Geometric Algebra 1

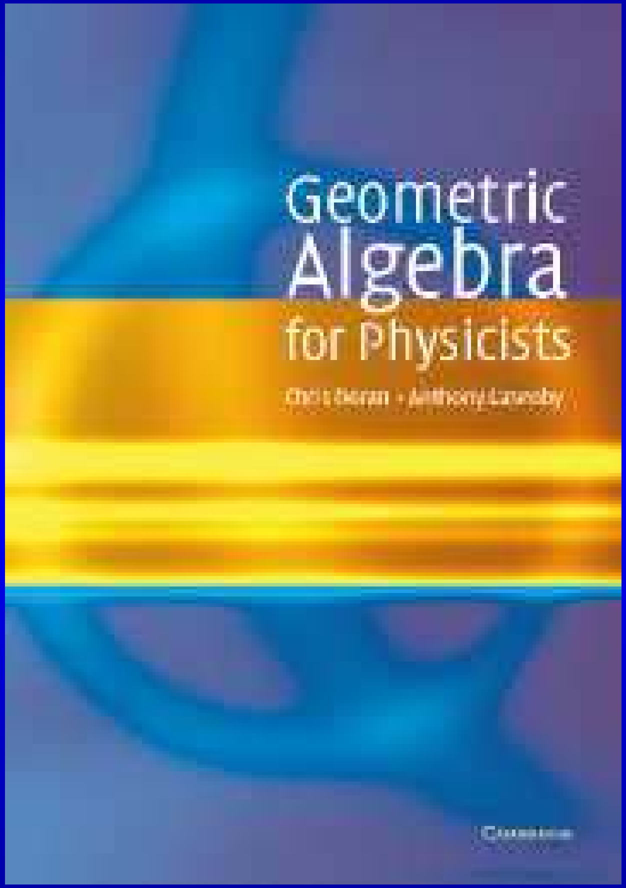
Chris Doran
Astrophysics Group
Cavendish Laboratory
Cambridge, UK

EPSRC

UNIVERSITY OF CAMBRIDGE

Resources

- A complete lecture course, including handouts, overheads and papers available from www.mrao.cam.ac.uk/~Clifford
- *Geometric Algebra for Physicists* out in March (C.U.P.)
- David Hestenes' website modelingnts.la.asu.edu



What is Geometric Algebra?

- Geometric Algebra is a universal Language for physics based on the mathematics of *Clifford Algebra*
- Provides a new product for vectors
- Generalizes complex numbers to arbitrary dimensions
- Treats points, lines, planes, etc. in a single algebra
- Simplifies the treatment of rotations
- Unites Euclidean, affine, projective, spherical, hyperbolic and conformal geometry

Grassmann

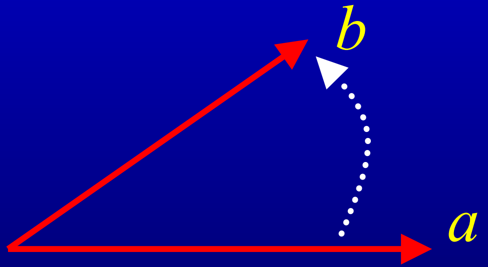
German schoolteacher
1809-1877

Published the *Lineale Ausdehnungslehre* in 1844

Introduced the *outer product*

$$a \wedge b = -b \wedge a$$

Encodes a plane segment



2D Outer Product

- Antisymmetry implies $a \wedge a = -a \wedge a = 0$
- Introduce basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$

$$\boldsymbol{a} = a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2 \qquad \boldsymbol{b} = b_1 \boldsymbol{e}_1 + b_2 \boldsymbol{e}_2$$

- Form product

$$\begin{aligned} \boldsymbol{a} \wedge \boldsymbol{b} &= a_1 b_2 \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 + a_2 b_1 \boldsymbol{e}_2 \wedge \boldsymbol{e}_1 \\ &= (a_1 b_2 - b_2 a_1) \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \end{aligned}$$

- Returns area of the plane + orientation.
- Result is a bivector
- Extends (antisymmetry) to arbitrary vectors

Complex Numbers

- Already have a product for vectors in 2D
- Length given by aa^*
- Suggests forming

$$
\begin{aligned}
ab^* &= (a_1 + a_2 i)(b_1 - b_2 i) \\
&= (a_1 b_1 + a_2 b_2) - (a_1 b_2 - a_2 b_1) i
\end{aligned}
$$

- Complex multiplication forms the inner and outer products of the underlying vectors!
- Clifford generalised this idea

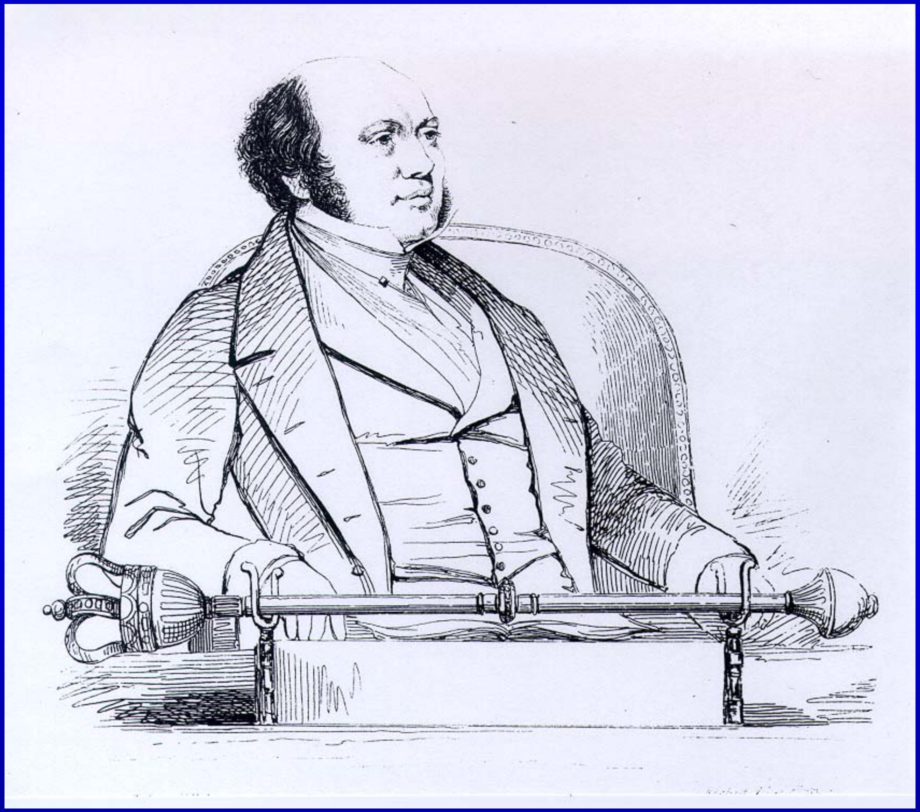
Hamilton

Introduced his quaternion algebra in 1844

$$i^2 = j^2 = k^2 = ijk = -1$$

Generalises complex arithmetic to 3 (4?) dimensions

Very useful for rotations, but confusion over the status of *vectors*



Quaternions

- Introduce the two quaternion 'vectors'

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$$
$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$$

- Product of these is $\boldsymbol{ab} = c_0 + \boldsymbol{c}$
- where c_0 is minus the scalar product and

$$\boldsymbol{c} = (a_2b_3 - b_2a_3)\boldsymbol{i} + (a_3b_1 - b_3a_1)\boldsymbol{j} + (a_1b_2 - b_1a_2)\boldsymbol{k}$$

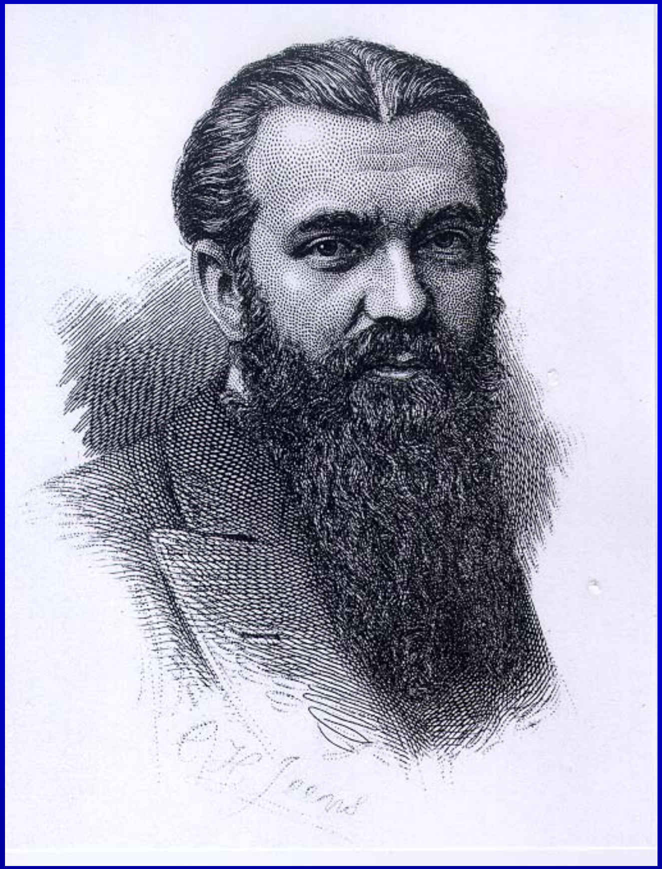
W.K. Clifford 1845-1879

Introduced the *geometric product*

$$ab = a \cdot b + a \wedge b$$

Product of two vectors returns the sum of a scalar and a bivector

Think of this sum as like the real and imaginary parts of a complex number



History

- Foundations of geometric algebra (GA) were laid in the 19th Century
- Key figures: Hamilton, Grassmann, Clifford and Gibbs
- Underused (associated with quaternions)
- Rediscovered by Pauli and Dirac for quantum theory of spin
- Developed by mathematicians (Atiyah etc.) in the 50s and 60s
- Reintroduced to physics in the 70s by David Hestenes

Properties

- Geometric product is associative and distributive

$$a(bc) = (ab)c = abc$$

$$a(b + c) = ab + ac$$

- Square of any vector is a scalar

$$(a + b)^2 = a^2 + b^2 + ab + ba$$

- Define the inner (scalar) and outer (exterior) products

$$a \cdot b = \frac{1}{2}(ab + ba) \qquad a \wedge b = \frac{1}{2}(ab - ba)$$

2D Algebra

- Orthonormal basis is 2D

$$e_1 \cdot e_1 = e_2 \cdot e_2 = 1 \qquad e_1 \cdot e_2 = 0$$

- Parallel vectors commute

$$e_1 e_1 = e_1 \cdot e_1 + e_1 \wedge e_1 = 1$$

- Orthogonal vectors anticommute since

$$e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = -e_2 \wedge e_1 = -e_2 e_1$$

- Unit bivector has negative square

$$(e_1 \wedge e_2)^2 = (e_1 e_2)(e_1 e_2) = e_1 e_2 (-e_2 e_1)$$
$$= -e_1 e_1 = -1$$

2D Basis

- Build into a basis for the algebra

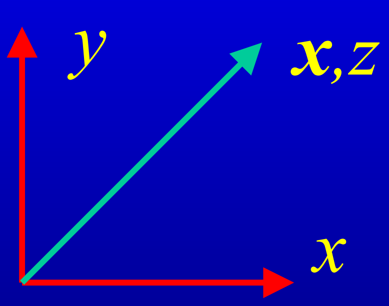
$$1 \qquad \{\boldsymbol{e}_1, \boldsymbol{e}_2\} \qquad \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 = I$$

1 scalar　　2 vectors　　1 bivector

- Even grade objects form complex numbers
- Map between vectors and complex numbers

$$x + Iy = \boldsymbol{e}_1(x\boldsymbol{e}_1 + y\boldsymbol{e}_2) = \boldsymbol{e}_1\boldsymbol{x}$$

$$z^* = x - Iy = \boldsymbol{x}\boldsymbol{e}_1$$



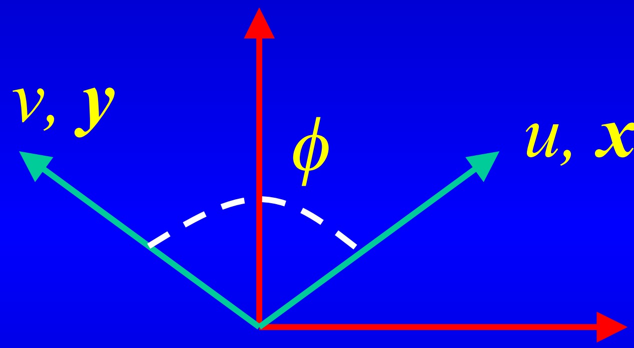
2D Rotations

- In 2D vectors can be rotated using complex phase rotations

$$v = \exp(i\phi)\, u$$

$$u = e_1 x$$

$$v = e_1 y$$



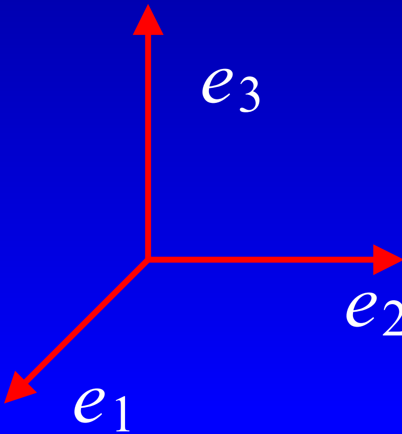
$$y = e_1 v = e_1 \exp(I\phi)\, e_1 x$$

- But $e_1 I e_1 = e_1(e_1 e_2)e_1 = e_2 e_1 = -I$
- Rotation $y = \exp(-I\phi)\, x = x \exp(I\phi)$

3 Dimensions

- Now introduce a third vector

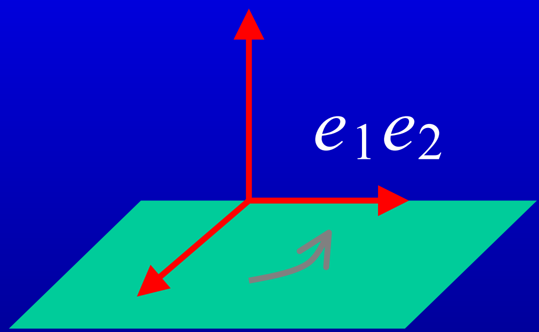
  $$\{e_1, e_2, e_3\}$$

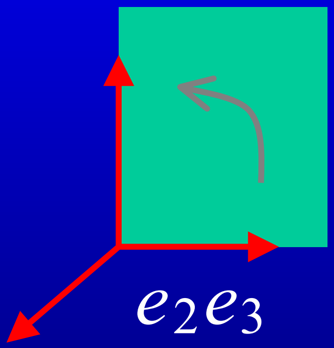


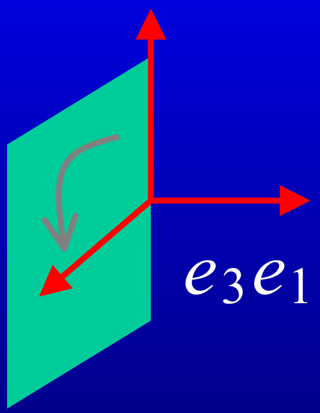
- These all anticommute

  $e_1 e_2 = -e_2 e_1$ *etc.*

- Have 3 bivectors now: $\{e_1 e_2, e_2 e_3, e_3 e_1\}$







Bivector Products

- Various new products to form in 3D
- Product of a vector and a bivector

$$e_1(e_1e_2) = e_2 \qquad e_1(e_2e_3) = e_1e_2e_3 = I$$

- Product of two perpendicular bivectors:

$$(e_2e_3)(e_3e_1) = e_2e_3e_3e_1 = e_2e_1 = -e_1e_2$$

- Set

$$i = e_2e_3, \qquad j = -e_3e_1, \qquad k = e_1e_2$$

- Recover quaternion relations

$$i^2 = j^2 = k^2 = ijk = -1$$

3D Pseudoscalar

- 3D Pseudoscalar defined by $I = e_1 e_2 e_3$
- Represents a directed volume element
- Has negative square

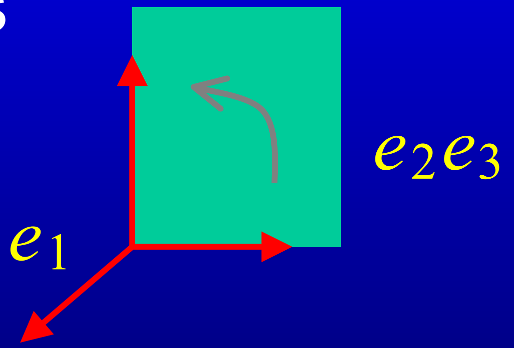
$$I^2 = e_1 e_2 e_3 e_1 e_2 e_3 = e_2 e_3 e_2 e_3 = -1$$

- Commutes with all vectors

$$e_1 I = e_1 e_1 e_2 e_3 = -e_1 e_2 e_1 e_3 = e_1 e_2 e_3 e_1 = I e_1$$

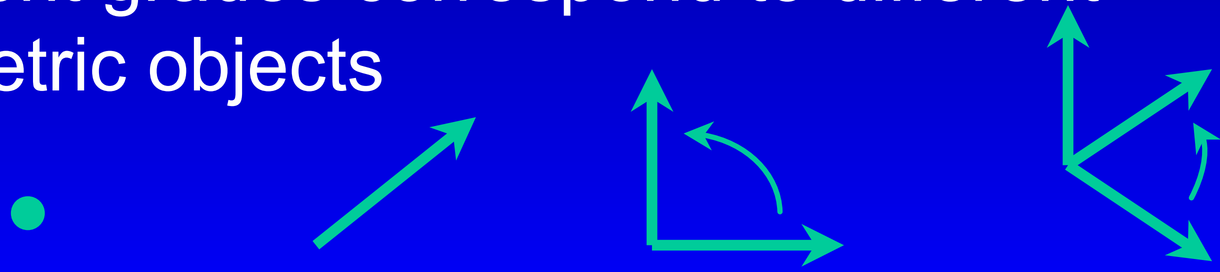
- Interchanges vectors and planes

$$e_1 I = e_2 e_3$$
$$I e_2 e_3 = -e_1$$



3D Basis

Different grades correspond to different geometric objects



Grade 0 Scalar	Grade 1 Vector	Grade 2 Bivector	Grade 3 Trivector
1	e_1, e_2, e_3	$e_1 e_2, e_2 e_3, e_3 e_1$	I

Generators satisfy Pauli relations $e_i e_j = \delta_{ij} + \epsilon_{ijk} I e_k$

Recover vector cross product $a \times b = -I a \wedge b$

Reflections

- Build rotations from reflections
- Good example of geometric product – arises in operations

$$a_{\parallel} = (a \cdot n)n$$
$$a_{\perp} = a - (a \cdot n)n$$

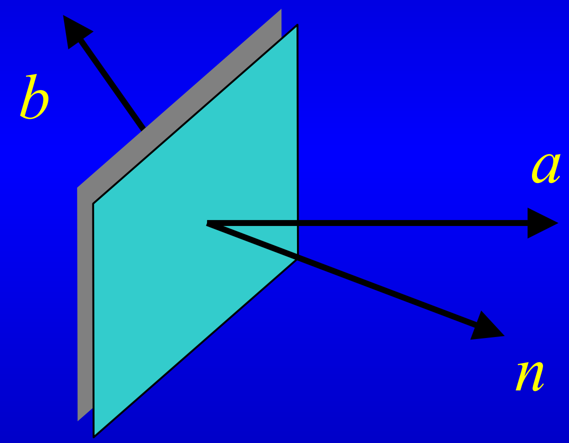


Image of reflection is

$$b = a_{\perp} - a_{\parallel} = a - 2(a \cdot n)n$$
$$= a - (an + na)n = -nan$$

Rotations

- Two rotations form a reflection

$$a \to -m(-nan)m = mnanm$$

- Define the rotor $R = mn$
- This is a geometric product! Rotations given by $a \to RaR^\dagger$ $\quad R^\dagger = nm$
- Works in spaces of any dimension or signature
- Works for all grades of multivectors $A \mapsto RAR^\dagger$
- More efficient than matrix multiplication

3D Rotations

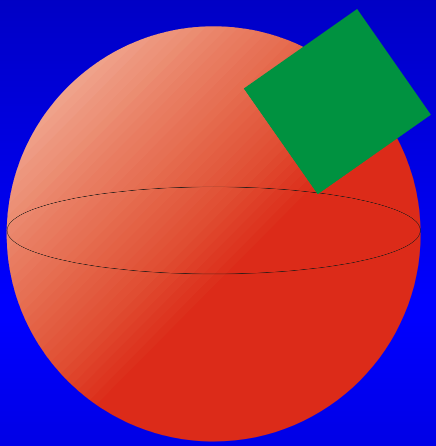
- Rotors even grade (scalar + bivector in 3D)
- Normalised: $RR^{\dagger} = mnnm = 1$
- Reduces d.o.f. from 4 to 3 – enough for a rotation
- In 3D a rotor is a normalised, even element

$$R = \alpha + B \qquad RR^{\dagger} = \alpha^2 - B^2 = 1$$

- Can also write $R = \exp(-B/2)$
- Rotation in plane B with orientation of B
- In terms of an axis $R = \exp(-\theta I n/2)$

Group Manifold

- Rotors are elements of a 4D space, normalised to 1
- They lie on a 3-sphere
- This is the group manifold
- Tangent space is 3D
- Can use Euler angles



$$R = \exp(-e_1 e_2 \phi/2)\exp(-e_2 e_3 \theta/2)\exp(-e_1 e_2 \psi/2)$$

- Rotors R and $-R$ define the same rotation
- Rotation group manifold is more complicated

Lie Groups

- Every rotor can be written as $R = \pm \exp(-B/2)$
- Rotors form a continuous Lie group
- Bivectors form a Lie algebra under the commutator product
- All finite Lie groups are rotor groups
- All finite Lie algebras are bivector algebras
- (Infinite case not fully clear, yet)
- In conformal case starting point of screw theory (Clifford, 1870s)!

Rotor Interpolation

- How do we interpolate between 2 rotations?
- Form path between rotors

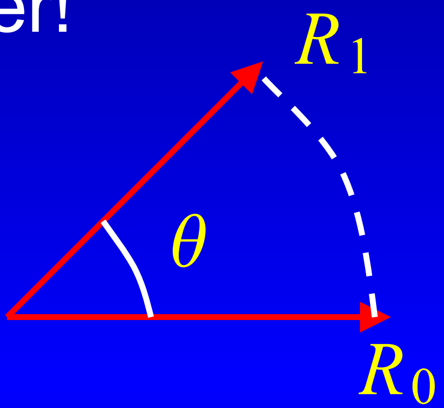
$$R(0) = R_0$$
$$R(1) = R_1$$

$$R(\lambda) = R_0 \exp(\lambda B)$$

- Find B from $\exp(B) = R_0^{\dagger} R_1$
- This path is invariant. If points transformed, path transforms the same way
- Midpoint simply $R(1/2) = R_0 \exp(-B/2)$
- Works for all Lie groups

Interpolation 2

- For rotors in 3D can do even better!
- View rotors as unit vectors in 4D
- Path is a circle in a plane
- Use simple trig' to get SLERP



$$R(\lambda) = \frac{1}{\sin(\theta)}\left(\sin((1-\lambda)\theta)R_0 + \sin(\lambda\theta)R_1\right)$$

- For midpoint add the rotors and normalise!

$$R(1/2) = \frac{\sin(\theta/2)}{\sin(\theta)}(R_0 + R_1)$$

Exercises

Verify the following

	1	e_1	e_2	e_3	e_2e_3	e_3e_1	e_1e_2	$e_1e_2e_3$
1	1	e_1	e_2	e_3	Ie_1	Ie_2	Ie_3	I
e_1	e_1	1	Ie_3	$-Ie_2$	I	$-e_3$	e_2	Ie_1
e_2	e_2	$-Ie_3$	1	Ie_1	e_3	I	$-e_1$	Ie_2
e_3	e_3	Ie_2	$-Ie_1$	1	$-e_2$	e_1	I	Ie_3
e_2e_3	Ie_1	I	$-e_3$	e_2	-1	$-Ie_3$	Ie_2	$-e_1$
e_3e_1	Ie_2	e_3	I	$-e_1$	Ie_3	-1	$-Ie_1$	$-e_2$
e_1e_2	Ie_3	$-e_2$	e_1	I	$-Ie_2$	Ie_1	-1	$-e_3$
$e_1e_2e_3$	I	Ie_1	Ie_2	Ie_3	$-e_1$	$-e_2$	$-e_3$	-1

$$\boldsymbol{a} \times \boldsymbol{b} = -I\boldsymbol{a} \wedge \boldsymbol{b}$$
$$= \boldsymbol{b} \cdot (I\boldsymbol{a}) = -\boldsymbol{a} \cdot (I\boldsymbol{b})$$