = \Omega_s \sin \theta_s$ follows from the condition of no slipping between the space and body cones as they roll around ω . But that's as far as we'll go here...

- The Earth is in force-free motion, but is slightly oblate and J is slightly inclined to the symmetry axis.
- The Earth's oblateness: $(I_3 - I_1)/I_1 \equiv \beta \approx \frac{\Omega^2 R^2}{g} \approx \frac{1}{300}$. The space-cone angle θ_s is negative for this case. Angles are tiny: $\theta_s \approx -\beta\theta_p \approx -\theta_p/300$.

- Space-cone tiny and *inside* the body-cone, which swings round it each day.
- In ≈ 300 days, ω moves in a cone round \hat{e}_3 , giving a latitude change: **Polar wander**, or **Chandler wobble**. The $J - \hat{e}_3$ angle is about 0.1 arcsec (a few metres at the surface).

affect the arguments.

- The period is actually 427-days, and irregular, the Earth is not rigid and its deformations



Chandler wobble. The $J - \hat{e}_3$ angle is about 0.1 arcsec (a few metres at the surface).

(See sections 5.3.2 and 5.4.2 of Handout.)

- Triaxial body has all principal moments of inertia different. Assume $I_1 < I_2 < I_3$.
- Free rotation is complicated, but we have the conservation laws:

J is conserved: $(I_1\omega_1, I_2\omega_2, I_3\omega_3)$;

T is conserved: $\frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$.

- **If the body spins about the 1-axis** it can't change ω at constant J without **decreasing** its energy. It is therefore STABLE (though read on...).

- **If the body spins about the 3-axis** it can't change ω at constant J without **increasing** its energy. It is therefore STABLE.

- **If the body spins about the 2-axis** it can change ω at constant J in many ways whilst conserving energy. It is therefore UNSTABLE, and the polhode will make large excursions around the inertia ellipsoid.

- To see this we can use Euler's equations for ω_1, ω_2 :
$$\dot{\omega}_1 = \frac{\omega_2}{I_2} \frac{d\omega_1}{\omega_2} = \frac{I_1 \omega_1 (I_3 - I_1)}{I_2 \omega_2 (I_2 - I_3)}$$

- Integrating, we find the locus is a conic section:
$$\frac{I_2(I_3 - I_2)}{\omega_1^2} + \frac{I_1(I_3 - I_1)}{\omega_2^2} = \text{constant}$$

- The locus can be (as here) an ellipse (stable) or a hyperbola (unstable), depending on the signs of $I_i - I_j$.

- Real objects are neither perfectly rigid nor perfectly elastic.
- During free precession, whilst the angular momentum is conserved, the centrifugal forces change as ω axis moves. The object therefore deforms, and loses energy.
- J can move w.r.t principal axes, but J^2 is constant, so as T is decreasing, the energy ellipsoid shrinks.
- To get the minimum energy for a certain J — align J with axis with largest I , i.e. the major axis.
- If J is not aligned with the major axis, energy can still be lost.
- **Major axis theorem** 'Any freely-rotating body that is not perfectly rigid, will lose kinetic energy and, whilst its angular momentum remains constant in magnitude and direction in space, its angular velocity vector moves with respect to the body coordinates until the body is rotating about its major axis'.

- **Celestial objects.** Freely-rotating celestial objects usually rotate about their major axes – asteroids, planets like the Earth, stars or spiral galaxies.

- **Chandler Wobble of the Earth.** The Earth is not rigid — free precession decays in about 68 years. The Earth's precession must be continuously excited/driven. The principal cause of the Chandler wobble^a is fluctuating pressure on the bottom of the ocean, caused by temperature and salinity changes and wind-driven changes in the circulation of the oceans.

- **Explorer 1 Satellite:** In 1958, a few months after Sputnik 1, the US launched Explorer 1. It was a long cylindrical object, with flexible radio antennae protruding from the sides.

- If the orientation of a satellite matters (e.g. for directing antennae or solar panels) it must be stabilised — a typical satellite with a solar panel will rotate 90° degrees in an hour from rest due to radiation pressure alone.

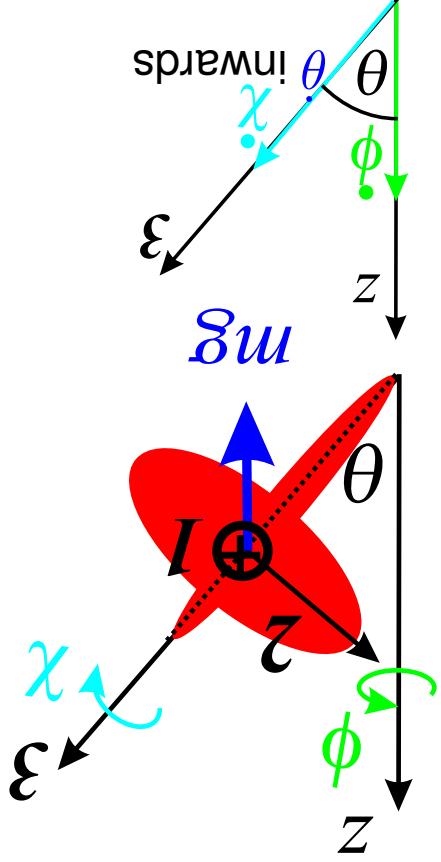
- They 'stabilised' it by spinning about its length — the minor axis. During first orbit the angular momentum vector moved to the major axis (perpendicular to the middle of the satellite). It spent the rest of its mission cart-wheeling through space. . .

- **Lewis Satellite** was lost in August 1997 shortly after launch. . .

^a'The excitation of the Chandler wobble' R.S. Gross, Geophysical Research Letters **27**, 2329-2332 (2000).

(See section 5.5 of Handout.)

- The most direct and systematic treatment of the motion with respect to fixed axes is via Lagrangian Mechanics (see TP1 next year). Here we obtain the same equations by inspection.
- We consider the motion of a symmetrical top, at first isolated, and then supported at its base under gravity (take $I_1 = I_2 \neq I_3$).
- Let (θ, ϕ) be the spherical polar coordinates of the symmetry 3-axis and χ the angle of rotation of the top about the 3-axis .
- (θ, ϕ, χ) are the **Euler angles** and are suitable coordinates to describe the attitude of the body.
- For a symmetric top we can make a convenient choice of axes: let the 1-axis be (instantaneously) horizontal; the z -axis is in the 2-3 plane. At this moment of time $\omega = (\omega_1, \omega_2, \omega_3) = (\dot{\theta}, \dot{\phi} \sin \theta, \dot{\chi} + \dot{\phi} \cos \theta)$ χ is measured wrt the moving z -3 plane.
- The angular momentum in body axes is given by $\mathbf{J} = (I_1 \dot{\theta}, I_1 \dot{\phi} \sin \theta, I_3 (\dot{\chi} + \dot{\phi} \cos \theta))$
- The symmetry $I_1 = I_2$ provides a considerable simplification; in general we have to resolve $\dot{\theta}$ and $\dot{\phi} \sin \theta$ onto the 1- and 2-axes which are rotated by the angle χ about the 3-axis.



(See section 5.5 of Handout.)

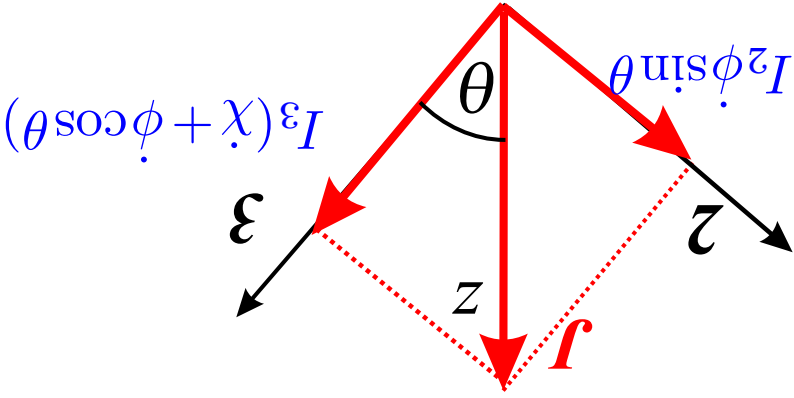
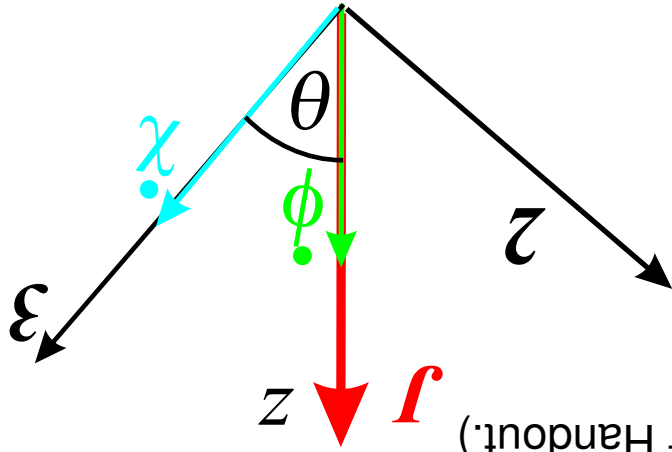
- The moment of inertia I_1 is taken with respect to the stationary point within the body (i.e. the C of Mass for the isolated body, base if supported under gravity). The gravitational couple \mathbf{G} , if present, is in the 1-direction.
(See section 5.5 of Handout.)
- There are three constants of motion, giving three equations

1. $J_3 = I_3(\dot{\chi} + \dot{\phi} \cos \theta)$ is constant. This follows from Euler's equation for $\dot{\omega}_3$ with $G_3 = 0$ and $I_1 = I_2$, implying $J_3 = I_3 \dot{\omega}_3 = 0$.
2. $J_z = J_3 \cos \theta + J_2 \sin \theta = J_3 \cos \theta + I_1 \dot{\phi} \sin^2 \theta$ is constant. This is because $G_z = 0$.
3. The total energy $E = T + U$ is constant.

The first two equations enable us to express $\dot{\phi}$ and $\dot{\chi}$ in terms of the angular momentum constants J_z and J_3 and the angle θ :

$$\dot{\phi} = \frac{J_z - J_3 \cos \theta}{I_1 \sin^2 \theta} \quad \dot{\chi} = \frac{J_3}{I_3} - \dot{\phi} \cos \theta = \frac{J_3}{I_3} + \frac{J_3 \cos^2 \theta - J_z \cos \theta}{I_1 \sin^2 \theta}$$

(See section 5.5.1 of Handout.)



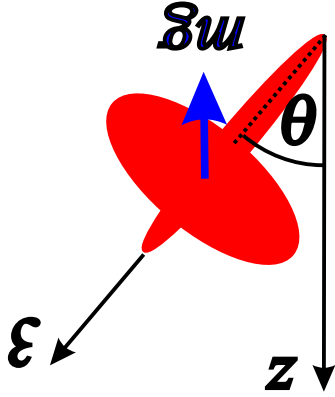
- Suppose that the body is isolated in free space. Take the \mathbf{J} direction as defining the z -axis.
- Then $J_z = J$ since the total \mathbf{J} is along z -axis and $J_z = J \cos \theta$. Therefore

$$\dot{\phi} = \frac{J(1 - \cos^2 \theta)}{I_1 \sin^2 \theta} = \frac{J}{I_1} = \dot{\Omega}_s,$$

$$\dot{\chi} = \frac{J \cos \theta}{I_3} - \frac{I_3}{J \cos \theta} \dot{\Omega}_s = I_3 \omega_3 \left(\frac{1}{I_1} - \frac{1}{I_3} \right) = \frac{I_1 - I_3}{I_1 I_3} \omega_3 = \dot{\Omega}_b$$

- The above constitutes another derivation of the space and body frequencies $\dot{\Omega}_s$ and $\dot{\Omega}_b$ (there is a third via Poinsot's construction using simple geometry, which is in the handout).
- The relationships between the different treatments are not at all obvious; text books can be baffling since few books give more than one treatment.

- (See section 5.5.2 of Handout.) Hereafter we consider the body supported at its base, so that there is a couple due to gravity. I_1 is now about the support, which is at h from the C of Mass. Conservation of J_z and J_3 give $\dot{\phi}$ and $\dot{\chi}$ as known functions of θ . Once θ is known as a function of time, ϕ and χ may in principle be found by integration.



- The angle θ may be found from the energy equation:

$$E = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2}I_3(\dot{\chi} + \dot{\phi} \cos \theta)^2 + mgh \cos \theta = \text{constant}$$

- Substituting for $\dot{\phi}$ and $\dot{\chi}$ gives $E = \frac{1}{2}I_1\dot{\theta}^2 + \frac{(J_z - J_3 \cos \theta)^2}{2I_1 \sin^2 \theta} + mgh \cos \theta + \frac{J_3^2}{2I_3} \underbrace{U^{\text{eff}}(\theta)}$

- The problem is now solved in principle, since θ is a known function of t and may be

integrated to give $\theta(t)$; ϕ and χ then follow.

- The integrations involve elliptic integrals and can be found in standard reference books.

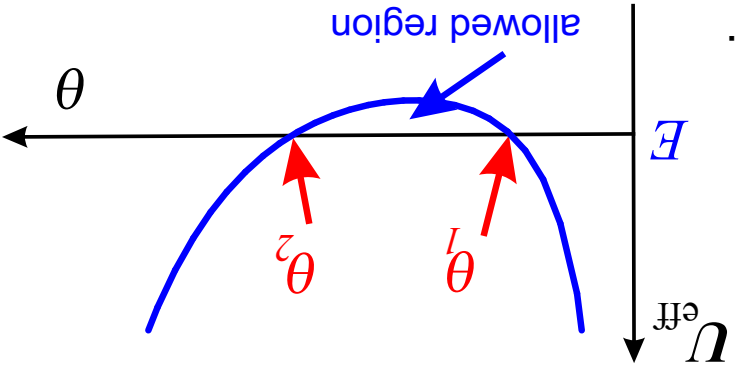
- The important physical results can be seen without recourse to the full mathematics by

treating the θ motion as oscillation in an effective potential U^{eff} as in the treatment of orbits.

THE MOTION OF A GYROSCOPE

(See section 5.5.2 of Handout.)

- Except in the special case $J_z = J_3$ ('sleeping top'), the effective potential has the form sketched.
- For energies E above the minimum in U_{eff} there is an allowed region of θ where $\dot{\theta}^2 \geq 0$.



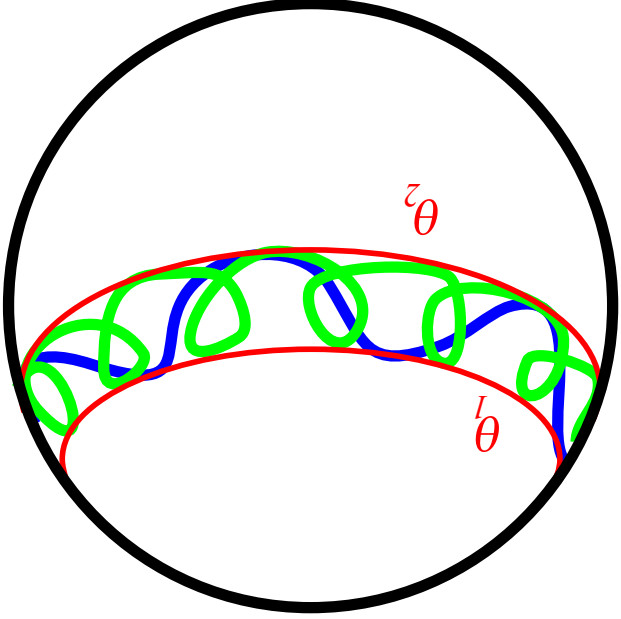
- In steady precession θ is at the 'equilibrium' position (minimum of U_{eff}) and constant ($E = U_0$). Conservation of the angular momenta imply that $\dot{\phi}$ and $\dot{\chi}$ are constant.

- The motion thus consists of steady rotation by $\dot{\chi}$ about the symmetry axis, with the axis itself steadily precessing by $\dot{\phi}$ at constant θ to the vertical.

- If E is slightly larger than the minimum of U_{eff} , then the θ motion can be treated as approximate SHM by Taylor expansion of U_{eff} about the minimum.
- The oscillations of θ give **nutaton**, i.e. oscillations in $\dot{\chi}$ and $\dot{\phi}$ about the steady precession values.

- For energies substantially higher than U_0 , the nutation can be quite complicated.

^a(with apologies for the artistic standard...)



(See section 5.5.3 of Handout.)

- The condition for steady precession is $dU_{\text{eff}}/d\theta = 0$, which gives an equation for θ in terms of the constants J_z and J_3 . After some algebra, it may be rewritten as

$$\dot{\phi}^2 I_1 \cos\theta - \dot{\phi} J_3 + mgh = 0.$$

- This gives us the steady precession speed $\dot{\phi}$ as a function of inclination θ . It is a quadratic, with solutions

$$\dot{\phi} = \frac{J_3 \pm \sqrt{J_3^2 - 4I_1 mgh \cos\theta}}{2I_1 \cos\theta}$$

This shows that, if $\cos\theta$ is positive (i.e. the top is standing above its base and not hanging below it), $\dot{\phi}$ is impossible unless $J_3^2 \geq 4I_1 mgh \cos\theta$. Thus steady precession requires the top to be spinning fast enough. In the 'gyroscopic limit' J is very large from rapid rotation about the symmetry axis and $J_3^2 \gg mgh I_1$.

- The quadratic for $\dot{\phi}$ shows there are two possible precession frequencies for given θ ; in the gyroscopic limit they are

$$\dot{\phi} \approx mgh/J_3, \text{ independent of } \theta; \text{ this is the 'slow precession', as derived in Part IA.}$$

$$\dot{\phi} \approx J_3/(I_1 \cos\theta), \text{ independent of } \theta; \text{ this is the 'free precession' as derived above:}$$

$$J \text{ is entirely in the } z\text{-direction and } \dot{\phi} \equiv \Omega_s.$$

(See section 5.5.4 of Handout.)

- The analysis of nutation about precession at general θ , even in the gyroscopic limit, is algebraically laborious. The case of nutation of a horizontal gyroscope is reasonably straightforward.

- Nutation of a gyroscope, with axis horizontal and supported at one end.

Put $\theta = \pi/2 + \epsilon$. For small ϵ , $\cos\theta \approx -\epsilon$, $\sin\theta \approx 1 - \epsilon^2/2$.

Then $U^{\text{eff}}(\theta) = \text{constant} + \epsilon \left(\frac{J_z J_3}{I_1} - mgh \right) + \epsilon^2 \left(\frac{J_2^2}{2I_1} + \frac{J_3^2}{2I_1} \right) + \dots$ for power series

expansion in ϵ . The term $\propto \epsilon$ is zero at θ_0 ; therefore $J_z = \frac{mgh I_1}{J_3}$; $\dot{\phi} = \frac{J_3}{mgh}$.

The gyroscope condition is $J_2^2 \gg mgh I_1$ and hence $J_2^2 \gg J_3^2$. The ϵ^2 -term gives the

'restoring force' term in U^{eff} and hence the equation of motion as follows:

$$U^{\text{eff}} = \text{constant} + \epsilon^2 \frac{J_2^2}{2I_1} \Leftrightarrow I_1 \ddot{\epsilon} + \frac{J_3^2}{I_1} \epsilon = 0. \text{ This gives SHM in } \epsilon \text{ at } \Omega \equiv \Omega_s = J_3/I_1.$$

- Another limiting case that can be handled relatively easily is the conical pendulum, i.e. as

above but with $I_3 = 0$. There is no gyroscopic action and the pendulum can only hang

below the support (for motion at steady θ). Note that compared to conventional treatments, θ is measured from the upward rather than downward vertical.

(See section 6 of Handout.)

- For a general Newtonian system, the equations of motion take the form
$$F = \left(\text{variables}, \frac{d}{dt} \text{variables} \right)$$

- These equations are often very complicated (e.g. rigid body motion).

- However, we are often interested in **small displacements** of a system from **equilibrium**.

- If we expand the variables (by long tradition we denote them as $\{q_i\}$ or $\{\bar{b}\}$) about their

equilibrium values \mathbf{q}^{eq} : variables $\mathbf{b} \equiv \mathbf{q}^{\text{eq}} + \delta \mathbf{b}$ we obtain the approximate equations

$$\ddot{\mathbf{q}} = \left. \frac{\partial F}{\partial \mathbf{q}} \right|_{\text{eq}} \cdot \delta \mathbf{b} + \left. \frac{\partial \dot{\mathbf{q}}}{\partial \dot{\mathbf{q}}} \right|_{\text{eq}} \cdot \dot{\mathbf{q}} \quad \text{i.e. linear equations}$$

- In general, **small displacements** about equilibrium lead to **linear equations**.

- This in turn leads to the **superposition of solutions**. [i.e. we can analyse the problem in pieces and add them all up afterwards. . . .]

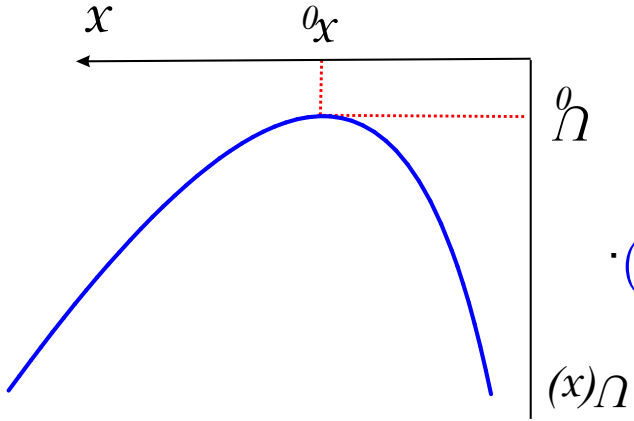
REMINDER: PREAMBLE TO NORMAL MODES

- Particle in a potential well $U(x)$.

- Total energy is conserved: $E = T + U = \frac{1}{2}m\dot{x}^2 + U(x)$.

- Energy method: $\frac{dE}{dt} = 0 \Leftrightarrow \dot{x}(m\ddot{x} + \frac{dU}{dx}) = 0$.

- The resulting equation of motion $m\ddot{x} + \frac{dU}{dx} = 0$ may well be nonlinear.



- Suppose there is an equilibrium position at x_0 where $\frac{dU}{dx} = 0$.

- To study small oscillations about x_0 , expand $U(x)$ in a Taylor series:

$$U(x) = U_0 + \frac{1}{2} \left. \frac{d^2U}{dx^2} \right|_{x_0} (x - x_0)^2 + \dots$$

- Defining the small displacement $\xi \equiv x - x_0$ and denoting $\left. \frac{d^2U}{dx^2} \right|_{x_0} \equiv U''_0$ (i.e. a constant)

$$m\ddot{\xi} + U''_0 \xi = 0$$

we get the linear equation of motion

which is SHM at angular frequency $\omega^2 = \frac{U''_0}{m}$.

- We will study the free (unforced) small oscillations of dynamical systems about equilibrium. The system will oscillate, but a system with many dynamical variables can oscillate in many different ways. The important concept is that of the **normal mode** of oscillation.
- Definition of a normal mode:

A **normal mode** of a system is an oscillation that has a **single frequency**.

- In a normal mode all parts of the system share the same periodic time dependence.

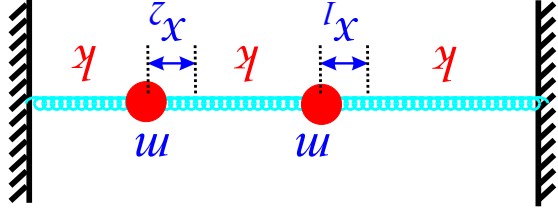
- All the more general free oscillations of the system can be expressed in terms of these

simple normal modes.

- It is worth emphasising once more that we are considering **free oscillations** in the absence

of external forces.

- Example: a system of two masses and 3 springs:
 (x_1, x_2) are the displacements of the masses



from equilibrium. The masses are both m and the spring constants are all k .

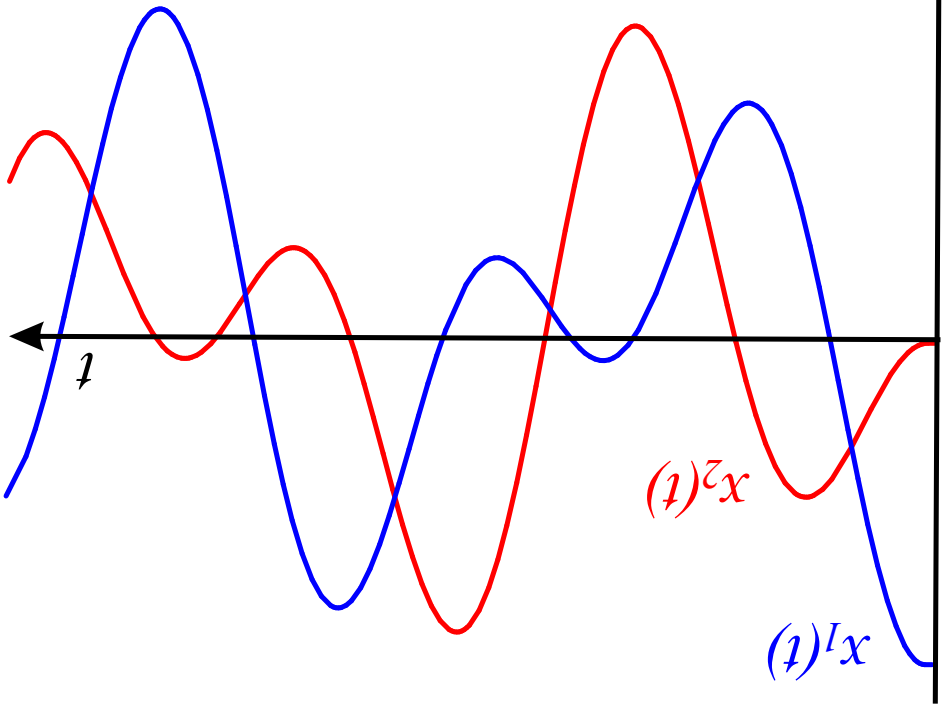
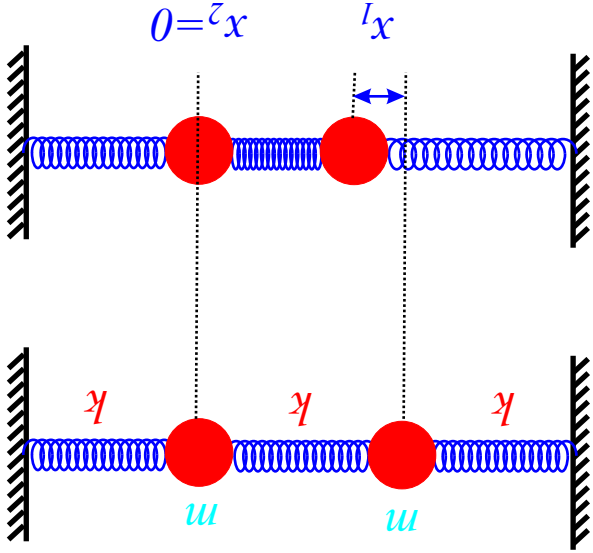
- The general motion is rather complicated.

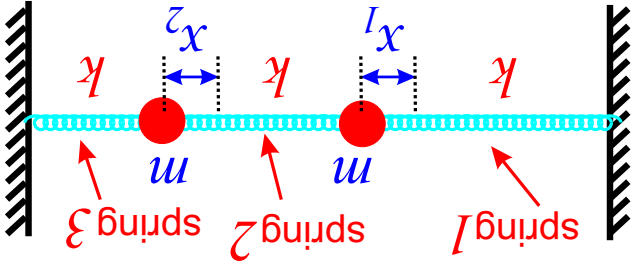
- However, the system has a **symmetry** as the masses and spring constants are equal.

- Suppose the mass at x_1 is displaced from equilibrium at $t = 0$, but $x_2 = 0$. The masses are then released and allowed to oscillate freely.

- We get the following motion:

- The oscillation looks complicated, but there are actually only two frequencies present in the oscillation (because there are two coordinates — this is a general result).





• Equations of motion for a two-mass system:

$$m\ddot{x}_1 = \underbrace{-kx_1}_{\text{spring 1}} + \underbrace{k(x_2 - x_1)}_{\text{spring 2}}$$

$$m\ddot{x}_2 = \underbrace{-k(x_2 - x_1)}_{\text{spring 2}} + \underbrace{-kx_2}_{\text{spring 3}}$$

- We can rewrite this compactly using matrix notation
- $$m \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- The equations of motion are coupled ODEs — we seek the Complementary Function by using a trial solution $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{i\omega t} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ where X_1, X_2 are constants.

- The result is a set of **homogeneous** linear equations for X_1, X_2 :

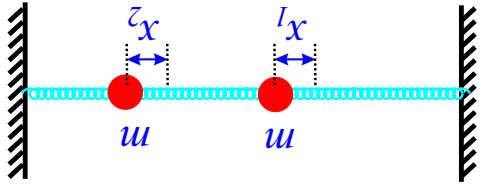
$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Homogeneous equations only have non-trivial solutions (i.e. other than zero) if the determinant of the matrix is zero. This implies $(2k - m\omega^2)^2 - k^2 = 0$ or

$$(3k - m\omega^2)(k - m\omega^2) = 0.$$

- We have two eigenvalues or characteristic frequencies: $\omega^2 = \frac{3k}{m}$ and $\omega^2 = \frac{k}{m}$.

- The matrix $\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix}$ has eigenvalues $\omega^2 = \frac{3k}{m}$ and $\omega^2 = \frac{k}{m}$.



$$(1) \quad \omega^2 = \frac{3k}{m} : \Leftrightarrow \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow X_1 + X_2 = 0.$$

The normal mode is any multiple of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$(2) \quad \omega^2 = \frac{k}{m} : \Leftrightarrow \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow X_1 - X_2 = 0.$$

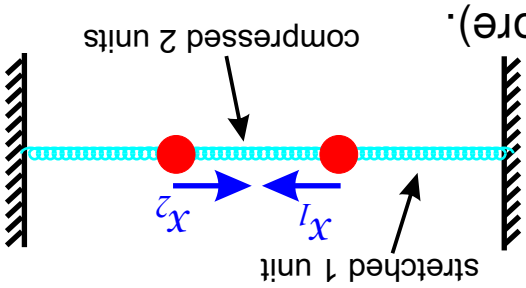
The normal mode is any multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

- Equations of motion:
$$\begin{aligned} m\ddot{x}_1 &= -2kx_1 + kx_2 \\ m\ddot{x}_2 &= kx_1 - 2kx_2 \end{aligned} \Leftrightarrow \begin{aligned} m(\ddot{x}_1 + \ddot{x}_2) &= -k(x_1 + x_2) \\ m(\ddot{x}_1 - \ddot{x}_2) &= -3k(x_1 - x_2) \end{aligned}$$
- We have simple (decoupled) equations for new variables $\xi_1 \equiv X_1 - X_2$ and $\xi_2 \equiv X_1 + X_2$.

- The variables (ξ_1, ξ_2) (or any multiples^a) are the **normal coordinates of the system**.

^aOne can use normalised coordinates $\xi_{1,2} = \frac{1}{\sqrt{2}}(1, \pm 1)$, but it is not compulsory in Classical physics.

- The symmetry of the two-mass system makes the normal modes obvious.



- **Symmetric mode:** has $X_2 = -X_1$, so that the central spring is compressed by twice as much as the outer springs are extended.

- We then have the equation of motion $m\ddot{x}_1 = -3kx_1$ (SHM as before).

- Knowing the shape (i.e. $X_2 = -X_1$) of the normal mode we can usually calculate the

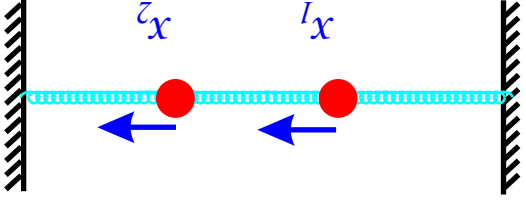
frequency using the **General rule**^a:

$$\omega^2 = \frac{\text{restoring force per unit extension}}{\text{mass}}$$

- For the symmetric mode the restoring force on mass 1 per unit displacement is $3k$ so that $\omega^2 = \frac{3k}{m}$ as before (same for mass 2, so we have a normal mode).

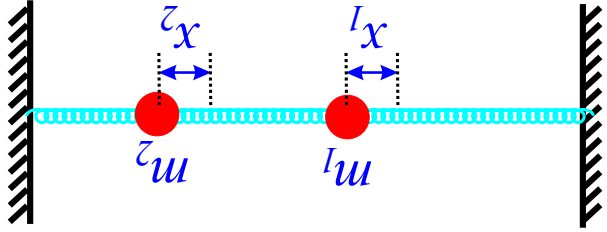
- **Antisymmetric mode:** has $X_2 = X_1$, so that the central spring is unchanged.

$$m\ddot{x}_1 = -kx_1, \text{ so } \omega^2 = \frac{k}{m}$$



- These are the normal modes of our two-mass system.

^aIn other circumstances it might be (e.g.) $\omega^2 = \frac{\text{restoring couple per unit angle}}{\text{moment of inertia}}$.



$$m_1 \ddot{x}_1 = -2kx_1 + kx_2$$

$$m_2 \ddot{x}_2 = kx_1 - 2kx_2$$

• Equations of motion:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t}$$

as before.

• This again gives a set of homogeneous linear (matrix) equations:

$$\begin{pmatrix} 2k - m_1\omega^2 & -k \\ -k & 2k - m_2\omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

• The frequencies of the normal modes are found by setting the determinant to zero

$$\Leftrightarrow 3k^2 - 2k(m_1 + m_2)\omega^2 + m_1m_2\omega^4 = 0 \text{ so that}$$

$$\omega^2 = \frac{m_1m_2}{k} \left(m_1 + m_2 \pm \sqrt{m_1^2 - m_1m_2 + m_2^2} \right)$$

• Having determined the normal frequencies we use them to find the shape of the normal

$$\text{modes: } \frac{X_1}{X_2} = \frac{m_2}{\left(m_2 - m_1 \pm \sqrt{m_1^2 - m_1m_2 + m_2^2} \right)}$$

• N.B. It is instructive to plot $\frac{X_2}{X_1}$ and the ratio of the two frequencies as a function of $\frac{m_1}{m_2}$.

• There are still **two normal modes**, in which the system oscillates at a single frequency.

• The general (free) oscillation is then an arbitrary sum of the two normal modes.

- If we know from physical grounds that the energy is conserved, we can always derive the equations of motion of systems that only have one degree of freedom (such as the SHO):

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E \Rightarrow \dot{x}(m\ddot{x} + kx) = 0 \Rightarrow m\ddot{x} + kx = 0$$
- This works because \dot{x} is in general non-zero.
- We have called this the **energy method**.

- We can often derive the equations of motion of more complicated systems with N variables in a similar way. It's certainly not rigorous, but works for small oscillations near equilibrium.
- The more advanced methods of Lagrangian and Hamiltonian mechanics derive the equations of motion of a system from a variational principle that involves the kinetic and potential energies in the form of the Lagrangian $\mathcal{L} = T - U$. This is rigorous, and always works.

- The advantage of Lagrangian and Hamiltonian methods is that we only have to work out the total kinetic and potential energies of a system. In Newtonian mechanics we need to evaluate all the forces acting, in order to calculate the accelerations.

- **Example:** for our two-mass system with three springs, we have $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$ and $U = \frac{1}{2}k(x_1^2 + (x_1 - x_2)^2 + x_2^2)$.

- Suppose the state of a system is specified by N coordinates $\{q_i\}$.

Examples:

1. system of M particles in 3-D — $\{r_i\} (N = 3M)$;
2. system of two masses and three springs in 1-D — $\{x_1, x_2\} (N = 2)$;
3. gyroscope — $\{\theta, \phi, \chi\} (N = 3)$.

- The variables $\{q_i\}$ or \bar{q} are usually called **generalised coordinates**.

- Suppose that the system has a position of equilibrium at $\bar{q} = \bar{0}$. Since the coordinates \bar{q} specify the state of the system can write the kinetic energy as $T = \frac{1}{2} \sum_{\text{system}} m | \dot{r} |^2$,

where $r = r(q_i)$ are the Cartesian coordinates of all of the parts of the system.

- Near equilibrium, we can make a Taylor expansion $r \approx \sum_i \dot{q}_i \left. \frac{\partial r}{\partial q_i} \right|_{\text{eq}}$, so that

$$T = \frac{1}{2} \sum_{ij} M_{ij} \dot{q}_i \dot{q}_j \equiv \frac{1}{2} \dot{\bar{q}} \cdot \bar{M} \cdot \dot{\bar{q}}$$

- The 'mass matrix' $M_{ij} \equiv \sum_{\text{system}} m \left. \frac{\partial r}{\partial q_i} \right|_{\text{eq}} \cdot \left. \frac{\partial r}{\partial q_j} \right|_{\text{eq}}$ is a **constant**.

- Thus, for a system near equilibrium, the kinetic energy is a quadratic function of the \bar{q} .

- The potential energy of a general system can also be written as a function of the generalised coordinates:

$$U = U(\bar{q}) \approx U_0 + \sum_i q_i \left. \frac{\partial U}{\partial q_i} \right|_{\text{eq}} + \frac{1}{2} \sum_{ij} q_i q_j \left. \frac{\partial^2 U}{\partial q_i \partial q_j} \right|_{\text{eq}} + \dots$$

in equilibrium constant

- The potential energy of a system near equilibrium is thus a quadratic function of the q .
- The total energy $E = U_0 + \frac{1}{2} \sum_{ij} M_{ij} \dot{q}_i \dot{q}_j + \frac{1}{2} \sum_{ij} K_{ij} q_i q_j$ is constant, so

$$\frac{dE}{dt} = 0 \Leftrightarrow \sum_{ij} \dot{q}_i (M_{ij} \ddot{q}_j + K_{ij} q_j) = 0$$

- **Health warning:**

- the following argument is not rigorous, but yields the correct equations of motion.
- The equations of motion for the system are then $\sum_j M_{ij} \ddot{q}_j + \sum_j K_{ij} q_j = 0$

or, equivalently, $\overline{M} \cdot \ddot{\bar{q}} + \overline{K} \cdot \bar{q} = \bar{0}$

More Rigorous Method.

- The properties of normal modes will be illustrated further by considering specific examples.

- For a general system specified by a set of N coordinates q_i , the equations of motion for small oscillations about equilibrium at $\bar{q} = \bar{0}$ are $\overline{M} \cdot \ddot{\bar{q}} + \overline{K} \cdot \bar{q} = \bar{0}$

- If the state of a system is specified by a set of N coordinates q_i , the Lagrangian is in general $\mathcal{L} \equiv T - U = \mathcal{L}(\{q_i\}, \{\dot{q}_i\}, t)$

- The **Lagrangian recipe** for mechanics (which is not emphasized in this course) then says that the equations of motion are derived from the Euler-Lagrange equations:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$

- For our system in equilibrium at $\bar{q} = \bar{0}$, the Lagrangian approach confirms our earlier result.

- To solve a problem involving normal modes we usually need to write down the 'mass matrix' $\overline{\overline{M}}$ and the 'spring constant' matrix $\overline{\overline{K}}$.

- **Example:** the two-mass system has coordinates $\overline{\mathbf{q}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and the energies are $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$ and $U = \frac{1}{2}k(x_1^2 + (x_1 - x_2)^2 + x_2^2)$.

- The matrix $\overline{\overline{M}}$ is straightforward since it contains only diagonal terms: The idea is to write

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) = \frac{1}{2} \begin{pmatrix} \dot{x}_1 & \dot{x}_2 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

- The matrices $\overline{\overline{M}}$ and $\overline{\overline{K}}$ are **by construction** symmetric (e.g. $K_{ij} = K_{ji}$). This is because

- (1) no antisymmetric part of K_{ij} could contribute to the quadratic form $\frac{1}{2} \sum_{ij} K_{ij} q_i q_j$ and

$$(2) \text{ the symmetry of } \frac{\partial^2 U}{\partial q_i \partial q_j} = \frac{\partial^2 U}{\partial q_j \partial q_i}.$$

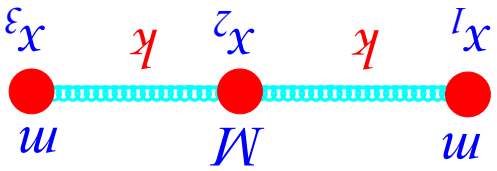
- The potential energy is $U = \frac{1}{2}(2kx_1^2 - 2kx_1x_2 + 2kx_2^2)$. When assembling the $\overline{\overline{K}}$ matrix we place the $2kx_1^2$ and $2kx_2^2$ on the diagonal, but have to split the off-diagonal term

$$-2kx_1x_2 \text{ equally on either side: i.e. } U = \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

In this way we get the correct result when the matrix products are evaluated.

EXAMPLE — THE TRIATOMIC MOLECULE

- We make a (Classical) model of a triatomic molecule consisting of three masses and two springs:



- Consider one-dimensional motion only: coordinates (x_1, x_2, x_3) .

• Kinetic energy: $T = \frac{1}{2}(m\dot{x}_1^2 + M\dot{x}_2^2 + m\dot{x}_3^2) = \frac{1}{2}\dot{\bar{x}}^T \cdot \bar{M} \cdot \dot{\bar{x}} \Rightarrow \bar{M} =$

$$\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

• Potential energy: $U = \frac{1}{2}(k(x_2 - x_1)^2 + k(x_3 - x_2)^2) = \frac{1}{2}\bar{x}^T \cdot \bar{K} \cdot \bar{x}$

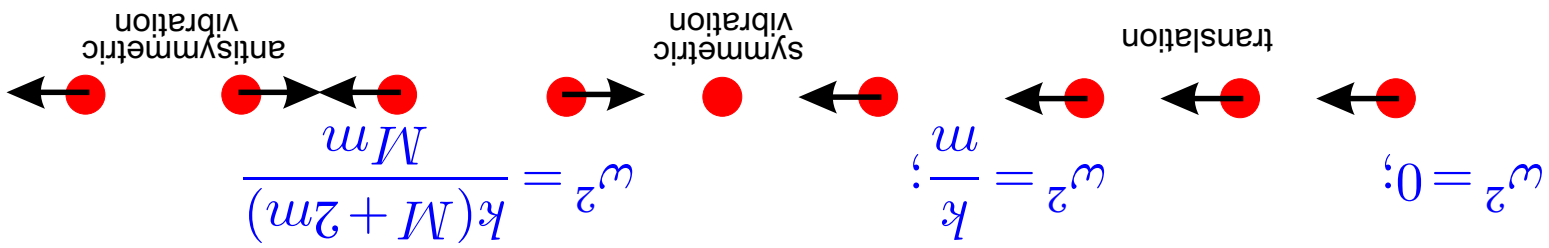
$$\bar{K} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

- Seek normal modes $\bar{x}(t) = \bar{X}e^{i\omega t} \Leftrightarrow (\bar{K} - \omega^2 \bar{M}) \cdot \bar{X} = \bar{0}$.

• Non-trivial solution only if $\det(\bar{K} - \omega^2 \bar{M}) = 0$

$$\det \begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{pmatrix} = \omega^2(k - m\omega^2)(mM\omega^2 - (2m + M)k) = 0.$$

- The normal modes of the system:



• The normal modes of the Classical model of a triatomic molecule:

1. $\omega^2 = 0$ mode is $\propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

This mode is free translation of the whole system along with the centre of mass. **Antisymmetric mode** with respect to inversion about central mass).

2. $\omega^2 = \frac{m}{k}$ mode is $\propto \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

This is a **symmetric** vibration (wrt inversion about central mass).

There is no motion of the C of Mass or the central atom.

3. $\omega^2 = \frac{k(M+2m)}{Mm}$ mode is $\propto \begin{pmatrix} 1 \\ -\frac{1}{2m} \\ \frac{1}{M} \end{pmatrix}$

This is a **antisymmetric** vibration. There is no motion of the C of Mass.

• **Alternative procedure**

1. Guess the shapes of the normal modes (all are obvious).
2. Substitute in to find the frequencies.

- Our equation for the normal modes $(\bar{K} - \omega^2 \bar{M}) \cdot \bar{Q} = \bar{0}$ is a simple generalisation of the eigenvalue equation $(\bar{A} - \lambda \bar{I}) \cdot \bar{x} = \bar{0}$ that you have already met.
- The matrices are symmetric, so the eigenvalues ω^2 are real, but the normal modes are not orthogonal in the usual sense. They are, however, orthogonal in an important physical sense: e.g. the asymmetric vibration of the triatomic molecule has no net linear momentum.
- You can make the normal mode system look more like the usual eigenvalue equation by writing it as $(\bar{M}^{-1} \bar{K} - \omega^2 \bar{I}) \cdot \bar{Q} = \bar{0}$, which an eigenvalue equation as taught in mathematics, but it has the same (possibly non-orthogonal) eigenvectors, and is no longer necessarily symmetric (as for the triatomic system) . . .
- Some books advise working with $\bar{M}^{-1} \bar{K}$ but, in my experience, the inversion of \bar{M} is simply a waste of time when working by hand. However, when using a computer algebra package, working with $\bar{M}^{-1} \bar{K}$ enables the use of standard eigenvector tools.
- The normal modes we have found \bar{Q}_i are actually orthogonal in the sense that $\bar{Q}_i^T \cdot \bar{M} \cdot \bar{Q}_j = 0$ for $i \neq j$. This is very easy to prove.
- This means that the “squashed” modes $\bar{M}^{1/2} \cdot \bar{Q}_j$ are orthogonal in the usual sense.

^aTo find the square root of a positive semi-definite matrix, diagonalise it, take the square root of the diagonal elements and rotate back.

- We have seen that the correct eigenvalue problem for normal modes is actually

$$\left(\overline{\overline{M}}^{-1/2} \overline{\overline{K}} \overline{\overline{M}}^{-1/2} - \omega^2 \overline{\overline{I}} \right) \cdot (\overline{\overline{M}}^{1/2} \cdot \overline{\overline{O}}) = \overline{\overline{0}}$$
 which retains all the helpful symmetry and orthogonality properties.
- We can find suitable 'squashed' coordinates that simultaneously diagonalise $\overline{\overline{M}}$ and $\overline{\overline{K}}$.
 - This **is** a proper procedure for getting orthonormal modes. In detail:
 1. diagonalise $\overline{\overline{M}}$;
 2. squash the coordinates so that $\overline{\overline{M}}$ becomes a unit matrix — this equivalent to taking the square root of $\overline{\overline{M}}$.
 3. diagonalise $\overline{\overline{K}}$ in the new coordinates.
 - In this way the $\overline{\overline{M}}$ remains diagonal during the final step.
 - In quantum mechanics it is known that you cannot simultaneously diagonalise two non-commuting matrices. Yet, the above is a procedure for doing exactly that! How have we achieved the impossible?
 - In quantum mechanics we have to use transformations that preserve the inner product (i.e. are **unitary**). There is no such restriction in Classical mechanics (the squashing step is non-unitary).

- The concept of a normal mode is very important, both theoretically and in practice.
- **Stability:** The normal frequencies ω^2 are real.

1. If all the ω^2 's are positive, the system is **stable**.

2. If any of the ω^2 's are negative ($\omega^2 = -\kappa^2$), then growing modes $\propto e^{\kappa t}$ exist and the

system is unstable.

3. Zero-frequency modes can exist, and usually correspond to translation or rotation of the

whole system. (Remember that the general solution to $\ddot{x} = 0$ is $x = At + B$).

- **Degeneracy:** two or more normal frequencies are equal. This often occurs because of some symmetry inherent in the system, but can also be **accidental** (i.e. not due to symmetry).

- You can use symmetry to guess normal modes: you can then usually find the frequencies

fairly easily via elementary arguments.

- Group representation theory is a very powerful way of deducing the nature of normal modes.

For example, applied to the triatomic molecule problem it says that there are two

antisymmetric modes and one symmetric mode.

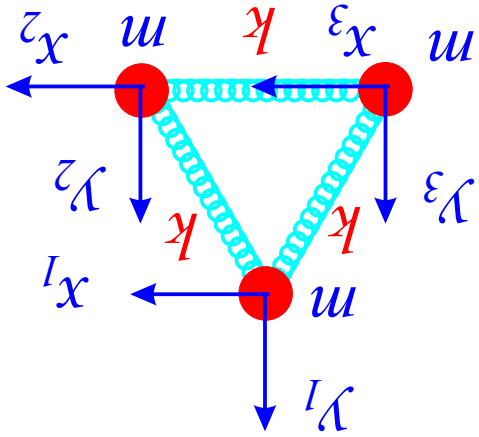
- Applied to the **equilateral** triatomic molecule problem it says that there is a two-dimensional

vibrational mode and a one-dimensional symmetric mode. It also says that there are three

zero-frequency modes, two of which are degenerate, and one other one-dimensional mode

that has a different **character** to the one-dimensional vibrational mode.

- Consider the classical system of three masses and three springs in an equilateral triangle as shown.
- It's a classical model for an ozone molecule^a
- The $\overline{\overline{M}}$ matrix is easy but the $\overline{\overline{K}}$ matrix is a little more tricky:



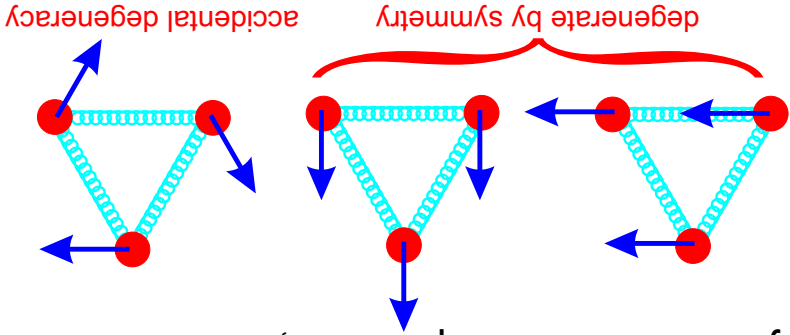
$$\overline{\overline{M}} = m \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overline{\overline{K}} = k \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{2}{3} & 0 & \frac{2}{\sqrt{3}} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{4} & \frac{1}{\sqrt{3}} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{3} & \frac{1}{4} & \frac{1}{3} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{3} & \frac{1}{4} \\ 0 & -\frac{1}{3} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 \end{pmatrix}$$

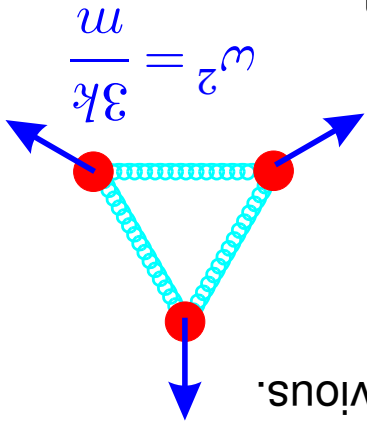
- We find the normal frequencies: $\omega^2 = \begin{pmatrix} 0, 0, 0, \frac{3k}{m}, \frac{3k}{m}, \frac{3k}{m} \end{pmatrix} = \dots$

^aChemists say that the ozone molecule is not actually equilateral, although we can't tell which atom has the odd angle out...

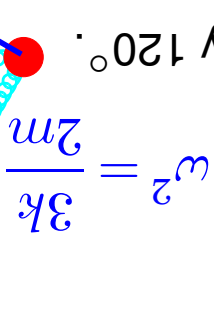
- The system looks complicated, but most of the normal modes are obvious.



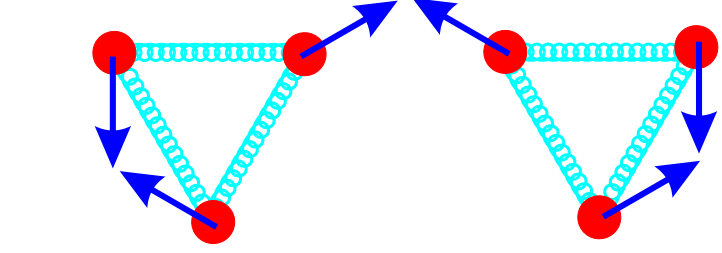
- The three modes with $\omega^2 = 0$ correspond to translations and rotation. There is also a symmetric vibrational ("breathing") mode with $\omega^2 = \frac{3k}{m}$.



- The remaining vibrational modes are degenerate, but must be orthogonal to the others, and so must have no net translation, no net rotation and no net outward motion. One possible mode is shown.



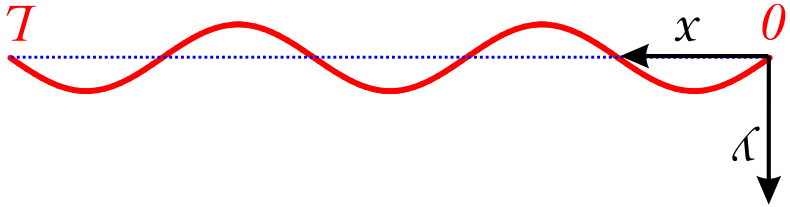
- We can find other modes with the same frequency by rotating this mode by 120° .



- There are only two independent modes with $\omega^2 = \frac{3k}{2m}$ since the sum of the three modes illustrated is zero and they are not linearly independent.

- The symmetry properties of the normal modes (i.e. two different one-dimensional modes and two sets two-dimensional modes) of can be determined by group representation theory.

- Standing waves on a string. Fixed ($y = 0$) at $x = 0$ and $x = L$.



- Tension T and mass per unit length ρ , gives equation of motion $T \frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2}$

- Seek wave solution $\propto \exp(i(\omega t - kx))$.

- Find relationship between k and ω : $\omega^2 = \frac{T}{\rho} k^2$.

- The boundary conditions fix the allowed k values: $k_n = \frac{n\pi}{L}$ (in this case they are harmonically related).

- The n th harmonic has n antinodes:

$$y(x, t) = A \sin k_n x \cos(\omega t + \chi)$$

constant amplitude A

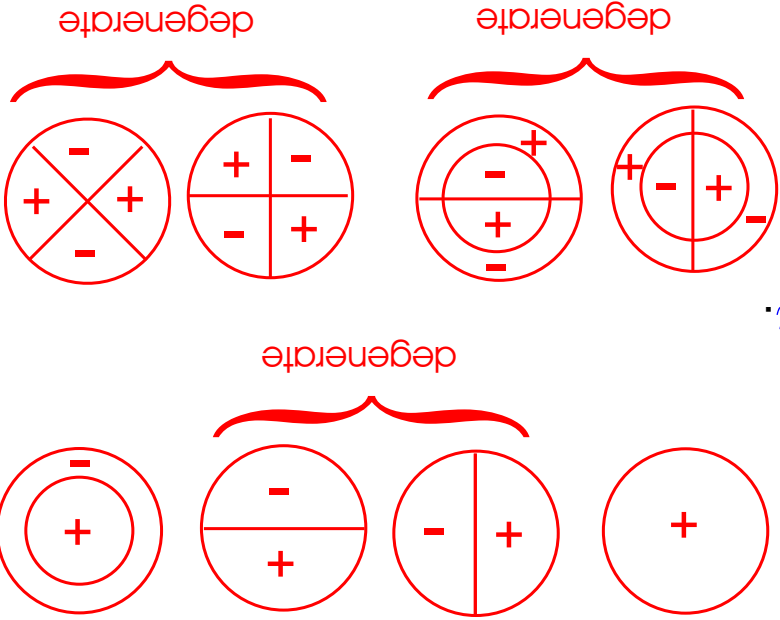
shape of normal mode $\sin k_n x$

time dependence (same for whole system) $\cos(\omega t + \chi)$

• Example 2.

Waves on a circular membrane — drum of radius a .
 Boundary conditions: $y(r=a) = 0$.

- The (n, m) th mode has n nodes in r and m nodes in ϕ :



$$y(x, t) = A \cos J_m(k_{nm}r) \sin \dot{\phi}(m\phi) \cos(\omega t + \chi)$$

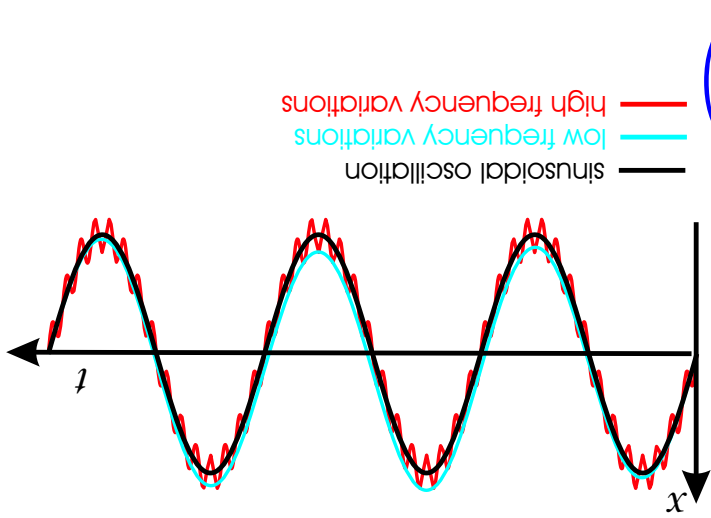
constant amplitude
 shape of normal mode
 time dependence (same for whole system)

where $J_m(k_{nm}a)$ is the n th zero of the Bessel function of order m .

- The frequencies are $\omega_{nm} = \sqrt{\frac{T}{\rho}} k_{nm}$ and are **not** harmonically related.

- The lowest mode has $\omega_{01} = 1.551 \sqrt{\frac{T}{\rho}}$.

- Lagrangian mechanics starts with **Hamilton's Principle**^a that the **action** $S = \int dt \mathcal{L}(q, \dot{q}, t)$ is stationary for small variations $\delta q_i(t)$ about the path $q_i(t)$.



- Consider the variation δS as the path varies from $q_i(t)$ to $q_i(t) + \delta q_i(t)$: $\delta S = \int_{t_2}^{t_1} dt \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} \delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i \right)$

- However, the variations of $\delta q_i(t)$ are determined by $\delta q_i(t)$, so we may integrate by parts:

$$\delta S = \sum_i \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i \right]_{t_2}^{t_1} + \int_{t_2}^{t_1} dt \sum_i \delta q_i \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right)$$

- Provided the integrated parts vanish, the condition that $\delta S = 0$ for all variations $\delta q_i(t)$ is

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \text{ for all } i$$

- These are **Lagrange's** equations of motion. In classical mechanics we can usually take as

Lagrangian $\mathcal{L} = T - U$, where T is the kinetic energy and U is the potential energy.

^aThis is often called the "Principle of Least Action". It's not a particularly good name since, although the action is stationary, it's not actually a minimum, except for the case of a free particle.

• Example illustrated.

Simple harmonic motion: $q_1 = x$. $T = \frac{1}{2}m\dot{x}^2$; $U = \frac{1}{2}kx^2$
 $\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$; $\frac{\partial \mathcal{L}}{\partial x} = -kx$ \Leftrightarrow Equation of motion: $m\ddot{x} = -kx$.

Solution: $x = A \cos \omega t + B \sin \omega t$, where $\omega^2 = k/m$.

Action on classical path $x_0(t_0)$ to $x(t)$: $\mathcal{S} = \frac{m\omega}{2} \frac{\sin \omega(t-t_0)}{\cos \omega(t-t_0) + x_0^2}$

• Possible variations (high and low frequencies) δx around the classical path are illustrated.

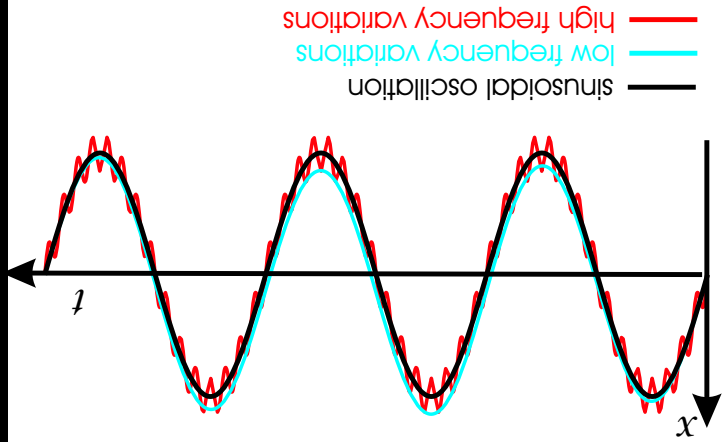
• The important quantities $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \equiv p_i$ are known as the **conjugate momenta** and play a

crucial role in Hamiltonian dynamics, and hence in the correspondence between classical dynamics and quantum mechanics.

• If the Lagrangian does not depend on one of the coordinates, say q_1 (i.e. $\frac{\partial \mathcal{L}}{\partial q_1} = 0$), then

from Lagrange's equations $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = 0$ and the conjugate momentum p_1 is a **constant**.

• **Symmetries** (here \mathcal{L} independent of q_1) thus lead to **conservation laws**.



- Lagrangian dynamics uses $\mathcal{L}(q_i, \dot{q}_i, t)$ where there is a dependence between q_i and \dot{q}_i .

The motivation for Hamiltonian dynamics is to find a function $H(q_i, p_i, t)$ that is not a

function of the velocities \dot{q}_i . This is easily done by forming the **Hamiltonian**

$$H \equiv \sum_i p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t).$$

- The total variation of H is $dH = \sum_i \left(\dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial p_i} dp_i - \frac{\partial \mathcal{L}}{\partial t} dt \right)$.

By the definition of p_i and Lagrange's equations this simplifies to

$$dH = \sum_i (p_i d\dot{q}_i - \dot{q}_i dp_i) - \frac{\partial \mathcal{L}}{\partial t} dt.$$

- This shows that H responds only to changes dp_i, dq_i, t and is **not** a function of \dot{q}_i . It is a

function $H(q_i, p_i, t)$ as required. By construction we have **Hamilton's equations:**

$$\dot{q}_i = \frac{\partial H}{\partial p_i} ; \dot{p}_i = - \frac{\partial H}{\partial q_i} ; \frac{\partial \mathcal{L}}{\partial t} = \frac{\partial H}{\partial t}.$$

- We therefore find $\frac{dH}{dt} = \sum_i (\dot{q}_i p_i - \dot{p}_i q_i) = \frac{\partial H}{\partial t}$

- This leads to the important result that, if the Lagrangian does not depend on time explicitly

(i.e. $\frac{\partial \mathcal{L}}{\partial t} = 0$), **the Hamiltonian is conserved.**

- **Simple harmonic motion:** $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$.

\mathcal{L} does not depend on t .

\Rightarrow The energy (Hamiltonian) $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ is conserved.

- **Orbits in central potential:** $\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$.

\mathcal{L} does not depend on ϕ or t :

\Rightarrow The angular momentum (p_ϕ) $J = mr^2\dot{\phi}$ is conserved;

\Rightarrow The energy (Hamiltonian) $E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r)$ is conserved.

- **Symmetric top:** $\mathcal{L} = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_3(\dot{\chi} + \dot{\phi}\cos\theta)^2 - mgh\cos\theta$.

\mathcal{L} does not depend on ϕ , χ or t :

\Rightarrow The angular momentum (p_ϕ) $J_z = I_3(\dot{\chi} + \dot{\phi}\cos\theta) + I_1\dot{\phi}\sin^2\theta$ is conserved;

\Rightarrow The angular momentum (p_χ) $J_\chi = I_3(\dot{\chi} + \dot{\phi}\cos\theta)$ is conserved;

\Rightarrow The energy $E = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_3(\dot{\chi} + \dot{\phi}\cos\theta)^2 + mgh\cos\theta$ is conserved.