Beyond Euclidean Geometry

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A Wealth of Geometries

- So far, dealt with Euclidean geometry in 2 and 3 dimensions
- But a wealth of alternatives exist
  - Affine
  - Projective
  - Spherical
  - Inversive
  - Hyperbolic
  - Conformal
- Will look at all of these this afternoon!

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What is a Geometry?

- A geometry consists of:
  - A set of objects (the elements)
  - A set of properties of these objects
  - A group of transformations which preserve these properties
- This is all fairly abstract!
- Used successfully in 19th Century to unify a set of disparate ideas

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Affine Geometry

- Points represented as displacements from a fixed origin
- Line through 2 points given by set

$$AB = a + \lambda(b - a)$$

- Affine transformation

$$t(x) = \mathsf{U}(x) + a$$

- U is an invertible linear transformation
- As it stands, an affine transformation is not linear

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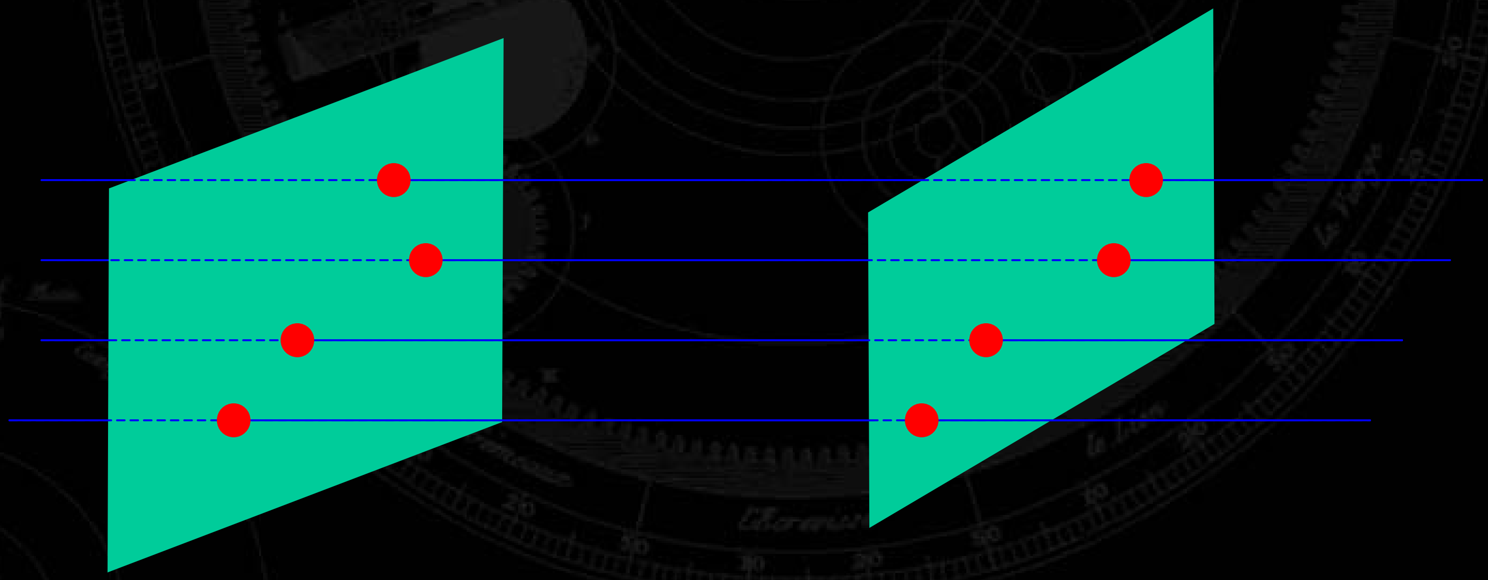
Parallel Lines

- Properties preserved under affine transformations:
  - Straight lines remain straight
  - Parallel lines remain parallel
  - Ratios of lengths along a straight line
- But lengths and angles are not preserved
- Any result proved in affine geometry is immediately true in Euclidean geometry

Geometric Picture

- Can view affine transformations in terms of parallel projections form one plane to another
- Planes need not be parallel



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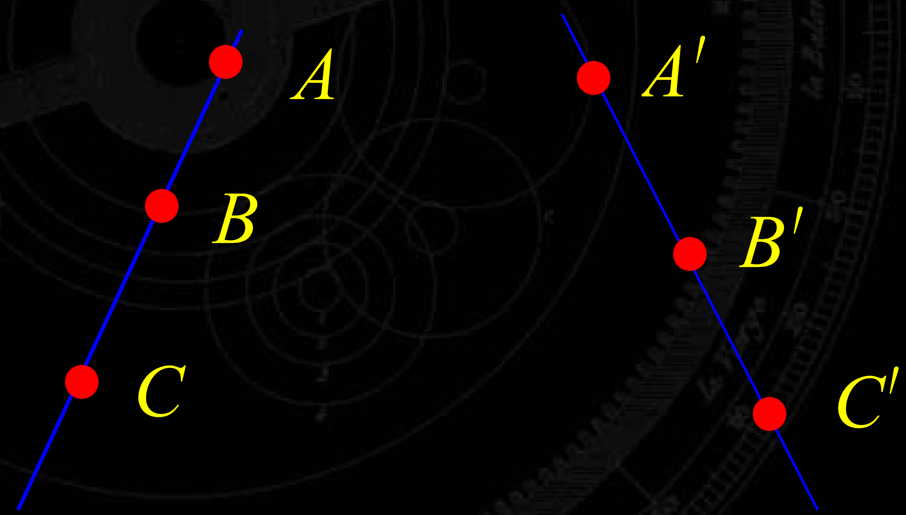
Line Ratios

- Ratio of distances along a line is preserved by an affine transformation

$$C = A + \lambda(B - A)$$

$$\frac{AC}{AB} = \frac{|\lambda(B - A)|}{|B - A|} = \lambda$$



$$C' = \mathbf{U}(A + \lambda(B - A)) + a$$

$$= A' + \lambda(B' - A')$$

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Projective Geometry

- Euclidean and affine models have a number of awkward features:
  - The origin is a special point
  - Parallel lines are special cases – they do not meet at a point
  - Transformations are not **linear**
- Projective geometry resolves all of these such that, for the plane
  - Any two points define a line
  - Any two lines define a point

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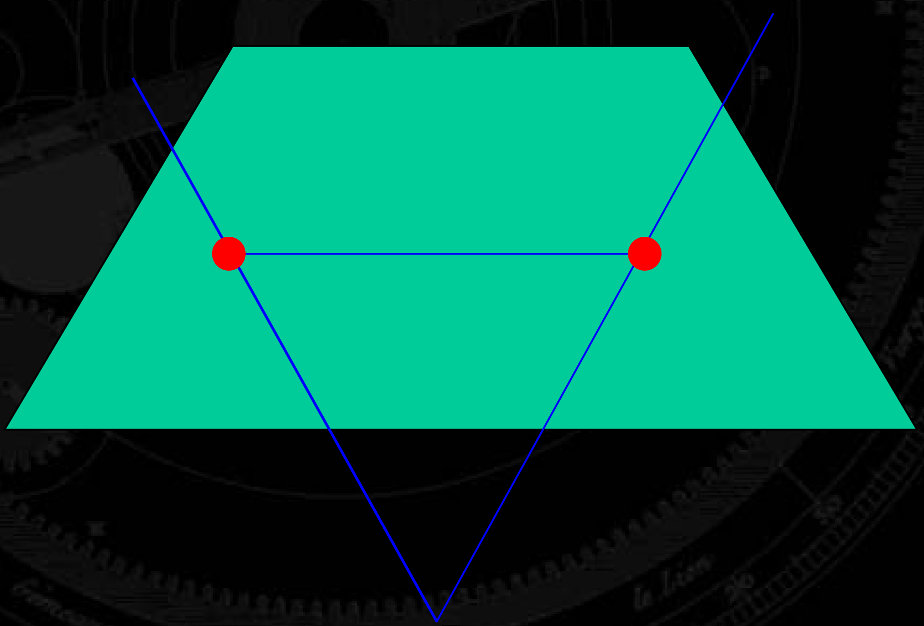
The Projective Plane

- Represent points in the plane with lines in 3D
- Defines **homogeneous coordinates**

$$(x, y) \mapsto [a, b, c]$$

$$x = \frac{a}{c} \quad y = \frac{b}{c}$$



- Any multiple of ray represents same point

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Projective Lines

- Points represented with grade-1 objects
- Lines represented with grade-2 objects
- If X lies on line joining A and B must have

$$X \wedge A \wedge B = 0$$

- All info about the line encoded in the bivector $A \wedge B$
- Any two points define a line as a **blade**
- Can dualise this equation to

$$X \cdot n = 0 \quad n = IA \wedge B$$

Intersecting Lines

- 2 lines meet at a point
- Need vector from 2 planes

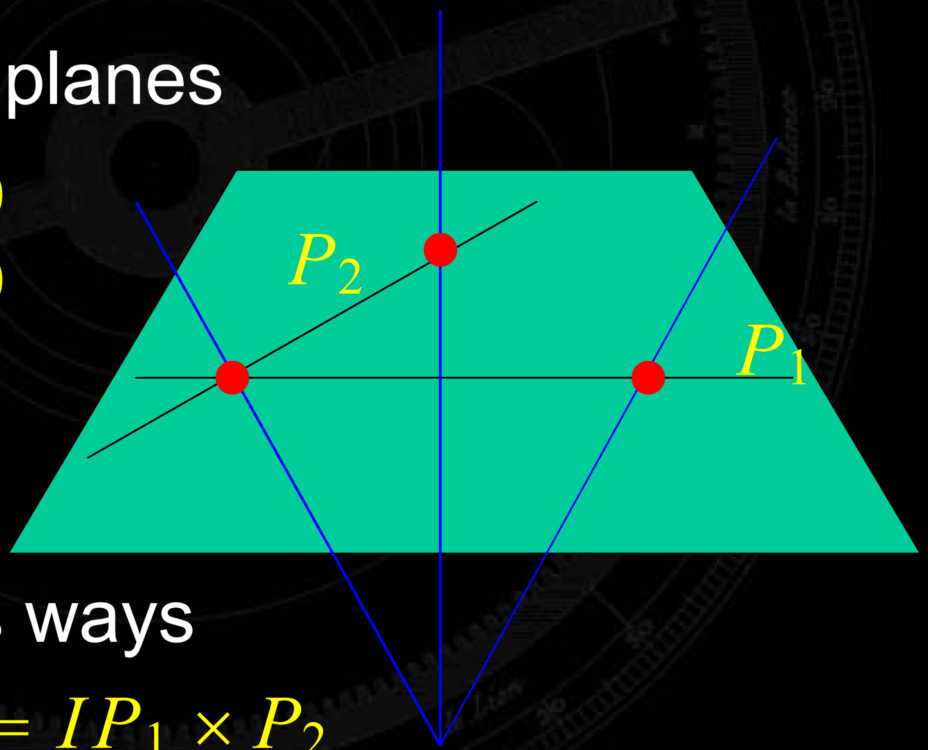
$$X \wedge P_1 = 0 \quad X \cdot p_1 = 0$$
$$X \wedge P_2 = 0 \quad X \cdot p_2 = 0$$

- Solution

$$X = I p_1 \wedge p_2$$

- Can write in various ways

$$X = P_1 \cdot p_2 = p_1 \cdot P_2 = I P_1 \times P_2$$



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Projective Transformations

- A general projective transformation takes

$$X \mapsto \mathsf{U}(X)$$

- U is an invertible linear function
- Includes all affine transformations

$$\begin{pmatrix} x+a \\ y+b \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Linearises translations
- Specified by 4 points

Invariant Properties

- Collinearity and incidence are preserved by projective transformations

$$X \wedge A \wedge B \mapsto \mathsf{F}(X) \wedge \mathsf{F}(A) \wedge \mathsf{F}(B) = \mathsf{F}(X \wedge A \wedge B)$$

- This defines the notation on the right
- But these are all pseudoscalar quantities, so related by a multiple. In fact

$$\mathsf{F}(I) = \mathsf{F}(e_1) \wedge \mathsf{F}(e_2) \wedge \mathsf{F}(e_3) = \det(\mathsf{F}) I$$

- So after the transformation

$$\mathsf{F}(X) \wedge \mathsf{F}(A) \wedge \mathsf{F}(B) = \det(\mathsf{F}) X \wedge A \wedge B = 0$$

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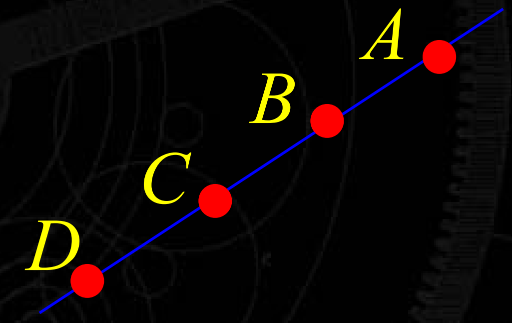
Cross Ratio

- Distances between 4 points on a line define a projective invariant

$$(ABCD) = \frac{AC\,DB}{AD\,CB}$$



- Recover distance using

$$\frac{A}{A \cdot n} - \frac{B}{B \cdot n} = \frac{1}{A \cdot n\, B \cdot n}(A \wedge B) \cdot n$$

- Vector part cancels, so cross ratio is

$$\frac{A \wedge C\; D \wedge B}{A \wedge D\; C \wedge B}$$

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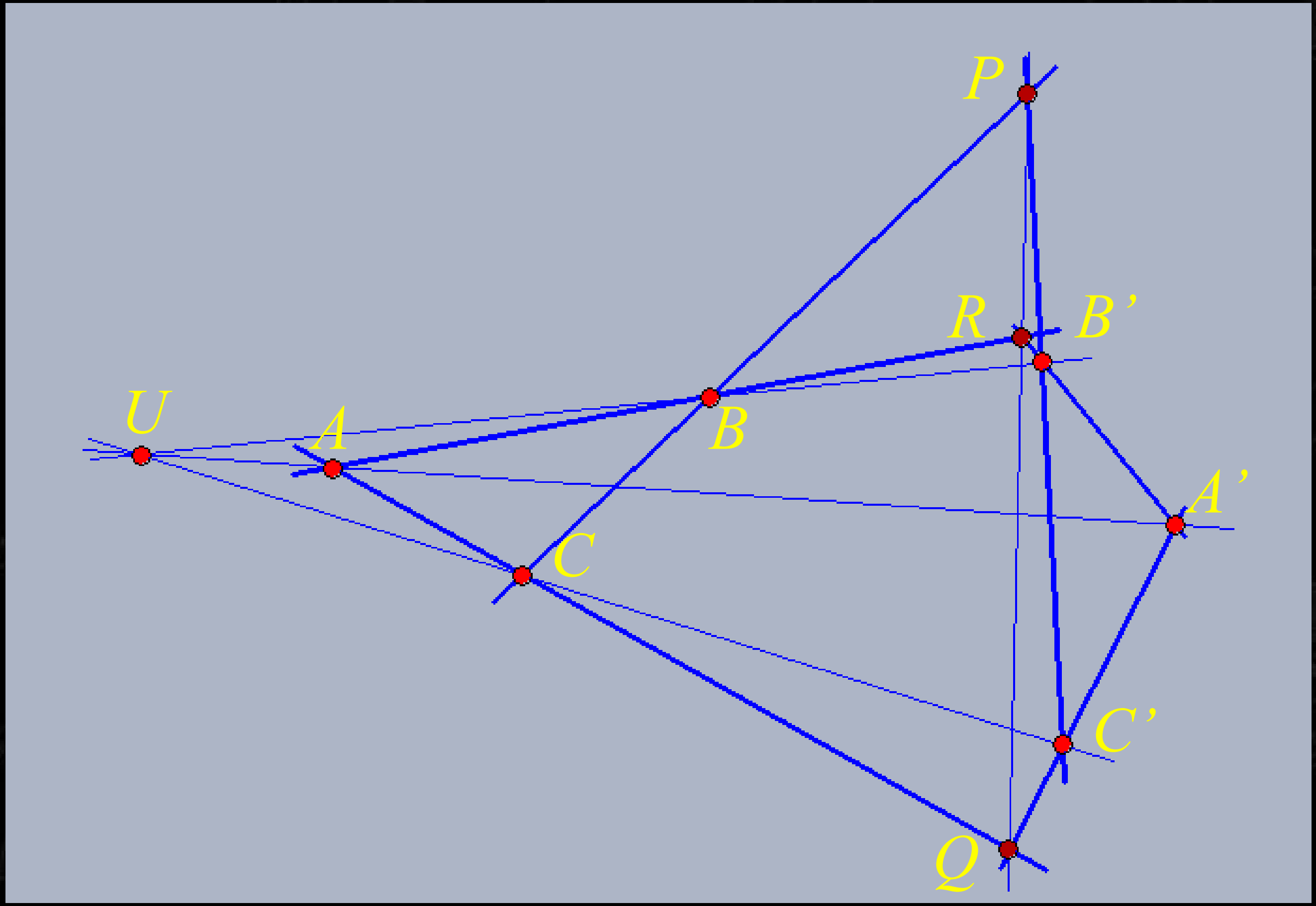
Desargues' Theorem

- Two projectively related triangles



P, Q, R collinear

Figure produced using Cinderella

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Proof

- Find scalars such that

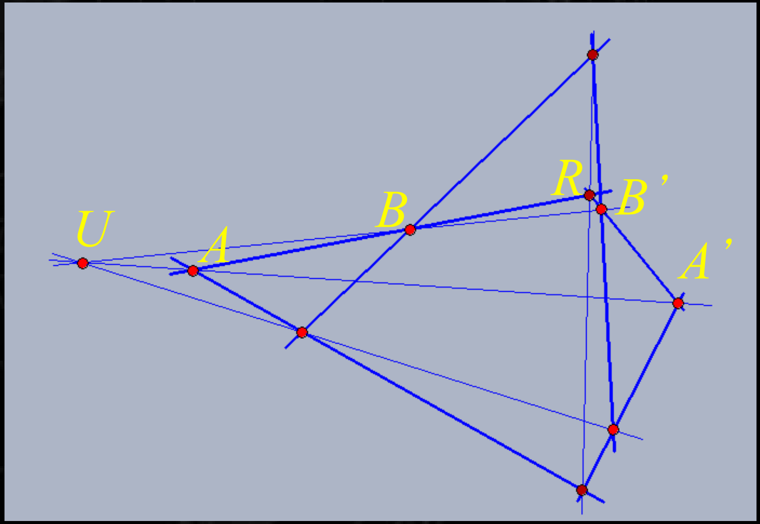
$$U = \alpha A + \alpha' A' = \beta B + \beta' B' = \gamma C + \gamma' C'$$

- Follows that

$$\alpha A - \beta B = \beta' B' - \alpha' A' = R$$

- Similarly

$$\beta B - \gamma C = P \quad \gamma C - \alpha A = Q$$



- Hence

$$P + Q + R = 0 \quad \Rightarrow \quad P \wedge Q \wedge R = 0$$

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3D Projective Geometry

- Points represented as vectors in 4D
- Form the 4D geometric algebra

$$1 \qquad e_i \qquad e_i e_j \qquad I e_i \qquad I$$

- 4 vectors, 6 bivectors, 4 trivectors and a pseudoscalar

$$I = e_1 e_2 e_3 e_4 \qquad I^2 = 1$$

- Use this algebra to handle points, lines and planes in 3D

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Line Coordinates

- Line between 2 points A and B still given by bivector $A \wedge B$
- In terms of coordinates

$$(a+e_4) \wedge (b+e_4) = a \wedge b + (a-b) \wedge e_4$$

- The 6 components of the bivector define the **Plucker** coordinates of a line
- Only 5 components are independent due to constraint

$$(A \wedge B) \wedge (A \wedge B) = 0$$

Plane Coordinates

- Take outer product of 3 vectors to encode the plane they all lie in

$$P = A \wedge B \wedge C$$

- Can write equation for a plane as

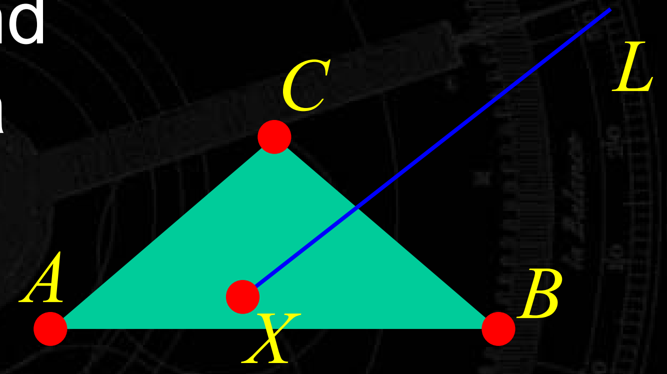
$$X \wedge P = 0 \qquad X \cdot (IP) = X \cdot p = 0$$

- Points and planes related by duality
- Lines are dual to other lines
- Use geometric product to simplify expressions with inner and outer products

Intersections

- Typical application is to find intersection of a line and a plane

$$X = (A \wedge B \wedge C) \vee L$$



- Replace meet with duality

$$X = (I\,A \wedge B \wedge C) \wedge (I\,L)\,I = \boxed{p \cdot L}$$

- Where $p = I\,A \wedge B \wedge C$
- Note the non-metric use of the inner product

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Intersections II

- Often want to know if a line cuts within a chosen simplex
- Find intersection point and solve

$$X = p \bullet L = \alpha A + \beta B + \gamma C$$

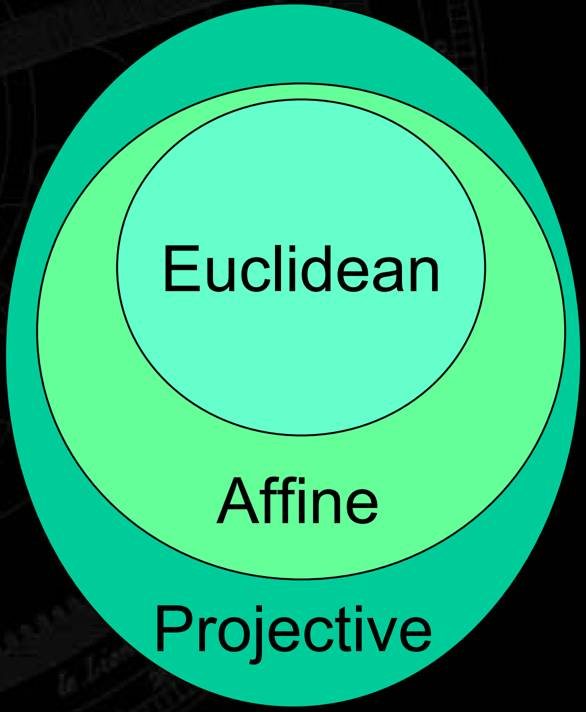
- Rescale all vectors so that 4th component is 1

$$\alpha + \beta + \gamma = 1$$

- If all of α, β, γ are positive, the line intersects the surface within the simplex

Euclidean Geometry Recovered

- Affine geometry is a subset of projective geometry
- Euclidean geometry is a subset of affine geometry
- How do we recover Euclidean geometry from projective?
- Need to find a way to impose a distance measure



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Fundamental Conic

- Only distance measure in projective geometry is the cross ratio
- Start with 2 points and form line through them
- Intersect this line with the **fundamental conic** to get 2 further points X and Y
- Form cross ratio

$$r = \frac{A \wedge X \, B \wedge Y}{A \wedge Y \, B \wedge X}$$

- Define distance by

$$d = \ln(\alpha r)$$

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Cayley-Klein Geometry

- Cayley & Klein found that different fundamental conics would give Euclidean, spherical and hyperbolic geometries
- United the main classical geometries
- But there is a major price to pay for this unification:
  - All points have complex coordinates!
- Would like to do better, and using GA we can!

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Further Information

- All papers on Cambridge GA group website: www.mrao.cam.ac.uk/~clifford
- Applications of GA to computer science and engineering are discussed in the proceedings of the AGACSE 2001 conference. www.mrao.cam.ac.uk/agacse2001
- IMA Conference in Cambridge, 9th Sept 2002
- 'Geometric Algebra for Physicists' (Doran + Lasenby). Published by CUP, soon.

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