Rotational Mechanics & Special Relativity

Prof. Steve Gull

Examples book 2014

\[ F = m \frac{d^2 r}{dt^2} \]

\[ E = \gamma mc^2 \]
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Special Relativity: Frames of reference: general ideas. Historical development: problems with classical ideas; the Aether; Michelson-Morley experiment. Inertial frames: Galilean transformation. Einstein’s postulates: statement; events, and intervals between them; consequences for time intervals and lengths; Lorentz transformation of intervals; simultaneity; proper time; twin paradox; causality; world lines and space–time diagrams. Velocities: addition; aberration of light; Doppler effect. Relativistic mechanics: momentum and energy; definitions; what is conserved; energy–momentum invariant. Nuclear binding energies, fission and fusion.

BOOKS

Understanding Physics, Mansfield & O’Sullivan (Praxis 2008)
Physics for Scientists and Engineers (with Modern Physics), Tipler P A & Mosca G (6th Edition Freeman 2008)
About this book of examples ....

Here are twenty-four examples (questions, problems) which have been carefully chosen to illustrate most of the material of the course, to aid your understanding, and to give you practice in analysis. You should aim to answer all of them by the beginning of the Easter the term if you can. To keep up you will need to attempt about six per week, but be guided by your supervisor. Keep all your written work as it will help you with revision in the Easter Term.

I have also included suggestions for further examples taken from past Tripos papers for those that want to try out more problems. These are strictly voluntary, and you should not feel under compulsion to tackle them. It will be sufficient to do the twenty-four examples written out in this book.

You may find some of these examples are more difficult than those you have met before. Don’t be alarmed if it takes you some time to learn the techniques required to answer them. Talking to others, your colleagues and in particular your supervisor, is a valuable way to explore different approaches to answering problems. Remember that you are not in competition with your colleagues. It is often very helpful, and quite satisfactory, to work with another student such as your supervision partner.

There follows a set of guidelines which you should adopt when you tackle any physics problem. Please use them – they really will help! Particular advice for answering relativity problems is provided in the Special Relativity section.

SFG
January 2014
How to solve Physics problems

Here is some general advice which you should apply to every Physics problem you tackle. Now is the time to develop the techniques which make the problems easier to understand and which can provide you with a framework within which to think about Physics.

1. **Draw a diagram**

   A diagram always helps to clarify your thoughts. **Use it to define the symbols you need to use** (see 3 below). Make it big enough and be tidy.

2. **Think about the Physics**

   Ask yourself what is going on, and write it down in words in just one or perhaps two sentences. Try to **understand** the problem qualitatively before writing down any equations. Do not just write down equations!

3. **Stay in symbols until the end**

   At school you may have been taught to make calculations numerically rather than algebraically. However, you usually give yourself a big advantage if you delay substitution of numerical values until the last line as it enables you to check dimensions at every stage, and quantities often cancel before the last line. An exception to this rule arises where some terms are dimensionless factors which are simple fractions.

4. **Check the dimensions**

   Think about the dimensions of every quantity even as you write it down. You will find this a discipline which helps enormously to avoid errors and helps understanding. Make sure that the dimensions of your final equation match on each side before you make a numerical substitution. Write down the units of your answer at the end e.g. 4.97 J kg⁻¹.

5. **Does the answer make sense?**

   You will probably have an idea of what looks about right, and what is clearly wrong. Many mistakes are simple arithmetic errors involving powers of ten. If in doubt, check your substitutions.
A question about calculating the position of the centre of mass etc. by integration.

1. (a) A uniform solid cone has a height $b$ and a base radius $a$. It stands on a horizontal table.

   (i) Draw a diagram showing the cone divided into thin horizontal discs, each of thickness $\delta h$. Find an expression for the volume of the disc at height $h$ above the base. Integrate over all the discs to show that the total volume, $V$, is given by $V = \frac{\pi}{3} a^2 b$.

   (ii) The height, $b_0$, of the centre of mass is defined by the relation

   $$ b_0 = \sum_i m_i h_i / M, $$

   where $M$ is the total mass, $m_i$ is the mass of the disc at height $h_i$ and the sum is taken over all the discs. Treat the sum as an integral, and hence show that $b_0 = b/4$.

(b) A uniform solid cylinder of radius $r$ and length $l$ is cut into two equal parts along its cylindrical axis. Find the position of the centre of mass of either part.

   \{ \frac{4r}{3\pi} \}

The rotational equivalent of the mass is the moment of inertia, and is the subject of the following example.

2. State the parallel and perpendicular axis theorems.

   (a) Calculate the moment of inertia of a uniform square plate of side $a$ and mass $m$ about an axis through its centre and parallel to a side.

   (b) Use the perpendicular axis theorem to find the moment of inertia through the centre and perpendicular to its plane.

   (c) More generally, show that the moment of inertia of a square plate about any axis in its plane through its centre is the same.

   (d) Use the theorems of parallel and perpendicular axes to find the moment of inertia of a hollow thin cubical box of side $a$ and total mass $M$ about an axis passing through the centres of two opposite faces.

   \{(a) ma^2/12; (b) ma^2/6; (d) 5Ma^2/18\}
Here is an example of the conservation of angular momentum, and the transformation of energy between different forms. Imagine that you are an astronaut in the spacecraft sitting on a bench in the middle and looking out of a window in the curved side of the spacecraft. You are holding the string of part (b). As you slowly let out the string, think about the tension you feel in the string, the work you do, and what you see out of the window.

3. A spacecraft can be regarded as a uniform cylinder, 1 m in diameter and of mass 250 kg. It is spinning about its cylindrical axis with a period of 3 s.

(a) What is the angular momentum of this isolated system? This quantity remains constant for all time.

(b) A mass of 50 g attached to a long string is slowly let out from the side of the cylinder until the period has increased to 10 minutes. Explain why the period increases and calculate the length of the string.

(c) The cylinder slows down when the string is let out, so it must feel a torque. Draw a diagram showing clearly how the torque arises.

(d) Calculate the initial and final kinetic energies of rotation of the system.

(e) An electric motor in the cylinder is now switched on to wind in the string slowly. Explain why it has to do work. Where does the work go?

\{a) 65.4 \text{ kg m}^2 \text{ s}^{-1}; (b) 353 \text{ m}; (d) 68.5 \text{ J}, 0.34 \text{ J}\}

Here is an example of an “angular collision”. Note that angular momentum is conserved only if the system is isolated.

4. Two gear wheels are cut from the same uniform sheet of metal, the mass per unit area of which is $\sigma$. One gear wheel has radius $a$ and the other radius $2a$ with twice as many teeth. They are mounted on parallel light axles through their centres and perpendicular to their faces just far enough apart not to mesh.

(a) Calculate their moments of inertia in terms of $\sigma$ and $a$.

(b) The larger wheel is now spun with angular speed $\omega$. What is the angular momentum in terms of $\sigma$, $a$ and $\omega$?

(c) The gear wheels are now suddenly meshed, but their axles remain in the same plane. During this process energy is lost, and the wheels exert equal and opposite
tangential impulses \( \int F(t) dt \) on each other. Consider the effect of the angular impulses \( \int rF(t) dt \) associated with these forces on the angular momentum of each wheel, where \( r \) is the radius of the wheel concerned. Hence show that the angular speed of the larger wheel falls by 20 per cent.

(d) Show that the total angular momentum falls by 30 per cent. Why is it not conserved in your calculation?

Break this question down into parts: first draw a diagram, and then find the frictional force on a circular element of the disc ... etc.

5. A uniform disc of mass \( m \) and radius \( a \), rotating at an angular speed of \( \omega_0 \) is placed flat on a horizontal flat surface. If the coefficient of friction is \( \mu \), find the frictional torque on the disc, and hence calculate the time it takes to come to rest.

\[
\mu \frac{3\omega a}{g} 
\]

6. The hollow cubical box of question 2 (d) is suspended from a horizontal frictionless hinge along one of its edges. The box is displaced slightly from equilibrium. Show that it undergoes simple harmonic motion with a period, \( T \), given by

\[
T = 2\pi \sqrt{\frac{7\sqrt{2} a}{9g}}
\]

These are examples of “solid body dynamics” in which there is both rotational and linear motion

7. A thin, uniform bar of length \( l \) and mass \( M \) is suspended horizontally at rest. It is suddenly released, and at the same instant, is struck by a sharp blow vertically upwards at one end — the duration of the impulse can be taken to be negligibly short.

(a) “The centre of mass moves as if acted upon by the (vector) sum of the external forces”. Describe the motion of the bar’s centre of mass.

(b) “The rotation about the centre of mass is caused by the sum of the moments of the external forces”. Describe the rotational motion of the bar.

(c) Hence, by combining your answers to (a) and (b), describe how the bar moves after being struck. Be as quantitative as possible, considering in particular (i) the
time taken by the centre of mass to reach maximum height, and (ii) the rate of rotation.

(d) In a particular experiment, the bar passes through its original position in the same orientation after a time $t$. Demonstrate that $t^2 = \frac{2\pi n l}{3g}$, where $n$ is an integer and $g$ is the magnitude of the acceleration due to gravity.

8. A thin uniform rod of mass $M$ is supported horizontally on knife edges at each end. If one of the supports is suddenly removed, show that the force on the other end is instantaneously reduced from $\frac{Mg}{2}$ to $\frac{Mg}{4}$, where $g$ is the magnitude of the gravitational acceleration.

9. A solid cylinder of mass $M$ and radius $a$ is free to roll on a horizontal surface and is connected to a light spring of constant $k$ as shown in the diagram. The system is displaced from equilibrium by rolling the cylinder so that the spring extends a small amount along its axis. Show that the system executes simple harmonic motion with a period, $T$, given by

$$T = 2\pi \sqrt{\frac{3M}{2k}}.$$
Several further questions on rotational mechanics from past Tripos papers (not compulsory):

2009 B7 (Answer: $\omega = \frac{6\sqrt{2gh}}{7l}; h/49$)

2007 B9 (Answer: $mu$, applied $2a/5$ above centre; $7mu^2/10; \sqrt[7]{\frac{10}{7}g(r + a)(1 - \cos \theta)}$)

2004 A1 (Answer: $h_u/2$)

2004 A2 (Answer: $34ma^2/3$)

2003 B7 (Answer: $\sqrt{3g/L}; m\sqrt{gL/3}; m\sqrt{3gL/4}$ at $2L/3$ from the axis)
Relativistic Kinematics is the application of the Special Theory of Relativity to space and time. Please read the following general advice on how to tackle kinematical relativity problems before proceeding.

Some or all of the following ‘rules’ can be applied to solve any kinematical problem in Special Relativity. If you apply the rules carefully, without first muddling yourself with too much potentially confusing thought about what contracts and what dilates etc., you will get the right answer. You can ponder about what it all means when you know that you have the right answer!

(i) **Identify the events.** Label them A, B, C etc. Thus event A may be the flash of a light, B the spaceship exploding, C the arrival of a message at the Earth etc.

(ii) **Draw diagrams showing the events in the relevant frames of reference.** Thus you might show events A and B as seen both in the Earth frame and in the rocket frame.

(iii) **Write down the intervals between the events in all frames.** Set these equal to known quantities where you can and put a question mark where you can’t. Thus you might write \( \Delta x'_{AB} = l_0; \ \Delta x_{AB} = ?; \ \Delta t'_{AB} = l_0/c; \ \Delta t_{AB} = ? \)

(iv) **Apply the Lorentz transformation to the intervals to find the unknown values.** If \( S' \) is the frame moving at speed \( v \) parallel to, and in the direction of, the positive \( x \) axis of frame \( S \), then the Lorentz transformation of the interval, \((\Delta x, \Delta y, \Delta z, \Delta t)\), between two events as observed in the \( S \) frame, and the interval \((\Delta x', \Delta y', \Delta z', \Delta t')\) between the same two events as observed in the \( S' \) frame is:

\[
\begin{align*}
\Delta t' &= \gamma (\Delta t - v\Delta x' / c^2); \\
\Delta x' &= \gamma (\Delta x - v\Delta t); \\
\Delta y' &= \Delta y; \\
\Delta z' &= \Delta z;
\end{align*}
\]

\[
\begin{align*}
\Delta t &= \gamma (\Delta t' + v\Delta x' / c^2); \\
\Delta x &= \gamma (\Delta x' + v\Delta t'); \\
\Delta y &= \Delta y'; \\
\Delta z &= \Delta z';
\end{align*}
\]

\[
\gamma = \sqrt{1 \over (1 - v^2 / c^2)}.
\]
**Special Relativity kinematics:** these two examples are about reference frames. The second example demonstrates the difference between the Galilean and Lorentz transforms. Remember that the former applies only when \( v \ll c \).

10. Explain in one page of writing what is meant by a frame of reference. Your answer should include a description of how you would measure positions and velocities, a description of Cartesian coordinates, and the manner in which you can transform your point of view from one frame to another. What special criterion needs to be satisfied for the frame of reference to be an inertial frame?

11. The space and time coordinates of two events as measured in a Galilean frame \( G \) are as follows:

   Event A: \( x_A = x_0, \ t_A = x_0/c \),

   Event B: \( x_B = 2x_0, \ t_B = x_0/(2c) \),

where \( y = z = 0 \) for both events.

(a) What is the speed and direction of travel of another Galilean frame \( G' \) in which both events occur at the same place? (Use the Galilean transformation – i.e. the classical, non-relativistic transformation. Assume the “standard configuration” for the frames \( G \) and \( G' \), with \( G' \) moving at speed \( v \) along the \(+x\)-axis, and the origins of \( G \) and \( G' \) coinciding at \( x = x' = 0, \ t = t' = 0 \).)

(b) Comment on the result. In particular, explain why the Galilean transformation is inappropriate in this example.

(c) Can the events be seen to occur at the same time in any Galilean frame?

(d) Now use the Lorentz transformation to calculate the speed and direction of travel of an inertial frame \( S' \) in which both events occur at the same time. (Take the “standard configuration” for the frames \( S \) and \( S' \), which is that \( S' \) moves at speed \( v \) along the \( x \)-axis, and the origins of \( S \) and \( S' \) coincide at the instant \( x = x' = 0, \ t = t' = 0 \).)

(e) Comment on the result. In particular, explain why the Lorentz transformation is appropriate in this example.

(f) When and where do the events occur as measured in \( S' \)?

\[ \{ (a) -2c \ (d) -c/2; \ (f) \ t' = \sqrt{3}(x_0 / c), \ x'_A = \sqrt{3}x_0, \ x'_B = 3\sqrt{3}x_0 / 2 \} \]
The next example is a variation of the so-called twin paradox. There is no paradox if you are careful to analyse the circumstances of each twin strictly according to the rules of special relativity. A paradox only arises if your analysis is faulty, or your thinking is woolly!

12. A spaceship sets off from Earth for a distant destination, travelling in a straight line at a uniform speed of $3c/5$. Ten years later, as measured on the Earth, a second spaceship sets off in the same direction with a speed of $4c/5$. The captains of the two vessels are twins.

(a) For how long, in the Earth’s frame, does each of the spaceships travel before the second spaceship catches up with the first?

Consider three events: (A) the slower spaceship leaves the Earth; (B) the faster spaceship leaves the Earth; and (C) the faster spaceship catches up with the slower spaceship.

(b) If the event A has coordinates $x = 0$, $t = 0$ in the Earth’s frame, what are the coordinates of the other two events, also as observed in the Earth’s frame?

(c) By transforming these events to frames moving with the slower and faster spaceships respectively, determine which of the twins is older, and by how much, when the faster spaceship catches up with the slower spaceship.

(a) 40 years, 30 years; (b) $x = 0$, $t = 10$ years and $x = 24$ light years, $t = 40$ years; (c) the first captain is older by 4 years

Two further examples of transformation of distances and times between different frames.

13. As a spaceship passes the Earth with a speed of $0.8c$, observers on this spaceship and on the Earth agree that the time is 12:00 in both places. Thirty minutes later, as measured on the spaceship’s clock, the spaceship passes an interplanetary navigation station fixed relative to the Earth. The clock on the interplanetary navigation station reads Earth time.

(a) What is the time on the navigation station clock as the spaceship passes?

(b) How far from the Earth, as measured in the Earth’s frame, is the navigation station?

(c) As the spaceship passes the navigation station, it reports back to Earth by radio. When, according to a clock on the Earth, is the signal received?

(d) There is an immediate reply from Earth. When, according to the spaceship’s clock, is the reply received at the spaceship?

{(a) 12:50; (b) $7.2 \times 10^{11}$ m; (c) 13:30; (d) 16:30}
14. A flash of light is emitted from the tail of a rocket of length $L_0$ (measured in its
rest frame) towards the nose. The flash is reflected by a mirror at the nose and
received back at the tail. If the rocket is moving at speed $v$ relative to the Earth,
what are the time intervals measured in the Earth's frame between the emission,
reflection, and reception of the flash?

$$\{ \sqrt{(c+v)/(c-v)}(L_0/c); \sqrt{(c-v)/(c+v)}(L_0/c) \}$$

A further question from a past Tripos paper (not compulsory):

2009 B8 (Answer: the star is 4 light-years away; clock on Ship B reads 18 years when
it gets the message; ship B arrives home when Earth time is 45 years; clock on ship A
is ahead of that on ship B by 5 years.)

The relativistic transformation of speeds

These are the formulas you must use when adding speeds together, or transforming
speeds from one inertial frame to another. For example, consider the case of a
passenger walking along the corridor of a carriage of a train in the same direction as
the train is travelling from a classical, non-relativistic, point of view. Inside the train,
the passenger’s speed is $u'_x$ relative to the carriage. If the train is also moving at speed
$v$ relative to the tracks, then the passenger’s speed relative to the tracks is $u_x = u'_x + v$.
This result applies when both $u'_x$ and $v$ are very much less than the speed of light, $c$,
i.e.:

$$u_x = u'_x + v$$

The classical approximation (if both $u'_x << c$ and $v << c$)

$$u_y = u'_y$$

$$u_z = u'_z$$

Now consider the case in which the passenger is an astronaut moving along a space
ship at speed $u'_x$ relative to the space ship in the same direction as the space ship is
travelling. Suppose that the space ship’s speed with respect to the Earth is $v$, and that
$v$ is comparable with $c$. In this case, you must use the relativistic transformation of
speeds formulas to find the astronaut’s speed, $u_x$, with respect to the Earth as follows:

$$u_x = \frac{u'_x + v}{1 + u'_x v / c^2}$$

Relativistic (always true)

$$u_y = \frac{u'_y}{\gamma(1 + u'_x v / c^2)}$$

$$u_z = \frac{u'_z}{\gamma(1 + u'_x v / c^2)}$$
Two examples of the way speeds transform between different frames

15. Two sub-atomic particles approach each other with velocities (relative to an observer at rest in the laboratory) of \(\frac{3}{5}c\) and \(-\frac{2}{5}c\). What is the speed of one particle as observed in the frame of reference of the other? With what velocity would the observer have to move to measure their velocities as equal and opposite?

\[ \frac{25c}{31}, 0.134c \]

16. A \(\pi^0\) meson, travelling with velocity \(\left(\frac{3}{4}c, 0, 0\right)\) in the laboratory frame, \(S\), decays into two photons in the \(xy\) plane. In \(S',\) the rest frame of the meson, one of the photons is emitted at an angle \(\theta' = 60^\circ\) to the \(x'\) axis. In frame \(S\), calculate the \(x\) and \(y\) components of the velocities of the two photons, and hence their angles of emission with respect to the \(x\) axis.

\[ \frac{10c}{11}; \frac{2c}{5}; 24.62^\circ; 66.42^\circ \]

A further question on addition of speeds from a past Tripos paper (not compulsory):

2006 B9 (Answer: 25 days after start, 15 light-days from Earth; 64 days after start; 4 days’ supply left.)

An example of the effects of simultaneity as viewed in different frames

17. A very fast train, of length \(L_0\) (measured in its own frame), rushes through a station which has a platform of length \(L\) (\(<L_0\)).

(a) What is the speed \(v\) of the train such that the back of the train is opposite one end of the platform at exactly the same instant as the front of the train is opposite the other end, according to observers on the platform?

(b) According to these observers, two porters standing at either end of the platform (a distance \(L\) apart) are foolish enough, but have quick enough reactions, to kick the train simultaneously as it passes, thereby making dents in it. When the train stops, the dents are found to be a distance \(L_0\) apart. Explain in words and with diagrams how the difference between \(L\) and \(L_0\) is explained by (i) an observer on the platform, and (ii) an observer travelling on the train?

\[ \{(a) \ v = c\sqrt{1-(L/L_0)^2} \} \]
A question about the relativistic Doppler effect. One way of thinking about this is to imagine that the reflected pulses are coming from the image of the rocket as seen in a plane mirror (the planet’s surface).

18. A rocket moving away from the Earth with speed $v$ emits light pulses with a frequency $f_0$ of one pulse per second as measured by a clock on the rocket.

(a) Show that the rate, $f$, at which the pulses are received on the Earth is given by

$$f = f_0 \sqrt{\frac{c-v}{c+v}}.$$

(b) The rocket travels to a distant planet, and the signals are received on Earth both directly and by reflection from the planet. The pulse rates for the two signals are found to be in the ratio $1:2$. Explain why this is so, and deduce the speed of the rocket.

(c) If the rocket transmits only during its flight and the number of pulses received directly is $10^4$, what is the distance of the planet from the Earth?

\{(b) $c/3$; (c) $1.06 \times 10^{12}$ m\}

Two further questions on kinematics from past Tripos papers (not compulsory):

2003 B8 (Answer: $f/3$; $3t$)  
2008 B8 (Answer: $3c/5$; $4c/5$; 1.5 GHz)
Relativistic Dynamics is the application of the Special Theory of Relativity to energy and momentum etc. Please read the following general advice on how to tackle relativistic dynamics problems before proceeding with the examples.

It is not so easy to identify a fixed set of ‘rules’ for solving relativistic dynamics problems. However, there are certain principles which can be acknowledged, some or all of which may be helpful in any given case:

(i) **Identify the events**, and label them A, B, C etc.

(ii) **Draw diagrams** showing the conditions before and after the events in a given frame (e.g. the energies, momentums, masses, and velocities of particles before and after a collision event in the laboratory frame).

(iii) If possible, **identify another frame** (such as the zero-momentum frame) in which it is easier to make calculations. Draw diagrams showing conditions before and after events in this frame.

(iv) **Apply the principle of the conservation of linear momentum** in any single frame before and after any event, i.e.

\[
\sum p_i = \text{constant},
\]

where \( p = \gamma m v \), \( m \) is the mass, \( v \) is the velocity in this frame, and \( \gamma = \frac{1}{\sqrt{1-v^2/c^2}} \).

The momentum can also be expressed in terms of \( \gamma v \) as

\[
p = \gamma m v = \gamma m \left( \frac{m c^2}{E} \right). \]

(v) **Apply the energy-momentum invariant**: 

\[
E_1^2 - p_1^2 c^2 = E_2^2 - p_2^2 c^2 = \text{constant},
\]

where \( E \) and \( p \) are the total energy and the total momentum in the system respectively, evaluated at time 1 or in frame 1 etc. By ‘system’ we mean either a single particle, or a group of particles, or the contents of a whole frame. Note that \( E^2 - p^2 c^2 \) is the same before and after *any* event when calculated in *any* inertial frame, provided that you don’t change the system. \( E \) is the total energy given by

\[
E = \sum \gamma_i m_i c^2,
\]

where \( \gamma_i \) is the value of \( \gamma \) appropriate to the \( i \)th particle whose mass is \( m_i \). It must include particles which are stationary (i.e. \( \gamma = 1 \)) as well as moving particles. Add up all the energies, then square the total to get \( E^2 \).
Note that the kinetic energy is **not** $mv^2/2$, nor is it $\gamma mv^2/2$. The kinetic energy, $K$, is simply the difference between the energy a particle has when it is moving and the energy it has at rest, i.e.

$$K = \gamma mc^2 - mc^2 = (\gamma - 1) mc^2.$$ 

Similarly, add up all the contributions to the momentum (as vectors) before squaring the magnitude of the result. For a single particle:

$$E^2 - p^2c^2 = m^2c^4.$$ 

**(vi) Apply the Lorentz transformation to energy and momentum.** You will hear mention of *four-vectors* which transform according to the Lorentz transformation. There are many four vectors, such as the four vector $(ct, x, y, z)$. Energy and momentum can also be combined to form the energy-momentum four-vector $(E/c, p_x, p_y, p_z)$. If a particle has an energy $E$ and a momentum given by $p = (p_x, p_y, p_z)$ as observed in frame $S$, then the corresponding quantities in frame $S'$ are given by

$$E' / c = \gamma(E / c - v p_x / c);$$
$$p_x' = \gamma(p_x - v E / c^2);$$
$$p_y' = p_y;$$
$$p_z' = p_z;$$

and

$$E / c = \gamma'(E' / c + v p_x' / c);$$
$$p_x = \gamma(p_x' + v E' / c^2);$$
$$p_y = p_y';$$
$$p_z = p_z';$$
The next few questions are about Relativistic Dynamics – i.e. how energy and momentum transform between moving frames. Please first read the advice on Pages 16 & 17.

19. A particle as observed in a certain reference frame has a total energy of 5 GeV and a momentum of 3 GeV/c.

(a) What is its mass in units of GeV/c^2?

(b) What is its energy in a frame in which its momentum is equal to 4 GeV/c?

(c) Use the formulae for the relativistic transformation of speeds, or the energy-momentum transformation, to find the relative speed of the two frames of reference, if the particles are moving in the same direction.

{(a) 4 GeV/c^2; (b) 4\sqrt{2} GeV; −0.186c}

20. A particle of mass m and kinetic energy K strikes and combines with a stationary particle of mass 2m, producing a single composite particle of mass \sqrt{17}m. Find the value of K.

{2mc^2}

21. If K represents the relativistic kinetic energy of a particle of mass m, show that

\[ K^2 + 2mc^2 = p^2c^2, \]

where p represents the momentum of the particle.

A particle of mass M disintegrates while at rest into two parts having masses of M/2 and M/4. Show that the relativistic kinetic energies of the parts are 3Mc^2/32 and 5Mc^2/32 respectively.

22. Estimate the energy required to accelerate an electron from rest to half the speed of light. How does this compare with the result of the same calculation carried out non-relativistically?
Another collision question. Try transforming into the ZMF to simplify the problem. The value of the E-p invariant is the same before and after a collision, as viewed in any inertial frame.

23. A high-energy proton hits a stationary proton and produces a neutral \( \pi^0 \) meson (a pion) with mass of 0.144 times the proton mass via the reaction

\[ p + p \rightarrow p + p + \pi^0. \]

If the incident proton has just enough energy to allow this reaction to occur, what is the velocity (speed \( v \) and direction) of the final protons and the pion in the laboratory frame of reference?

The neutral pion is observed to decay into two photons of equal energy in the laboratory frame. Show that the angle \( \theta \) between the \( \pi^0 \) direction and the direction of either of the photons (as observed in the laboratory frame) is given by

\[ \cos(\theta) = \frac{v}{c}, \]

and hence determine the opening angle between the two photons. If the \( \pi^0 \) meson decays with a proper lifetime of \( 8.4 \times 10^{-17} \) s (as measured in its rest frame), how far on average does it travel before decaying?

\[ \{v = 0.36c; 137.8^\circ; 9.7 \text{ nm}\} \]

24. An \( \alpha \)-particle has twice the charge and four times the mass of a proton. If an accelerator imparts \( \sqrt{6} \) times as much momentum to an initially-stationary \( \alpha \)-particle as it does to an initially-stationary proton, what is the accelerating voltage?

\[ \{1.87 \text{ GV}\} \]

Three further questions on relativistic dynamics from past Tripos papers (not compulsory):

- **2007 B7** (Answer: 0.061\( r/c; 0.71fMc/(0.75fM+m) \))
- **2005 B7** (Answer: \( 3m/\sqrt{2} = 2.12m; (3/2)m^2, mc^2; (3/4)m^2, 0.75; (3/4)m/c, 71^\circ \))
- **2000 C13** (Answer: \( 4c/5, \sqrt{7}c/4 \))

Please email me (steve@mrao.cam.ac.uk) with any corrections or suggestions.