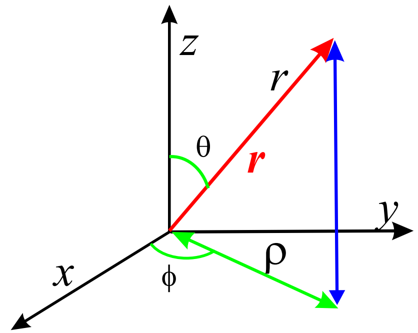
- Newtonian dynamics: $\dfrac{d\boldsymbol{p}}{dt} = \boldsymbol{F}$, where the momentum $\boldsymbol{p} = m\boldsymbol{v}$
- Simple harmonic oscillator. The energy method — example.
- **Revision of dynamics of many-particle system**

  Definitions Centre of mass. Total momentum and angular momentum. Total external force and couple. Kinetic energy, potential energy and total energy of system. Intrinsic angular momentum in CoM frame.
- Total momentum acts as if it were acted upon by the total external force.
- Total angular momentum acts as if it were acted upon by the total external couple.
- Galilean transformation from centre of mass frame S' to frame S moving at velocity $\boldsymbol{V}$ (CoM at $\boldsymbol{R}' = \boldsymbol{V}t$):
  - **Momentum**: $\boldsymbol{P} = \boldsymbol{P}' + M\boldsymbol{V}$
  - **Angular Momentum**: $\boldsymbol{J} = \boldsymbol{J}' + M\boldsymbol{R}' \times \boldsymbol{V}$
  - **Kinetic energy**: $\boldsymbol{T} = \boldsymbol{T}' + \frac{1}{2}M\boldsymbol{V}^2$

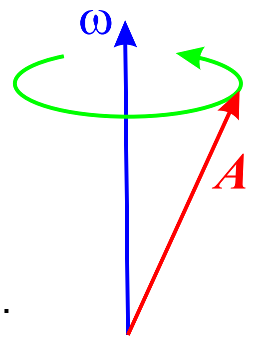


- Position vector $\boldsymbol{r}$ has Cartesian coordinates (x, y, z), cylindrical polar coordinates (ρ, ϕ, z) and spherical polar coordinates (r, θ, ϕ).
- Relations: $x = \rho\cos\phi = r\sin\theta\cos\phi$; $y = \rho\sin\phi = r\sin\theta\sin\phi$; $z = z = r\cos\theta$.
- Unit vectors $\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta$ vary with position: e.g. $\dot{\hat{\mathbf{e}}}_\rho = \dot{\phi}\,\hat{\mathbf{e}}_\phi$ and $\dot{\hat{\mathbf{e}}}_\phi = -\dot{\phi}\,\hat{\mathbf{e}}_\rho$.
- Velocity in polar coordinates: $\boldsymbol{v} = \dot{\rho}\,\hat{\mathbf{e}}_\rho + \rho\dot{\phi}\,\hat{\mathbf{e}}_\phi$.
- Radial velocity $v_\rho = \dot{\rho}$ and transverse velocity $v_\phi = \rho\dot{\phi}$.
- Acceleration in polar coordinates:

$$\ddot{\boldsymbol{r}} = \dot{\boldsymbol{v}} = \boldsymbol{a} = \underbrace{(\ddot{\rho} - \rho\dot{\phi}^2)}_{\text{radial}}\hat{\mathbf{e}}_\rho + \underbrace{(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})}_{\text{transverse}}\hat{\mathbf{e}}_\phi$$

- In inertial frame S_0 we have $m\ddot{\boldsymbol{r}}_0 = \boldsymbol{F}$. What is the apparent equation of motion in a moving frame S?
- For a non-rotating accelerated frame S such that $\boldsymbol{r} = \boldsymbol{r}_0 - \boldsymbol{R}(t)$ we find $m\ddot{\boldsymbol{r}} = \boldsymbol{F} - m\ddot{\boldsymbol{R}}$.
- To get the correct equation in S we add an inertial (or fictitious) force, proportional to the mass.
- Special case: steady motion $\ddot{\boldsymbol{R}} = 0$ — no inertial forces (Galilean transformation).
- Unaccelerated frame S rotating at fixed angular velocity $\boldsymbol{\omega}$
  - For any vector $\mathbf{A}$ the rates of change in frame S_0 and

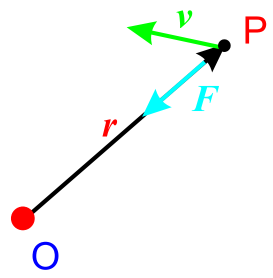
    in frame S are related by $\left[\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right]_{S_0} = \left[\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right]_S + \boldsymbol{\omega}\times\boldsymbol{A}$
  - Apply this operator relation twice to $\boldsymbol{r}$ ($\boldsymbol{r} = \boldsymbol{r}_0$ at $t = 0$).



- $\Rightarrow \boldsymbol{a}_0 = \boldsymbol{a} + 2\boldsymbol{\omega}\times\boldsymbol{v} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ – rearrange to get EoM in S:

$$m\boldsymbol{a} = \boldsymbol{F} \quad \underbrace{-2m(\boldsymbol{\omega}\times\boldsymbol{v})}_{\text{Coriolis force}} \quad \underbrace{-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})}_{\text{Centrifugal force}}$$

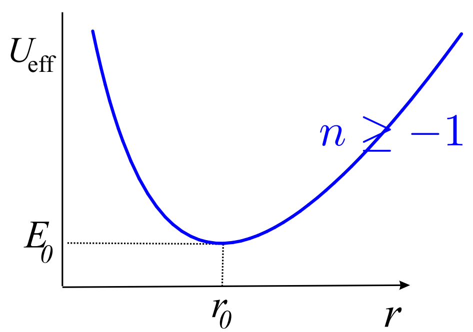
- Applications: oblateness of planets; weather patterns; Foucault pendulum; problems 4–7. Restricted 3-body problem (to come).

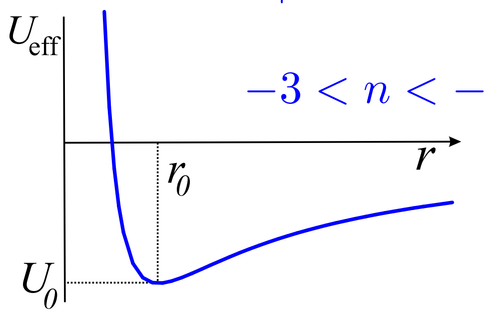


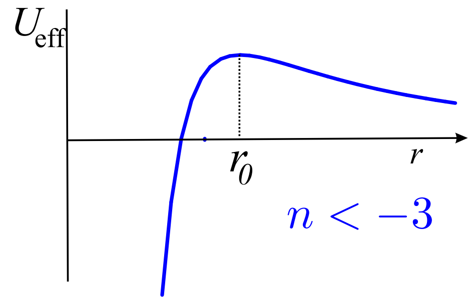
- Particle moving in central potential $U(r)$.
- Angular momentum conserved: $mr^2\dot{\phi} = J =$ constant
- Total energy is conserved: $E = U(r) + \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) = \frac{m}{2}\dot{r}^2 + U(r) + \frac{J^2}{2mr^2}$

item **Effective potential** $U_{\text{eff}}(r) = U(r) + \frac{J^2}{2mr^2}$

- Power law force $F = -Ar^n$ gives $U_{\text{eff}}(r) = \frac{Ar^{n+1}}{n+1} + \frac{J^2}{2mr^2}$







- The potential is qualitatively different for different values of n:

$n > -1$:	Orbit at r_0 stable.	All orbits bound.
$-3 < n < -1$:	Orbit at r_0 stable.	Unbound orbits for $E > 0$.
$n < -3$:	Orbit at r_0 unstable.	

- Nearly-circular orbits: expand $U_{\text{eff}}(r)$ around r_0: small radial oscillations at period $\tau = \frac{T}{\sqrt{n+3}}$, where T is the orbital period.
- Special cases: $n = 1$ (SHM) and $n = -2$ (inverse square law) give closed elliptical orbits.

SUMMARY SHEET — ORBITS IN INVERSE SQUARE LAW— S4

- **Kepler's Laws**
  - **First Law:** Planetary orbits are ellipses with the Sun at one focus.
  - **Second Law:** The line joining the planets to the Sun sweeps out equal areas in equal times.
  - **Third Law:** The square of the period of a planet is proportional to the cube of the major axis of the orbit.

- Conservation of angular momentum and energy lead to equation of orbit $r_0 = r(1 + e\cos\phi)$.

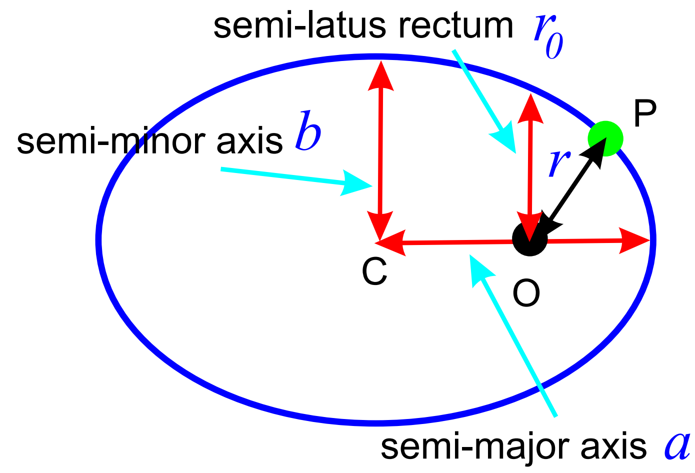
  Method 1. Use $u = 1/r$ and change to ϕ as independent variable
  $\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2 = \frac{2m}{J^2}(E + Au)$. Use Kepler II to get period.

  Method 2. Use $\boldsymbol{J}\times\dot{\boldsymbol{v}} = -A\dot{\hat{\mathbf{e}}}_r$ and integrate: $\boldsymbol{J}\times\boldsymbol{v} + A(\hat{\mathbf{e}}_r + \boldsymbol{e}) = 0$ where e is a constant vector.

  Method 3. Change independent variable to $r\mathrm{d}s = \mathrm{d}t$ to get
  $\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - \frac{2E}{m}r^2 - \frac{2A}{m}r + J^2 = 0 \Rightarrow r = a(1 - e\cos\Omega s)$, where $\Omega^2 = \frac{-2E}{m}$. Also
  $t = as - \frac{a}{\Omega}\sin\Omega s$.

  Method 4. Use $x + iy = z = \zeta^2$ where $\zeta = \xi + i\eta$ and $r\mathrm{d}s = \mathrm{d}t$ to transform Kepler problem to the two-dimensional SHO.

- Methods 3 and 4 non-examinable.



- The equation of an ellipse in polar coordinates

$$r_0 = r(1 + e\cos\phi)$$

- The distances of closest and furthest approach follow from this: $r_{\min} = \dfrac{r_0}{1+e}$ and $r_{\max} = \dfrac{r_0}{1-e}$

- The semi-major axis a satisfies $2a = r_{\min} + r_{\max}$

$\Rightarrow$ $a = \dfrac{r_0}{1-e^2}$ so that $r_{\max,\min} = a(1 \pm e)$. Also $b = \dfrac{r_0}{\sqrt{1-e^2}}$

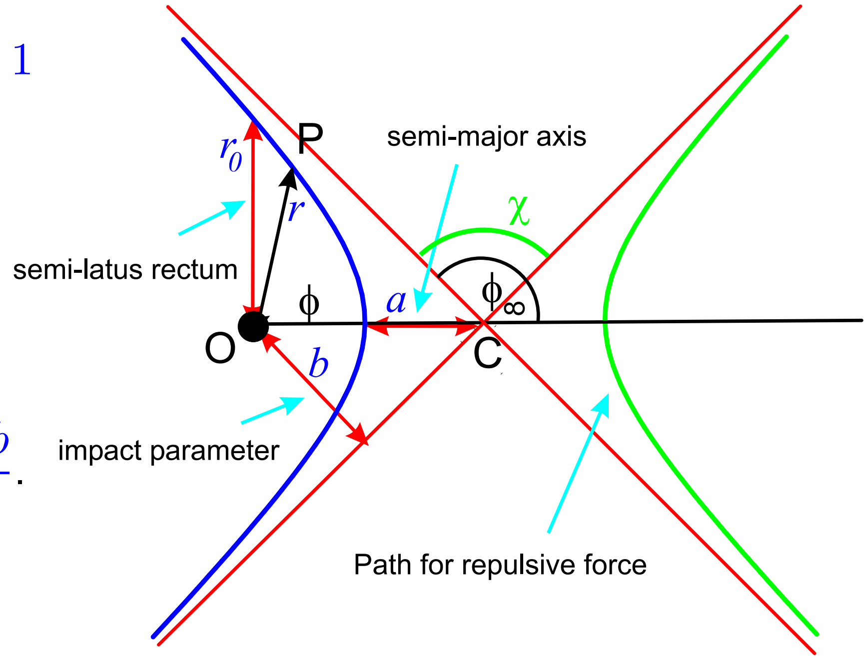
- The semi-major axis a determines the **energy** and the **period** of the orbit

$$E = -\frac{A}{2a}\ ; \quad T = \frac{2\pi}{\omega}\ ; \quad \omega^2 = \frac{A}{ma^3}$$

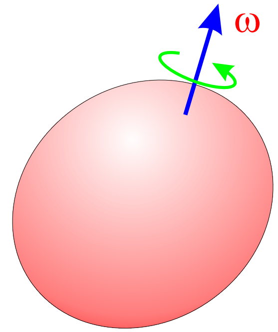
- The semi-latus rectum r_0 determines the **angular momentum** of the orbit: $J^2 = Amr_0$

- If you need to derive any of these formulae in a hurry **consider a silly case**.

  Use the simple balance of forces argument for circular motion: $\dfrac{A}{r^2} = m\omega^2 r$.



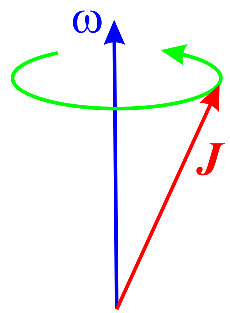
- **Attractive potential**: all previous formulae still valid, but $e > 1$ so $a < 0$ and energy $E = -\dfrac{A}{2a} = \dfrac{(e^2-1)A}{2r_0} > 0$.
- Impact parameter b and velocity at infinity v_∞ determine angular momentum $J = mbv_\infty$ and energy $E = \frac{1}{2}mv_\infty^2$.
- Total angle of deflection $\chi = 2\phi_\infty - \pi$ (χ positive).
- Asymptotes at $\pm\phi_\infty$: $\sec\phi_\infty = -e \Rightarrow \tan\phi_\infty = \dfrac{mv_\infty^2 b}{A}$.
- **Repulsive potential**: (a) Change $\dfrac{A}{r^2} \to -\dfrac{B}{r^2}$, define $J^2 = Bmr_0$ and use branch $r_0 = r(e\cos\phi - 1)$.
- a is positive again and the total angle of deflection is now $\chi = \pi - 2\phi_\infty$ (χ negative).
- Asymptotes satisfy $\tan\phi_\infty = \dfrac{mv_\infty^2 b}{B}$. and the distance of closest approach is $a(1+e)$.
- (b) **Alternative.** Use the results for the attractive potential case ($r_0 = r(1 + e\cos\phi)$) and let ϕ exceed ϕ_∞. The radius r is **negative** and the particle traces out the repulsive branch, getting closest to O at $\phi = \pi$, so that $r(\pi) = a(1+e)$, (a negative). Potential energy $-\dfrac{A}{r}$ is positive when $r < 0$ (repulsive potential).
- **3-body problem** 3 bodies will attract each other and interact strongly. Eventually the interaction will form a close binary (negative gravitational binding energy) releasing kinetic energy and the bodies to escape to infinity.



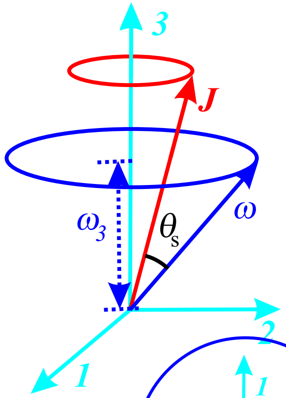
- Rigid body spinning about z-axis has $J = I\omega$ where $I = \sum m(x^2+y^2)$ is the moment of inertia. Also, $T = \frac{1}{2}I\omega^2 = \frac{1}{2}J\omega$.
- General case: $\boldsymbol{J} = \sum \boldsymbol{r}\times\boldsymbol{p} = \sum m\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})$.
- $\boldsymbol{J}$ is proportional to angular velocity $\boldsymbol{\omega}$ but not necessarily parallel to it,
  $\boldsymbol{J} = \boldsymbol{I}\cdot\boldsymbol{\omega}$ or $J_i = I_{ij}\omega_j$, where I_{ij} is the **inertia tensor**.

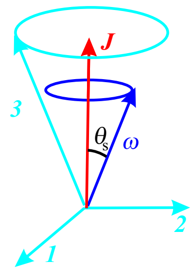
$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

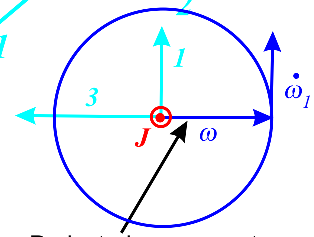
- Inertia tensor is symmetric and diagonal *wrt* 3 perpendicular **principal axes** $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ fixed in the body.
- The eigenvalues $\{I_1, I_2, I_3\}$ are the **Principal Moments of Inertia**.
- $\boldsymbol{J}$ is simple *wrt* the principal (body) axes: $\boldsymbol{J} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$.
- The kinetic energy is $T = \frac{1}{2}\boldsymbol{J}\cdot\boldsymbol{\omega} = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$.
- In $\boldsymbol{\omega}$-space, a surface of constant $T = T(\boldsymbol{\omega})$ is an ellipsoid, which is fixed to the body —the **Inertia Ellipsoid**;



- **Dynamics**: In an inertial frame S_0 we have $\left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_{S_0} = \boldsymbol{G}$.
- The equation of motion in the rotating frame S is : $\boldsymbol{G} = \left[\frac{d\boldsymbol{J}}{dt}\right]_S + \boldsymbol{\omega}\times\boldsymbol{J}$
- **Euler's equations** for the motion of a rigid body are $G_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$ and G_2, G_3 similarly by cyclic permutation.





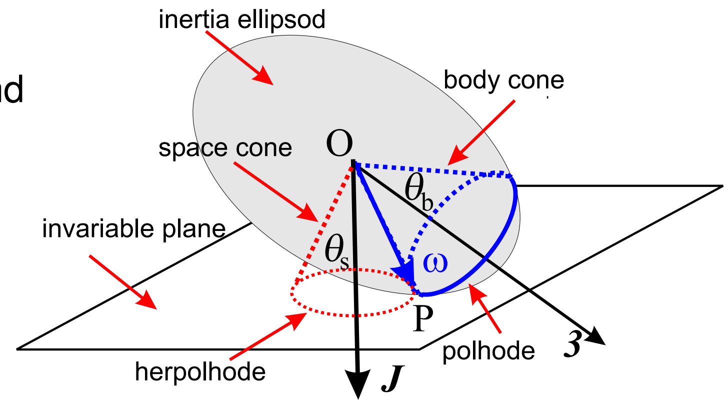


- Free Precession of Symmetric Body: $I_1 = I_2 \neq I_3$.

- Euler's equations show that in the body frame $\boldsymbol{\omega}$ precesses around the 3-axis with angular velocity $\Omega_b = \dfrac{I_1 - I_3}{I_1}\omega_3$ (body frequency), tracing out the body cone of half-angle θ_b.

- Inertial observer sees $\boldsymbol{\omega}$ rotating around $\boldsymbol{J}$ at the space frequency $\Omega_s = J/I_1$, tracing out the space cone of half-angle θ_s.

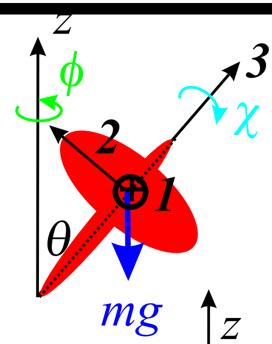
- Related via: $\Omega_s \sin\theta_s = \Omega_b \sin\theta_b$.



- Poinsot's construction: inertia ellipsoid rolls on the plane $T = \frac{1}{2}\boldsymbol{J}\cdot\boldsymbol{\omega}$, with $\boldsymbol{\omega}$ tracing out the polhode on the ellipsoid and the herpolhode on the plane.

- Free precession of triaxial bodies:
  $I_1 < I_2 < I_3$: rotation about 1- and 3-axes stable,
  but rotation about intermediate 2-axis is unstable.

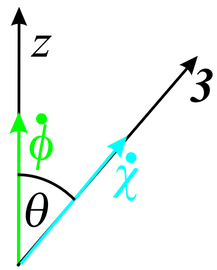
- Energy dissipation in non-rigid bodies eventually leads to stable rotation about the axis with the largest moment of inertia (major axis theorem).



- Lagrange's approach via Euler angles (θ, ϕ, χ) (symmetric body take 1-axis inwards).
- Resolve rates of change $(\dot\theta, \dot\phi, \dot\chi)$ onto body axes:

$$(\omega_1, \omega_2, \omega_3) = (\dot\theta, \dot\phi \sin\theta, \dot\chi + \dot\phi\cos\theta)$$

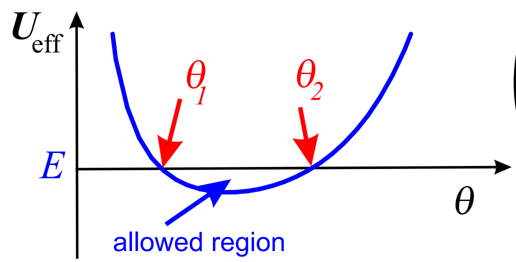
- Can now write $\boldsymbol{J}$ and T in terms of space coordinates.

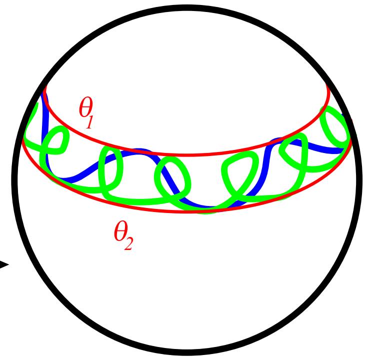


- Conservation laws of J_3, J_z and E enable us to write an energy equation for $\dot\theta$:

$$E = \tfrac{1}{2} I_1 \dot\theta^2 + \underbrace{\frac{(J_z - J_3\cos\theta)^2}{2I_1\sin^2\theta} + mgh\cos\theta + \frac{J_3^2}{2I_3}}_{U_{\text{eff}}(\theta)}$$

- Allowed region $\theta_1 < \theta < \theta_2$. $\theta_1 = \theta_2$ is steady precession.
- Some other possible paths are illustrated (nutation).





- Steady precession: $\dot\phi = \dfrac{J_3 \pm \sqrt{J_3^2 - 4I_1 mgh\cos\theta}}{2I_1\cos\theta}$
- There are two possible precession frequencies for given θ; in the 'gyroscopic limit' $J_3^2 \gg mghI_1$ they are

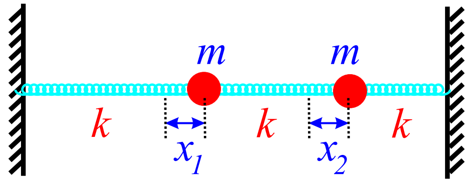
  $\dot\phi \approx mgh/J_3$, independent of θ; this is 'slow precession'

  $\dot\phi \approx J_3/(I_1\cos\theta)$, independent of $\boldsymbol{G}$; this is 'free precession' again.

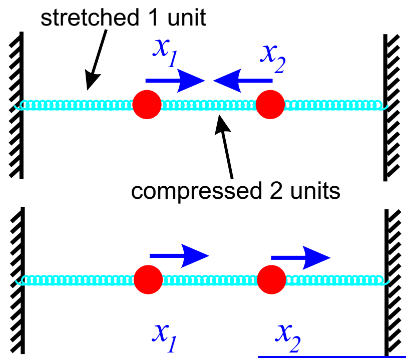
- Detailed analysis of nutation about precession at general θ, even in the gyroscopic limit, is algebraically laborious. The case of nutation of a horizontal gyroscope is reasonably straightforward.

Normal Modes of systems undergoing small oscillations near equilibrium.

- **Small displacements** about equilibrium lead to **linear equations**.
- A **normal mode** is an oscillation that has a **single frequency**.
- Example: system of 2 masses and 3 springs:



- Coupled equations of motion: $\begin{pmatrix} m\ddot{x}_1 \\ m\ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
- Seek solution $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t}$: $\Rightarrow$ Linear equations $\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- The determinant must vanish for non-trivial solutions, giving two normal frequencies: $\omega^2 = \dfrac{3k}{m}$ and $\omega^2 = \dfrac{k}{m}$.
- The symmetry of the system makes the normal modes obvious.



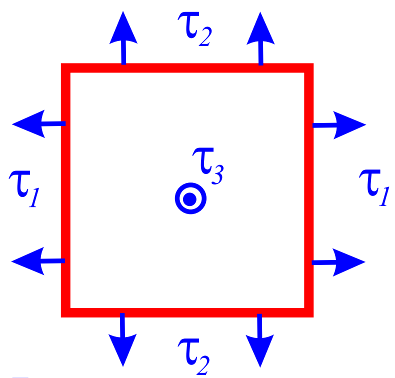
- **Symmetric mode**: $\omega^2 = \dfrac{3k}{m}$ has $X_2 = -X_1$
- **Antisymmetric mode**: $\omega^2 = \dfrac{k}{m}$ has $X_2 = X_1$.
- Unequal masses m_1 and m_2: the symmetry is broken but there are still two normal frequencies $\omega^2 = \dfrac{k}{m_1 m_2}\left(m_1 + m_2 \pm \sqrt{m_1^2 - m_1 m_2 + m_2^2}\right)$

  and normal modes $\dfrac{X_2}{X_1} = \dfrac{\left(m_2 - m_1 \mp \sqrt{m_1^2 - m_1 m_2 + m_2^2}\right)}{m_2}$.

- General system: **generalised coordinates** $\{q_i\}, i = 1, N$.
  Examples: system of M particles, $N = 3M$; rigid body rotation, angles (θ, ϕ, χ), $N = 3$; system of two masses, $N = 2$.
- System in equilibrium at $\underline{q} = \underline{0}$. For small oscillations about equilibrium the kinetic energy is a quadratic function of the $\dot{q}_i$ and the potential energy is a quadratic function of the q_i:
  $$T = \tfrac{1}{2}\sum_{ij} \dot{q}_i M_{ij} \dot{q}_j \;; \quad U = U_0 + \tfrac{1}{2}\sum_{ij} q_i K_{ij} q_j .$$
- Equation of motion (best from Lagrange's equations): $\sum_j (M_{ij}\ddot{q}_j + K_{ij} q_j) = 0$.
- Find **normal modes** of free oscillations by substituting $q_i(t) = Q_i e^{i\omega t}$ where Q_i is a constant normal mode shape.
- We get a set of homogeneous linear equations: $(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) \cdot \underline{Q} = \underline{0}$.
- There is a non-zero solution only if $\det(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) = 0$.
- There are N normal frequencies ω_i^2. For each value of ω_i^2 there is then a normal mode $\underline{Q}_i$, which can have an arbitrary amplitude.
- The modes are orthogonal (or can be made so) in an important physical sense: $\underline{Q}_J^T \cdot \underline{\underline{M}} \cdot \underline{Q}_I = 0$ for $I \neq J$.
- We can often use symmetry arguments to find obvious normal modes, then find the frequencies
  $$\omega^2 = \frac{\text{restoring force per unit extension}}{\text{mass}} .$$
- Examples: triatomic molecule; ozone; waves on strings and drums.

- General system: **generalised coordinates** $\{q_i\}, i = 1, N$.
- **Hamilton's Principle**: The **action** $S = \int \mathrm{d}t\, \mathcal{L}(q_i, \dot{q}_i, t)$ is stationary for small variations $\delta q_i(t)$ about the path $q_i(t)$. The Lagrangian $\mathcal{L} \equiv T - U$, where T is the kinetic energy and U is the potential energy.
- The condition that $\delta S = 0$ for all variations $\delta q_i(t)$ is $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$ for all i (**Lagrange's equations**).
- The quantities $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \equiv p_i$ are the **conjugate momenta**. If the Lagrangian does not depend on one of the coordinates, say q_1 (i.e. $\frac{\partial \mathcal{L}}{\partial q_1} = 0$), then from Lagrange's equations the conjugate momentum p_1 is a **constant**.
- **Symmetries** (here $\mathcal{L}$ independent of q_1) thus lead to **conservation laws**.
- If the Lagrangian does not depend on time explicitly (i.e. $\frac{\partial \mathcal{L}}{\partial t} = 0$), the **Hamiltonian** $H(q_i, p_i, t) \equiv \sum_i p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t)$ **is conserved** (Usually $H = T + U$).
- **Simple harmonic motion**: $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$. $\mathcal{L}$ does not depend on t.
  $\Rightarrow$The Hamiltonian (energy) $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2m}p_x^2 + \frac{1}{2}kx^2$ is conserved.
- **Orbits in central potential**: $\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$. $\mathcal{L}$ does not depend on ϕ or t:
  $\Rightarrow$ The angular momentum (p_ϕ) $J = mr^2\dot{\phi}$ is conserved;
  $\Rightarrow$ The energy (Hamiltonian) $E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r)$ is conserved.
- **Symmetric top**: $\mathcal{L} = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\chi} + \dot{\phi}\cos\theta)^2 - mgh\cos\theta$. $\mathcal{L}$ does not depend on ϕ, χ or t:
  $\Rightarrow$ The angular momentum (p_ϕ) $J_z = I_3(\dot{\chi} + \dot{\phi}\cos\theta)\cos\theta + I_1\dot{\phi}\sin^2\theta$ is conserved;
  $\Rightarrow$ The angular momentum (p_χ) $J_3 = I_3(\dot{\chi} + \dot{\phi}\cos\theta)$ is conserved;
  $\Rightarrow$ The energy $E = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\chi} + \dot{\phi}\cos\theta)^2 + mgh\cos\theta$ is conserved.

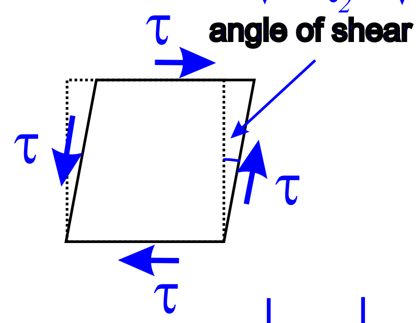
- Elastic behaviour: Stress $\propto$ Strain
- Longitudinal stress: $\dfrac{\text{Force}}{\text{Area}} = E\dfrac{\delta l}{l}$, where E is Young's modulus.
- Strains in the other directions are $-\sigma\dfrac{\delta l}{l}$, where σ is Poisson's ratio.
- E and σ determine the elastic properties of a uniform, isotropic medium.
- Normal stresses: τ_1, τ_2, τ_3 produce normal strains e_1, e_2, e_3 and are related by $Ee_1 = \tau_1 - \sigma\tau_2 - \sigma\tau_3$ and cyclically.



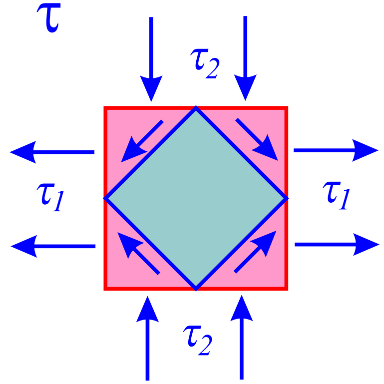
- Bulk modulus: medium under uniform pressure: $P = -B\dfrac{\delta V}{V}$: $\tau_1 = \tau_2 = \tau_3 = -P$.

  $Ee_1 = -P(1-2\sigma)$ and $\dfrac{\delta V}{V} \approx 3e_1$ give $B = \dfrac{E}{3(1-2\sigma)}$.

  An elastic medium must have $\sigma < \frac{1}{2}$.



- Shear modulus: $G = \dfrac{\text{shear stress}}{\text{shear angle}} = \dfrac{\tau_{xy}}{e_{xy}}$.
- Shear τ_{xy} can be produced by normal stresses $\tau_1 = \tau, \tau_2 = -\tau, \tau_3 = 0$.
- Shear stress τ transferred to inner square.

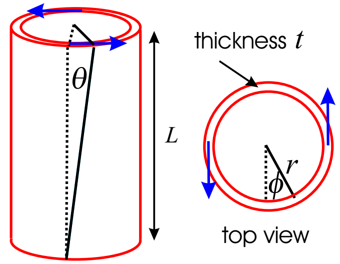


- Shear angle is $2e_1$, where $Ee_1 = \tau(1+\sigma)$ (see detailed diagram in notes).
- Shear modulus $G = \dfrac{E}{2(1+\sigma)}$. (Must have $\sigma > -1$.)

Summary Sheet — Elasticity II — S14

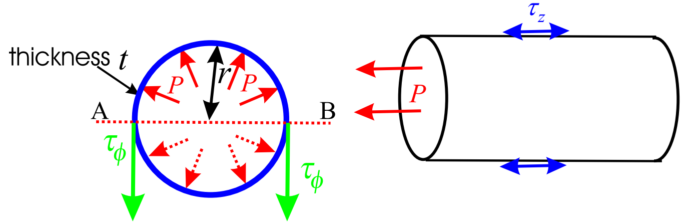
- Stress tensor τ relates force to area element $\mathrm{d}\boldsymbol{F} = \underline{\underline{\tau}}\cdot\mathrm{d}\boldsymbol{S}$.
- Stress tensor is symmetric $\tau_{xy} = \tau_{yx}$, etc and can be made diagonal:
- Arbitrary stresses can be made up from normal stresses $\{\tau_1, \tau_2, \tau_3\}$.
- Distortion of the medium takes a point at $\boldsymbol{x}$ to $\boldsymbol{x} + \boldsymbol{X}(\boldsymbol{x})$. The derivatives of the distortion field $\boldsymbol{X}(\boldsymbol{x})$ determine the **strain**.
- The strain tensor is defined as $e_{ij} = \frac{1}{2}\left(\frac{\partial X_i}{\partial x_j} + \frac{\partial X_j}{\partial x_i}\right)$ The distortion due to strain: $\delta \underline{X} = \underline{\underline{e}}\cdot\delta\underline{x}$.
- The strain tensor is also symmetric and there are a set of perpendicular normal strains $\{e_1, e_2, e_3\}$.
- In an isotropic elastic medium the principal axes of the strain tensor are the same as the principal axes of the stress tensor.
- Stress in terms of strain: $\tau_1 = (B - \frac{2}{3}G)(e_1 + e_2 + e_3) + 2Ge_1$, etc.
- There a part of the stress that is proportional to the strain (the G term), plus an additional isotropic pressure proportional to the net change of volume $e_1 + e_2 + e_3 = \mathrm{Tr}(\underline{\underline{e}}) = \boldsymbol{\nabla}\cdot\boldsymbol{X}$ (the $(B - \frac{2}{3}G)$ term). We can also write $\underline{\underline{\tau}} = (B - \frac{2}{3}G)\mathrm{Tr}(\underline{\underline{e}})\underline{\underline{I}} + 2G\underline{\underline{e}}$.
- Elastic energy: $U(\underline{\underline{e}}) = \frac{1}{2}\left((B - \frac{2}{3}G)(e_1 + e_2 + e_3)^2 + 2G(e_1^2 + e_2^2 + e_3^2)\right)$

- Torsion of thin tube: $C = \dfrac{2\pi G r^3 t}{L}\phi$. Torsion of cylinder: $C = \dfrac{\pi a^4}{2L}\phi$.



- Tube under pressure P: hoop stresses $\tau_\phi = \dfrac{Pr}{t}(\gg P)$.

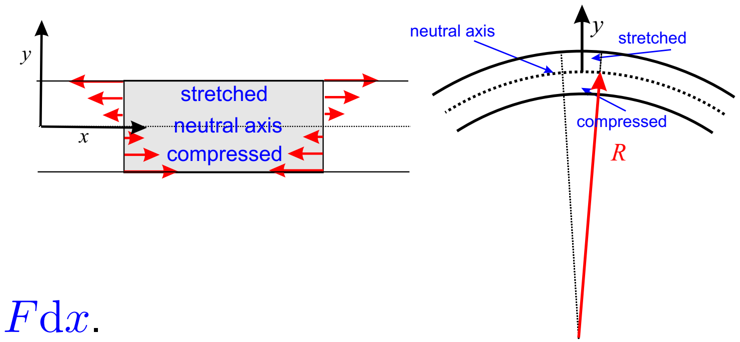
  Longitudinal stress $\tau_z = \dfrac{Pr}{2t}$.



- Bending of **thin** beams:

  bending moment $B = \dfrac{EI}{R}$,

  where R is radius of curvature and $I = \displaystyle\int y^2 \mathrm{dArea}$.



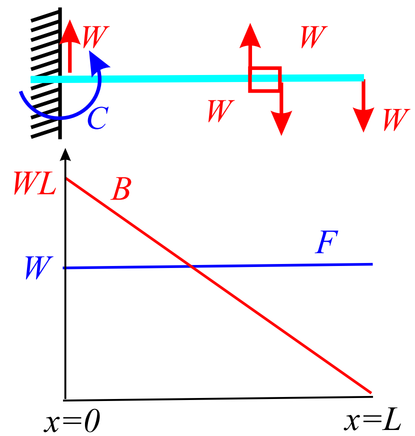
- Force F on element $\mathrm{d}x$ changes bending moment $\mathrm{d}B = -F\mathrm{d}x$.
- **Draw a diagram of the forces and bending moments** along the beam and then use $B = EI/R \approx EIy''$.
- **Example**: cantilevered beam, loaded at free end.

  Force $F = W$ all along the beam.

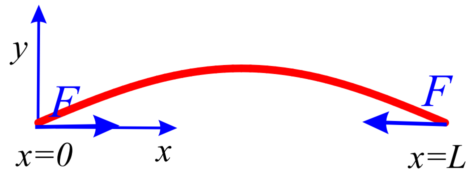
  Bending moment $B = W(L - x) = EIy''$

  decreases linearly. Integrating and $y = y' = 0$

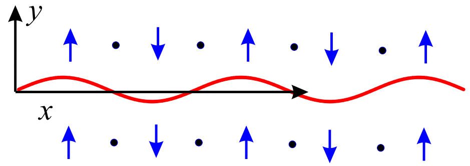
  at $x = 0$ gives $y(x) = \dfrac{W}{EI}\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right)$.



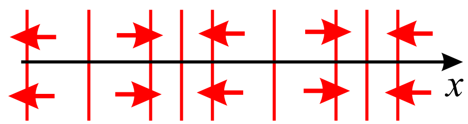
- **Reciprocity theorem**: deflection y_{PQ} of a beam at point P due to a load at Q **is the same** as the deflection y_{QP} of a beam at point Q due to a load at P. Reciprocity is very general in physics.



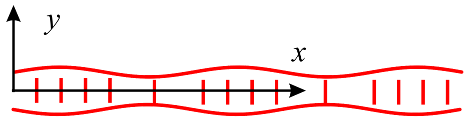
- Euler strut: buckled beam has bending moment $B = -Fy = EIy''$.
  Solution $y = A\sin\sqrt{EI/F}x$ with $\sqrt{EI/F}L = \pi$.
  Euler force $F_\mathrm{E} = \frac{\pi^2 EI}{L^2}$, independent of A.
- **Dynamics of elastic media.** Acceleration due to variation of stresses: $\rho\frac{\partial^2 X_i}{\partial t^2} = \sum_j \frac{\partial \tau_{ij}}{\partial x_j}$.
- Stress related to strain as $\underline{\underline{\tau}} = (B - \frac{2}{3}G)\mathrm{Tr}(\underline{\underline{e}})\underline{\underline{I}} + 2G\underline{\underline{e}}$. This leads to a **vector wave equation** $\rho\frac{\partial^2 \boldsymbol{X}}{\partial t^2} = (B + \frac{1}{3}G)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{X}) + G\boldsymbol{\nabla}^2\boldsymbol{X}$, which permits wave solutions $\boldsymbol{X}_0 e^{i(\omega t - kx)}$



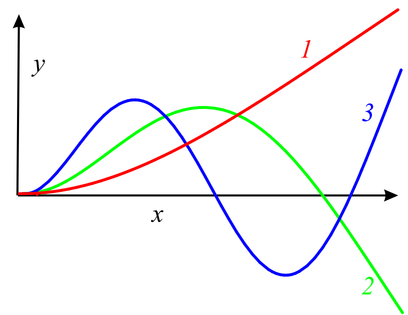
- For transverse **shear**, or **S-waves**: waves polarised in y- or z-directions
  we find $\rho\omega^2 = Gk^2$, so velocity $v_\mathrm{S}^2 = G/\rho$.
- Longitudinal **compressional**, or **P-waves** have velocity $v_\mathrm{P}^2 = (B + \frac{4}{3}G)/\rho$.



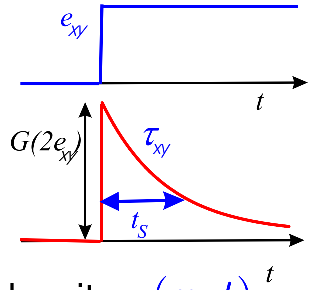
- Surface effects are important. Waves on thin rod: $v^2 = E/\rho$.
  Waves on thin plate: $v^2 = E/(1-\sigma^2)\rho$ (bulges sideways) .



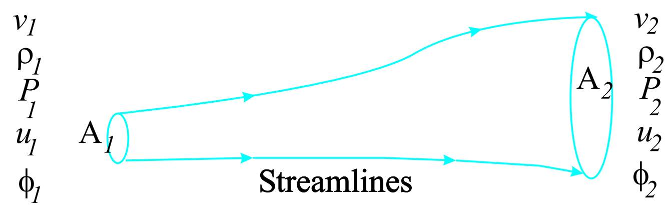
- Waves $y(x,t)$ on string, density ρ, tension T.
  Energy : $\int \mathrm{d}x\ \frac{1}{2}(\rho\dot{y}^2 + T(y')^2)$. Energy flow is $-\dot{y}(Ty')$.
  Examples: energy in travelling waves and standing waves. Energy flow in elastic waves: $-\underline{\underline{\tau}}\cdot\dot{\boldsymbol{X}}$.



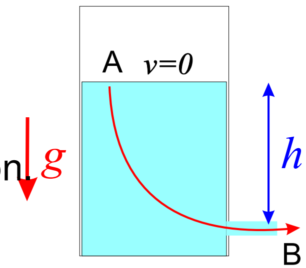
- Normal modes of a cantilevered bar $\propto e^{i\omega t}$.
- $y = Ae^{ikx} + Be^{-ikx} + Ce^{kx} + De^{-kx}$, where $\omega^2 = (EI/\rho)k^4$.
  The first few modes have $k_n L = (1.875, 4.694, 7.854, 10.995\ldots)$
  Not harmonic: $\omega \propto n^2$.

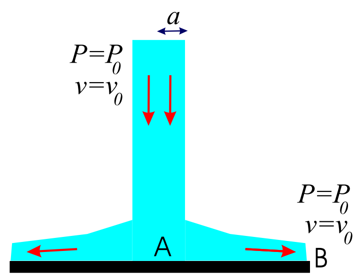


- Liquids and gases (**fluids**) cannot maintain shear stresses. A sudden shear produces a stress that rapidly decays. In fluids the only stresses are an **isotropic pressure** P.
- There is a very wide variety of fluid behaviour on Earth and in astrophysics!
- **Fluid elements** are regions large enough to have well-defined macroscopic properties: density $\rho(\boldsymbol{x},t)$; velocity $\boldsymbol{v}(\boldsymbol{x},t)$; pressure $P(\boldsymbol{x},t)$ and (for compressible fluids) energy density $u(\boldsymbol{x},t)$.
- A fluid element is acted on by pressure forces and gravity $(\boldsymbol{g} = -\boldsymbol{\nabla}\phi_{\rm g})$.
- Flow visualisation: a **streamline** follows the velocity vector $\boldsymbol{v}(\boldsymbol{x})$ at a given time. The concepts of **particle paths** and **streaklines** are sometimes useful.
- The continuity equation for compressible flow $\dfrac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0$.
- For incompressible flow the density ρ is a constant everywhere and $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, except at **sources** and **sinks**.
- The vector equation of motion for a fluid element is $\rho\dfrac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla}P + \rho\boldsymbol{g}$, where the **convective derivative** $\dfrac{D\boldsymbol{v}}{Dt} = \dfrac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}$ is the acceleration of the fluid element.
- For **steady flow**, consider streamlines connecting areas A_1 and A_2:

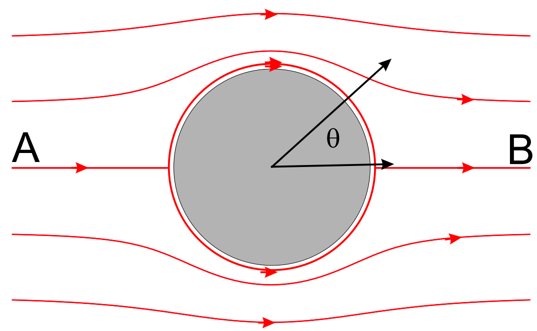


- **Compressible Bernoulli equation**: along a streamline the quantity $(u+P)/\rho + \frac{1}{2}\boldsymbol{v}^2 + \phi_{\rm g} =$ constant. ($h \equiv u + P$ is the specific enthalpy ($h = \gamma/(\gamma-1)$ for a perfect gas.)).
- For **incompressible, steady flow** ρ is constant and $u = 0$, so $P + \frac{1}{2}\rho\boldsymbol{v}^2 + \rho\phi_{\rm g} =$ constant along a streamline.

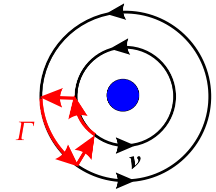




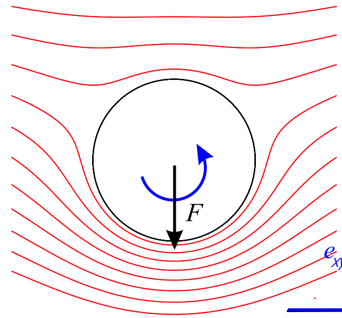
- Applications of Bernoulli: 1) Flow from water tank: $\boldsymbol{v}^2 = 2gh$.
  2) Water jet hitting wall (no gravity) $v_{\rm out} = v_{\rm in}$.
- Circulation around a loop Γ $K = \oint_\Gamma \mathrm{d}\boldsymbol{l}\cdot\boldsymbol{v}$ defines the amount of rotation.
- Related to the vorticity $\boldsymbol{\omega} \equiv \boldsymbol{\nabla}\times\boldsymbol{v}$ by $K = \int \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{\omega}$.
- Kelvin's circulation theorem for incompressible flow: $\dfrac{DK}{Dt} = 0$: "vortex lines are conserved and move with the fluid".
- Generalised Bernoulli: for incompressible and **irrotational** flow ($\boldsymbol{\nabla}\times\boldsymbol{v} = 0$):
  $(\rho\dot{\Phi}+)P + \frac{1}{2}\rho\boldsymbol{v}^2 + \rho\phi_{\rm g}$ **is a constant everywhere**.
- If $\boldsymbol{\nabla}\times\boldsymbol{v} = 0$, there is a **scalar velocity potential** Φ such that $\boldsymbol{v} = \boldsymbol{\nabla}\Phi$.
- For incompressible flow $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ so $\boldsymbol{\nabla}^2\Phi = 0$ and we can use potential to find the velocity distribution and use Bernoulli to find the pressure afterwards.
- Flow rate Q from an isotropic source has velocity potential and flow
  $\Phi = -\dfrac{Q}{4\pi R}$; $\boldsymbol{v} = \dfrac{Q}{4\pi R^2}\hat{\boldsymbol{u}}_R$. Source near plate — use method of images.
- Two sources or sinks attract: $F = \dfrac{Q_1 Q_2 \rho}{4\pi d^2}$. Sources and sinks repel.
- Potential flow around spheres: ($v_r(a) = 0$) $\Phi = V_0\cos\theta\left(r + \dfrac{a^3}{2r^2}\right)$
- Bernoulli gives pressure on sphere:
  $P(\theta) = P_0 + \frac{1}{2}\rho V_0^2 - \frac{9}{8}\rho V_0^2 \sin^2\theta$

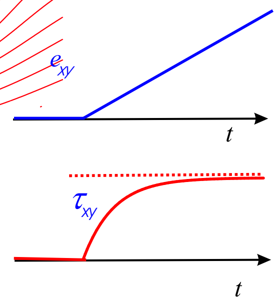


SUMMARY SHEET — VORTICITY AND VISCOSITY — S19

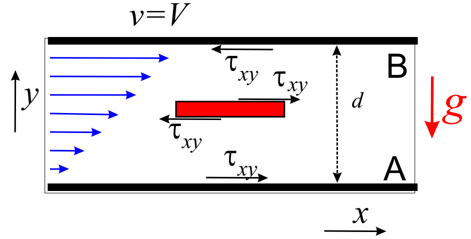


- Potential flow around cylinder: $\Phi = V_0 \cos\theta \left(r + \frac{a^2}{r}\right)$
- Irrotational flow around rotating cylinder: $\Phi = \frac{K\theta}{2\pi}$: $v_\theta = \frac{K}{2\pi r}$
- Rotating cylinder in external flow: $\Phi = V_0 \cos\theta \left(r + \frac{a^2}{r}\right) + \frac{K\theta}{2\pi}$
- Magnus effect $\boldsymbol{F} = \rho \boldsymbol{v} \times \boldsymbol{K}$
- Vortex in free fluid has core of vorticity rotating like a solid body.
- Opposite vortices "blow each other along"; like vortices circle each other.
- Vortex rings: $v = \frac{K}{4\pi a} \log(a/r_{\text{core}})$. Many interesting properties.



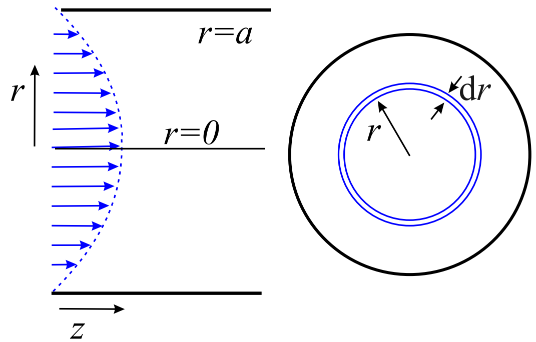


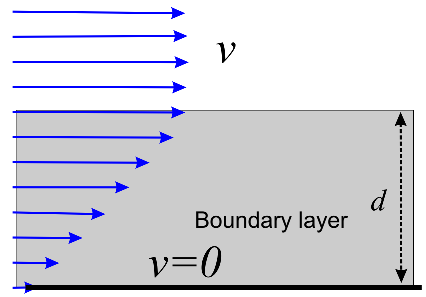
- Viscous shear stress in Newtonian fluids proportional to rate of shear.
- Viscosity: $\underline{\underline{\tau}} = \eta \frac{\mathrm{d}(2\underline{\underline{e}})}{\mathrm{d}t} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$.
- Equation of motion for incompressible flow including viscosity: $\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} P + \rho \boldsymbol{g} + \eta \boldsymbol{\nabla}^2 \boldsymbol{v}$.
- Shear layer (Couette flow): viscous stress $\tau_{xy} = \eta \frac{\mathrm{d}v_x}{\mathrm{d}y} = \eta V/d$ is constant in y and is transmitted to both plates.



- Poiseuille flow in circular pipe with longitudinal pressure gradient.

  Volume flow rate: $Q = \frac{\pi a^4}{8\eta} \left|\frac{\mathrm{d}P}{\mathrm{d}z}\right|$.
- Friction factor f: $\left|\frac{\mathrm{d}P}{\mathrm{d}z}\right| \equiv f \frac{\rho \overline{v}^2}{2a}$ (Mean flow velocity $\overline{v} = \frac{Q}{\pi a^2}$).
- For lamina flow: $f = 32/N_{\mathrm{R}}$, where the dimensionless **Reynolds number** is $N_{\mathrm{R}} = \rho(2a)\overline{v}/\eta$.





- Boundary condition for surface of solid body: no radial or tangential velocity (no slip).
- Flow past a solid body must develop a boundary layer with a velocity gradient.

  This is a region with vorticity ($\boldsymbol{\nabla}\times\boldsymbol{v} \neq 0$), which can then enter the fluid flow.
- The **Reynolds number** $N_{\rm R} = \dfrac{\rho v L}{\eta}$ is the ratio of the inertial stress ρv^2 to the viscous stress $\eta v/L$, where v is a characteristic velocity and L a characteristic length scale.
- In steady **laminar flow**, the kinematic viscosity is given by $\eta/\rho \sim \lambda_{\rm c} v_{\rm T}$, where $\lambda_{\rm c}$ is the mean-free path and $v_{\rm T}$ is the thermal velocity.
- If the inertial stress is too high in a flow random transverse motions will cause turbulent mixing, which increases the effective viscosity to include an “eddy viscosity” $(\eta/\rho)_{\rm effective} \sim \lambda_{\rm c} v_{\rm T} + L_{\rm eddy} v_{\rm eddy}$. Energy is transported by nonlinear processes to progressively smaller scales, where viscous dissipation can occur.
- Examples:

  Realistic flows around spheres, golf balls and cylinders.

  Turbulent flows in pipes.

  Subsonic turbulent jets.

  Extragalactic radio galaxies.